# Operator Theory

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April 15, 2021

# Chapter II FUNDAMENTAL PROPERTIES OF BOUNDED LINEAR OPERATORS

# 2.4 Spectrum of an operator

# 2.4.1 Two kinds of classifications of Spectrum

**Definition**

An operator T on a Hilbert space H is said to be an invertible operator if there exists an operator S such that ST=TS=I, where I is the identity operator. We write $S = T^{-1}$ and $T^{-1}$ is called the inverse of T.

## Theorem

If T is an operator and c is a positive number such that $\|Tx\| \geq c\|x\|$ for every vector $x \in H$, then R(T), the range of T is closed.

## Proof

Let T be an operator and c is positive number such that
$\|Tx\| \geq c\|x\|$ for every vector $x \in H$ (1)
To prove that the range of T, R(T) is closed.
Let $y_0$ be a limit point of R(T), then there exists sequence $\{x_n\}$ in H,
such that $y_n = Tx_n,\ n = 1, 2 \dots$ and $y_n \to y_0$ (2)
consider

$$\begin{aligned} \|y_n - y_m\| &= \|Tx_n - Tx_m\| \\ &= \|T(x_n - x_m)\| \\ &\geq c\|x_n - x_m\|[by(1)] \\ \Rightarrow \|x_n - x_m\| &\leq \frac{1}{c}\|y_n - y_m\| \to 0[by(2)] \end{aligned}$$

$\therefore \{x_n\}$is a cauchy sequence in H.

Since H is a Hilbert space, there exists $x_0 \in H$ such that $x_n \to x_0$.
Consider

$$\begin{aligned} \|y_0 - Tx_0\| &\leq \|y_0 - Tx_n\| + \|Tx_n - Tx_0\| \\ &= \|y_0 - y_n\| + \|T(x_n - x_0)\| \\ &\leq \|y_0 - y_n\| + \|T\|\|x_n - x_0\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

$(\because y_n \to y_0 \ \& \ x_n \to x_0)$

$\Rightarrow y_0 = Tx_0$
$\Rightarrow y_0 \in R(T)$
Hence R(T) contains all its limit points.
Hence R(T) is closed.

## Theorem

An operator T on a Hilbert space H is invertible if and only if the following (i)and(ii) hold

(i) There exists a positive number c such that

$$\|Tx\| \geq c\|x\| \text{holds for any } x \in H.$$

(ii) R(T), the range of T is dense in H, i.e $\overline{R(T)} = H$.

## Proof

Assume that the operator T on H is invertible.
For any $x \in H$, $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\|$
$\Rightarrow \|Tx\| \geq \dfrac{1}{\|T^{-1}\|}\|x\|$
Hence (i) holds.
Let $y \in H$ and $x = T^{-1}y$
then $y = Tx \in R(T)$
Hence H=R(T)
Hence (ii) holds.

Conversely assume that, the following (i)and (ii) holds

(i) There exists a positive number c such that

$$\|Tx\| \geq c\|x\| \text{holds for any } x \in H.$$

(ii) R(T), the range of T is dense in H, i.e $\overline{R(T)} = H$.

To prove that T is invertible

By (ii), $\overline{R(T)} = H$ (1)

By (i) and the following theorem

" If T is an operator and c is a positive number such that $\|Tx\| \geq c\|x\|$ for every vector $x \in H$, then R(T), the range of T is closed."

$\overline{R(T)} = R(T)$ (2)

From (1) and (2),

$$R(T) = H$$

Hence T is onto.

Let $Tx_1 = Tx_2$ for $x_1,\ x_2 \in H$

Then $0 = \|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \geq c\|x_1 - x_2\|$ [by (i)]

$\Rightarrow \|x_1 - x_2\| = 0 \quad \Rightarrow x_1 = x_2$. Hence T is one to one.

Therefore , for every vector $y \in H$, there exists a unique $x \in H$ such that $y = Tx$
Define $S : H \to H$ as $Sy = x$.
Let $y_1, y_2 \in H$ and $\alpha$ be scalar. Then there exists $x_1, x_2 \in H$ such that $y_1 = Tx_1$ &$y_2 = Tx_2$ then $Sy_1 = x_1$ and $Sy_2 = x_2$

$$\begin{aligned} \therefore S(y_1 + y_2) &= S(Tx_1 + Tx_2) \\ &= ST(x_1 + x_2) \\ &= x_1 + x_2 \\ &= Sy_1 + Sy_2 \end{aligned}$$

$$\begin{aligned} S(\alpha y_1) &= S(\alpha Tx_1) \\ &= S(T(\alpha x_1)) \\ &= \alpha x_1 \\ &= \alpha Sy_1 \end{aligned}$$

Hence S is linear.

Let $y \in H$ then there exists $x \in H$ such that $Tx = y$ and $Sy = x$

$\therefore \|y\| = \|Tx\| \geq c\|x\| = c\|Sy\|$ [by (1)]

$\Rightarrow \|Sy\| \leq \frac{1}{c}\|y\|, \forall y \in H.$

Hence S is bounded.

Hence S is an operator such that $\|S\| \leq \frac{1}{c}$ and $STx = Sy = x, \ \forall x \in H.$

and $TSy = Tx = y, \ \forall y \in H$

Hence ST=TS=I

Hence S is the inverse of T.

i.e T is invertible.

Hence the theorem

## Corollary

If $T \geq cI$ for some $c > 0$, then T is invertible

## Proof

Let $T \geq cI$ for some $c > 0$
By theorem,
"An operator T on a Hilbert space H is invertible if and only if the following (i)and(ii) hold

(i) There exists a positive number c such that

$$\|Tx\| \geq c\|x\| \text{holds for any } x \in H.$$

(ii) R(T), the range of T is dense in H, i.e $\overline{R(T)} = H$, "

it is enough to prove (i) and (ii) for T.
By Schwarz inequality,
$< Tx, x > \leq \|Tx\|\|x\|, \forall x \in H.$

$$\begin{aligned}
\therefore \|Tx\|\|x\| &\geq < Tx, x > \\
&\geq < cIx, x > \\
&= c < x, x > \\
&= c\|x\|^2 \\
\Rightarrow \|Tx\| &\geq c\|x\|, \forall x \in H.
\end{aligned}$$

Hence (i) holds.
Let y be orthogonal to R(T).

$$\begin{aligned}
\Rightarrow < y, Tx > &= 0, \ \forall x \in H \\
\Rightarrow < Ty, Tx > &= 0, \ \forall x \in H \\
\Rightarrow Ty &= 0 \\
\therefore 0 = < Ty, y > &\geq < cy, y > = c\|y\|^2 \\
&\Rightarrow \|y\| = 0 \\
&\Rightarrow y = 0
\end{aligned}$$

Hence if y is orthogonal to R(T), then y = 0
Hence R(T) is dense in H
i.e (ii) holds
Hence T is invertible.
Hence the theorem.

**Definition**

Let T be an operator on a Hilbert space H.

1. $\sigma(\mathrm{T})$ of T is defined as

$$\sigma(\mathrm{T}) = \{\lambda \in \mathrm{C}/\mathrm{T} - \lambda \text{is not invertible}\}$$

and $\sigma(\mathrm{T})$ is said to be the spectrum of T.

2. $\rho(\mathrm{T})$ of T is defined as

$$\rho(\mathrm{T}) = \mathrm{C} - \sigma(\mathrm{T})$$

and $\rho(\mathrm{T})$ is said to be the resolvent of T.

## Definition

$\sigma(T)$ of T can be divided into the following three parts

(i)

$$P_\sigma(T) = \{\lambda \in C / \text{there exists } x \neq 0 \text{such that} Tx = \lambda x\}$$

and $P_\sigma(T)$ is said to be the point spectrum of T.

(ii)

$$C_\sigma(T) = \{\lambda \in C / (T - \lambda)^{-1} \text{is unbounded and } \overline{R(T - \lambda)} = H\}$$

and $C_\sigma(T)$ is said to be the continuous spectrum of T.

(iii)

$$R_\sigma(T) = \{\lambda \in C / (T - \lambda)^{-1} \text{exists and } \overline{R(T - \lambda)} \subseteq H\}$$

and $R_\sigma(T)$ is said to be the residual spectrum of T.

$$\begin{cases} (a)(T-\lambda)^{-1}\text{does not exist} \iff \lambda \in P_\sigma(T) \\ (b)(T-\lambda)^{-1}\text{exists} \begin{cases} (b_1)\ (T-\lambda)^{-1}\text{is bounded} \begin{cases} (b_{11})\ \overline{R(T-\lambda)} = H \iff \lambda \in \\ (b_{12})\ \overline{R(T-\lambda)} \subsetneq H \iff \lambda \in \end{cases} \\ (b_2)\ (T-\lambda)^{-1}\text{is unbounded} \begin{cases} (b_{21})\ \overline{R(T-\lambda)} = H \iff \lambda \\ (b_{22})\ \overline{R(T-\lambda)} \subsetneq H \iff \lambda \end{cases} \end{cases} \end{cases}$$

### Proposition 1

$\sigma(T) = P_\sigma(T) \cup C_\sigma(T) \cup R_\sigma(T)$ where $P_\sigma(T),\ C_\sigma(T),\ R_\sigma(T)$ are mutually disjoint parts of $\sigma(T)$

## Definition

$A_\sigma(T) = \{\lambda \in C/\text{there exists a sequence of unit vectors}\{x_n\}\text{such that } \|x_n\| = 1 \text{ and } \|Tx_n - \lambda x_n\| \to 0\}$ and $A_\sigma(T)$ is said to be the approximation point spectrum of T.

$$\Gamma(T) = \{\lambda \in C/\overline{R(T-\lambda)} \subsetneq H\}$$

and $\Gamma(T)$ is said to be the compression spectrum of T.

## Proposition 2

$\sigma(T) = A_\sigma(T) \cup \Gamma(T)$ holds, where $A_\sigma(T)$ and $\Gamma(T)$ are not necessarily disjoint parts of $\sigma(T)$

By the theorem,
" An operator T on a Hilbert space H is invertible if and only if the following (i)and(ii) hold

(i) There exists a positive number c such that

$$\|Tx\| \geq c\|x\| \text{holds for any } x \in H.$$

(ii) R(T), the range of T is dense in H, i.e $\overline{R(T)} = H$. "

$T - \lambda$ is invertible $\iff$ (i) There exists a positive number c such that

$$\|(T - \lambda)x\| \geq c\|x\| \text{holds for any } x \in H.$$

and (ii) $R(T - \lambda)$, the range of $T - \lambda$ is dense in H, i.e $\overline{R(T - \lambda)} = H$.
$T - \lambda$ is not invertible $\iff$ either (i)or(ii) is not satisfied.
Hence $\sigma(T) = A_\sigma(T) \cup \Gamma(T)$.

## Theorem

If T is an operator such that $\|I - T\| < 1$, then T is invertible.

## Proof

Let T be an operator such that $\|I - T\| < 1$
Let $\|I - T\| = 1 - \alpha$, where $0 < \alpha < 1$ then

$$\begin{aligned} \|Tx\| &= \|x - (x - Tx)\| \\ &\geq \|x\| - \|(I - T)x\| \\ &\geq \|x\| - (1 - \alpha)\|x\| \\ &= \alpha\|x\| \end{aligned}$$

Therefore $\|Tx\| \geq \alpha\|x\|, \ \forall x \in H$ (1)
Let $y \in H$ and $\delta = \inf\{\|y - x\|/x \in R(T)\}$
If $T > 0$, then there exists a vector $x \in R(T)$, such that
$(1 - \alpha)\|y - x\| < \delta$
Since $x, \ T(y - x) \in R(T), \ x + T(y - x) \in R(T)$

$$\begin{aligned} \therefore \delta &\leq \|y - \{x + T(y - x)\} \\ &= \|(y - x) - T(y - x)\| \\ &\leq \|(I - T)(y - x)\| = (1 - \alpha)\|y - x\| < \delta \end{aligned}$$

Which is a contradiction.
Hence $\delta = 0$
$\inf\{|y - x\|/x \in R(T)\} = 0$
$\Rightarrow \ y \in \overline{R(T)} = H$ (2)

Hence by theorem,
An operator T on a Hilbert space H is invertible if and only if the following (i)and(ii) hold

(i) There exists a positive number c such that

$$\|Tx\| \geq c\|x\| \text{holds for any } x \in H.$$

(ii) R(T), the range of T is dense in H, i.e $\overline{R(T)} = H$. "

T is invertible.

## Theorem

If T is an operator, then $\sigma(T)$ is a compact subset of the complex plane if $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$.

## Proof

$\sigma(T) = \{\lambda \in C / T - \lambda \text{is not invertible}\}$
Claim: $\sigma(T)$ is compact subset of C.
To prove this, it is sufficient to prove that $\sigma(T)$ is closed subset of C
(or) $\rho(T) = C - \sigma(T)$ is an open subset of C.
Let $\lambda_0 \in \rho(T) = C - \sigma(T)$
$\Rightarrow \ \lambda_0 \notin \sigma(T)$
$\Rightarrow T - \lambda_0$ is invertible. Then

$$\begin{aligned}
\|I - (T-\lambda_0)^{-1}(T-\lambda)\| &= \|(T-\lambda_0)^{-1}(T-\lambda_0) - (T-\lambda_0)^{-1}(T-\lambda)\| \\
&= \|(T-\lambda_0)^{-1}\{(T-\lambda_0) - (T-\lambda)\}\| \\
&= \|(T-\lambda_0)^{-1}(\lambda-\lambda_0)\| \\
&= \|(T-\lambda_0)^{-1}\||(\lambda-\lambda_0)|
\end{aligned}$$

$\therefore$ whenever $|\lambda - \lambda_0| < \dfrac{1}{\|(\mathrm{T} - \lambda_0)^{-1}\|}$,
$\|\mathrm{I} - (\mathrm{T} - \lambda_0)^{-1}(\mathrm{T} - \lambda)\| < 1$
$\Rightarrow (\mathrm{T} - \lambda_0)^{-1}(\mathrm{T} - \lambda)$ is invertible.
$(\mathrm{T} - \lambda)$ is also invertible whenever $|\lambda - \lambda_0|$ is sufficiently small
Hence $\rho(\mathrm{T})$ is an open subset of C.
Hence $\sigma(\mathrm{T})$ is closed subset of C
Hence $\sigma(\mathrm{T})$ is compact.

To prove that $|\lambda| \leq \|T\|$, for $\lambda \in \sigma(T)$
Let $\lambda \in \sigma(T)$
If $|\lambda| > \|T\|$, then $\|(\frac{T}{\lambda})\| <$ i.e $\|I - (I - \frac{T}{\lambda})\| < 1$
$\Rightarrow I - \frac{T}{\lambda}$ is invertible.
$\Rightarrow \frac{1}{\lambda}(\lambda - T)$ is invertible.
$\Rightarrow T - \lambda$ is invertible.
$\lambda \notin \sigma(T)$
which is a contradiction
Hence if $\lambda \in \sigma(T)$, then $|\lambda\| \leq \|T\|$

## Theorem

If T is an operator, then $A_\sigma(T)$ is a compact subset of the complex plane.

## Proof

Let $\lambda_0 \notin A_\sigma(T)$
Then there exists a positive number $\epsilon$ such that $\|Tx - \lambda_0 x\| \geq \epsilon$, for all unit vector x
$\therefore$ If x is a unit vector and if $|\lambda - \lambda_0| < \epsilon/2$, then

$$\begin{aligned}
\|Tx - \lambda\| &= \|Tx - \lambda_0 x + \lambda_0 x - \lambda x\| \\
&= \|Tx - \lambda_0 x + (\lambda_0 - \lambda)x\| \\
&\geq \|Tx - \lambda_0 x\| - \|(\lambda_0 - \lambda)x\| \\
&= \|Tx - \lambda_0 x\| - |\lambda_0 - \lambda|\|x\| \\
&= \|Tx - \lambda_0 x\| - |\lambda_0 - \lambda| \\
&\geq \epsilon - \epsilon/2 = \epsilon/2
\end{aligned}$$

$\Rightarrow \|Tx - \lambda x\| \geq \epsilon/2$, whenever $|\lambda - \lambda_0| < \epsilon/2$
i.e $\lambda \notin A_\sigma(T)$, whenever $|\lambda - \lambda_0| < \epsilon/2$
Hence complement of $A_\sigma(T)$ is open.
Hence $A_\sigma(T)$ is closed subset of the complex plane.
Hence $A_\sigma(T)$ is compact.

**Theorem**

If T is a self adjoint operator on a Hilbert space H, then all the eigen values of T are real number.

**Proof**

Let T be a self-adjoint operator on a Hilbert space H.
Let $\lambda$ be an eigen value of T, then $Tx = \lambda x$, for some $x \in H$ with $x \neq 0$
Consider

$$\begin{aligned} \lambda < x, x > &= < \lambda x, x > \\ &= < Tx, x > \\ &= < x, T^* x > \\ &= < x, Tx > \\ &= < x, \lambda x > \\ &= \bar{\lambda} < x, x > \\ \lambda &= \bar{\lambda} \end{aligned}$$

i.e $\lambda$ is real. Hence all the eigenvalues of T are real numbers.

## Theorem

If T is a self-adjoint operator on a Hilbert space H. Then $T + iI$ has a bounded inverse operator

Let T be a self-adjoint operator on a Hilbert space H.
Claim 1: To prove that $\|(T + iI)x\| \geq \|x\|, \ \forall x \in H$.
Consider

$$\begin{aligned}
\|(T + iI)x\|^2 &= <(T + iI)x, (T + iI)x> \\
&= <Tx + iIx, Tx + iIx> \\
&= <Tx, Tx> + <ix, Tx> + <Tx, ix> + <ix, ix> \\
&= <Tx, Tx> + i <x, Tx> - i <Tx, x> + i(-i) <x, x> \\
&= \|Tx\|^2 + i\{<x, Tx> - <Tx, x>\} + \|x\|^2 \\
&= \|Tx\|^2 + i\{<T^*x, x> - <Tx, x>\} + \|x\|^2 \\
&= \|Tx\|^2 + i\{<Tx, x> - <Tx, x>\} + \|x\|^2 \\
&= \|Tx\|^2 + \|x\|^2 \\
&\geq \|x\|^2
\end{aligned}$$

Hence $\|(T + iI)x\| \geq \|x\|, \ \forall x \in H$.

## Proof

Claim 2: To prove that $\overline{R(T+iI)} = H$
Let $y \in H$ such that $y \perp R(T+iI)$

$$\begin{aligned}
&\Rightarrow \quad < y, (T+iI)x >= 0, \forall x \in H. \\
&\Rightarrow \quad < (T+iI)^* y, x >= 0, \forall x \in H. \\
&\Rightarrow \quad < (T-iI)y, x >= 0, \forall x \in H. \\
&\Rightarrow \quad (Ty - iy) = 0 \\
&\Rightarrow \quad Ty - iy = 0 \\
&\Rightarrow \quad Ty = iy
\end{aligned}$$

which is a contradiction, since T is on a self-adjoint operator, all its eigenvalues must be real. Hence $y = 0$
Hence $\overline{R(T+iI)} = H$.

By claim 1 and claim 2 and by theorem, " An operator T on a Hilbert space H is invertible if and only if the following (i)and(ii) hold

(i) There exists a positive number c such that

$$\|Tx\| \geq c\|x\| \text{holds for any } x \in H.$$

(ii) R(T), the range of T is dense in H, i.e $\overline{R(T)} = H$."

$T + iI$ is invertible.
Hence $T + iI$ has a bounded inverse.

## Theorem

If T is any operator on a Hilbert space, then the following (i)and (ii) hold

(i) $H = \overline{R(T)} \oplus N(T^*)$

(ii) $H = \overline{R(T^*)} \oplus N(T)$

## Proof

Since $\overline{R(T)}$ is a closed subspace of H, $H = \overline{R(T)} \oplus \overline{R(T)}^{\perp}$

If $y \in N(T^*)$, then $< Tx, y >=< x, T^*y >=< x, 0 >= 0$

$\Rightarrow R(T) \perp N(T^*)$

& $\overline{R(T)}^{\perp} = N(T^*)$

$\therefore H = \overline{R(T)} \oplus \underline{N(T^*)}$

Similarly $H = \overline{R(T^*)} \oplus N(T)$

## Theorem

If $\lambda \in \Gamma(T)$, then $\overline{\lambda} \in P_\sigma(T^*)$

## Proof

Let $\lambda \in \Gamma$
$\Rightarrow \overline{R(T-\lambda)} \subsetneq H$. [by the definition of $\Gamma(T)$]
$\therefore H = \overline{R(T-\lambda)} \oplus N((T-\lambda)^*)$
$\therefore$ there exists non zero vector $x \in N(T-\lambda)^*$

$$\begin{aligned} &\Rightarrow (T-\lambda)^* x = 0 \\ &\Rightarrow (T^* - \overline{\lambda})x = 0 \\ &\Rightarrow T^* x = \overline{\lambda} x \\ &\Rightarrow \overline{\lambda} \in P_\sigma(T^*) \end{aligned}$$

$\therefore$ If $\lambda \in \Gamma(T)$, then $\overline{\lambda} \in P_\sigma(T^*)$

## Corollary

If $\lambda \in R_\sigma(T)$, then $\overline{\lambda} \in P_\sigma(T^*)$

## Proof

Let $\lambda \in R_\sigma(T)$
$\Rightarrow (T-\lambda)^{-1}$ exists and $\overline{R(T-\lambda)} \subsetneq H$
Since $\overline{R(T-\lambda)} \subsetneq H$, $\lambda \in \Gamma(T)$
Hence by theorem,
"If $\lambda \in \Gamma(T)$, then $\overline{\lambda} \in P_\sigma(T^*)$"
$\lambda \in P_\sigma(T^*)$
$\therefore$ if $\lambda \in R_\sigma(T)$, then $\overline{\lambda} \in P_\sigma(T^*)$

## Theorem

If an operator T is normal, then $\sigma(T) = A_\sigma(T)$ holds.

## Proof

Since $A_\sigma(T) \subset \sigma(T)$ is always true, it is enough to prove that, $\sigma(T) \subset A_\sigma(T)$ for normal operator T.
Let T be a normal operator and $\lambda \in \sigma(T)$
If $\lambda \notin A_\sigma(T)$, then there exist $\epsilon > 0$ such that
$\|Tx - \lambda x\| \geq \epsilon, \ \forall x \in H$ with $\|x\| = 1$

$$\Rightarrow \|Ty - \lambda y\| \geq \epsilon\|y\|, \ \forall \ y \in H \quad (1)$$

$$\Rightarrow \|(T - \lambda)^* y\| = \|(T - \lambda)y\| \geq \epsilon\|y\| \ \forall y \in H \ [\because T \text{ is normal}] \quad (2)$$

## Proof

Claim: $\overline{R(T-\lambda)} = H$

Let $y \perp R(T-\lambda)$

$$
\begin{aligned}
&\Rightarrow \quad <(T-\lambda)x, y> = 0, \ \forall x \in H \\
&\Rightarrow \quad <x, (T-\lambda)^* y> = 0, \ \forall x \in H \\
&\Rightarrow \quad (T-\lambda)^* y = 0, \\
&\Rightarrow \quad (T^* - \overline{\lambda}) y = 0 \\
&\Rightarrow \quad T^* y = \overline{\lambda} y
\end{aligned}
$$

Substituting in(2)

$\|y\| \leq 0$

$\Rightarrow \|y\| = 0$

$\Rightarrow y = 0$

Hence $\overline{R(T-\lambda)} = H$ (3)

From (1) and (3), $(T-\lambda)$ satisfies the two conditions (i) there exist a constant $\epsilon > 0$ such that $\|(T-\lambda)y\| \geq \epsilon\|y\|, \ \forall\ y \in H$

(ii)$\overline{R(T-\lambda)} = H$

Hence $(T-\lambda)$ is invertible

$\Rightarrow \lambda \notin \sigma(T)$,

Which is a contradiction.

Hence $\lambda \in A_\sigma(T)$

Hence we have proved that, $\sigma(T) \subset A_\sigma(T)$

Hence if T is normal, $\sigma(T) = A_\sigma(T)$

## Theorem

If an operator T is normal, then $R_\sigma(T) = \emptyset$

## Proof

Let T be a normal operator and $\lambda \in R_\sigma(T)$ then $\overline{\lambda} \in P_\sigma(T^*)$ [By result, If $\lambda \in R_\sigma(T)$, then $\overline{\lambda} \in P_\sigma(T^*)$]
$\Rightarrow \exists$ non zero $y \in H$ such that $T^*y = \overline{\lambda}y$
$\Rightarrow Ty = \lambda y$ [$\because$ T is normal, $\|T^*y - \overline{\lambda}y\| = \|Ty - \lambda y\|$]
$\Rightarrow \lambda \in P_\sigma(T)$,
Which is a contradiction, since $R_\sigma(T) \cap P_\sigma(T) = \emptyset$.
Hence $R_\sigma(T) = \emptyset$

## Theorem

If an operator T is self-adjoint, then $\sigma(\mathrm{T})$is a subset of the real line.

## Proof

Let T be a self-adjoint operator
Let $\lambda \in \sigma(\mathrm{T})$ and $\lambda$ is not a real number then $\lambda \neq \overline{\lambda}$
$\therefore$ for all non-zero vector x,

$$\begin{aligned}
0 &< |\lambda - \overline{\lambda}| \|x\|^2 \\
&= |\lambda - \overline{\lambda}| < x, x > \\
&= | < (\overline{\lambda} - \lambda)x, x > | \\
&= | < (Tx - \lambda x) - (Tx - \overline{\lambda}x), x > | \\
&= | < (T - \lambda)x, x > - < (T - \lambda)^* x, x > | \ [\because T = T^*] \\
&= | < (T - \lambda)x, x > - < x, (T - \lambda)x > | \\
&\leq | < (T - \lambda)x, x > + | < x, (T - \lambda)x > | \\
&\leq \|(T - \lambda)x\| \|x\| + \|x\| \|(T - \lambda)x\| \\
&= 2\|Tx - \lambda x\| \|x\|
\end{aligned}$$

$\therefore \ \|Tx - \lambda x\|\|x\| > 0, \forall$ nonzero $x \in H \Rightarrow \lambda \notin A_\sigma(T)$
$\Rightarrow \lambda \notin \sigma(T)$, [$\because$ for self adjoint operator, $\sigma(T) = A_\sigma(T)$]
Which is a contradiction.
Hence $\sigma(T)$ of a normal operator is a subset of the real line

## Theorem

Let T be a normal operator, $Tx = \lambda x$ and $Ty = \mu y$, where $\lambda \neq \mu$. Then $< x, y >= 0$

## Proof

Let T be a normal operator, $Tx = \lambda x$ and $Ty = \mu y$, where $\lambda \neq \mu$. Then $< x, y >= 0$

## Theorem

The following two conditions on an operator T are equivalent:

(i) T has an approximate point spectrum $\mu$ such that $|\mu| = \|T\|$

(ii) $\sup\{| < Tx, x > |/\|x\| = 1\} = \|T\|$

## Proof

To prove that (i)⇒(ii)

Assume that T has an approximate point spectrum $\mu$ such that $|\mu| = \|T\|$

$\Rightarrow \exists$ a sequence$\{x_n\}$ of unit vectors such that $\|Tx_n - \mu x_n\| \to 0$ and $|\mu| = \|T\|$

then

$$\begin{aligned} | < Tx_n, x_n > -\mu| &= | < Tx_n, x_n > -\mu < x_n, x_n > | \\ &= |(< Tx_n - \mu x_n), x_n > | \\ &\leq \|Tx_n - \mu x_n\|\|x_n\| \\ &= \|Tx_n - \mu x_n\| \to 0 \text{ as } n \to \infty \end{aligned}$$

$< Tx_n, x_n > \to \mu$ as $n \to \infty$
$| < Tx_n, x_n > | \to |\mu|$ as $n \to \infty$
Since $\|T\| \geq \sup\{| < Tx, x > |/\|x\| = 1\}$
$\geq | < Tx_n, x_n > | \to |\mu| = \|T\|$
$\Rightarrow \sup\{| < Tx, x > |/\|x\| = 1\} = \|T\|$.
Hence (i)$\Rightarrow$ (ii)

To prove that (ii)⇒(i)
Assume that $\sup\{|<Tx,x>|/\|x\|=1\}=\|T\|$
$\exists$ a sequence of vectors $\{x_n\}$ such that $\|x_n|=1$ and
$|<Tx_n,x_n>|\to\|T\|$
Assume that $<Tx_n,x_n>\to\mu \ni |\mu|=\|T\|$
Claim: $\mu\in A_\sigma(T)$
Consider

$$\begin{aligned}\|Tx_n-\mu x_n\|^2 &= <Tx_n-\mu x_n, Tx_n-\mu x_n> \\ &= \|Tx_n\|^2-\overline{\mu}<Tx_n,x_n>-\mu<x_n,Tx_n>+\mu\overline{\mu}<x_n,x_n> \\ &= \|Tx_n\|^2-\overline{\mu}<Tx_n,x_n>-\mu<\overline{Tx_n,x_n}>+|\mu|^2 \\ &\to |\mu|^2|-\mu|^2-|\mu|^2+|\mu|^2=0\end{aligned}$$

$\therefore\ \mu\in A_\sigma(T)$
Hence T has an approximate point spectrum $\mu$ such that $|\mu|=\|T\|$
Hence (ii)⇒(i)
Hence (i) and (ii) are equivalent.

## Theorem

For any operator A and B, $\sigma(AB) - \{0\} = \sigma(BA) - \{0\}$ holds. i.e the nonzero elements of $\sigma(AB)$ and $\sigma(BA)$ are the same.

## Proof

Let A and B be any two operators .
To prove that $\sigma(AB) - \{0\} = \sigma(BA) - \{0\}$
To prove this , we have to show that if $\lambda \neq 0$, then $AB - \lambda$ is invertible $\iff$ $BA - \lambda$ is invertble.
Without loss of generality, it is sufficient to show that, if $I - AB$ is invertible, then $BA - I$ is invertible. Let $I - AB$ be invertible and C be its inverse, then
$(I - AB)C = C(I - AB) = I$
$\Rightarrow C - ABC = C - CAB = I$
$\Rightarrow ABC = CAB = C - I \quad (1)$

Consider

$$\begin{aligned}(I + BCA)(I - BA) &= I - BA + BCA - BCABA \\ &= I - BA + BCA - B(C - I)A \\ &= I - BA + BCA - BCA + BA \\ &= I\end{aligned}$$

Similarly $(I - BA)(I + BCA) = I$

i.e $(I - BA)(I + BCA) = (I + BCA)(I - BA) = I$

Hence $(I - BA)$ is invertible and $(I - BA)^{-1} = I + BCA$

# 2.4.2 Spectral mapping theorem

**Theorem (Spectral mapping theorem)**

Let $\sigma(T)$ be the spectrum of an operator T and p(t) be any polynomial of a complex number t. Then $\sigma(p(T)) = p(\sigma(T))$

**Proof:**

Let $\sigma(T)$ be the spectrum of an operator T and p(t) be any polynomial of a complex number t.
Let $\lambda_0 \in \sigma(T)$
$\Rightarrow T - \lambda_0 I$ is not invertible.
Since there exists $g(\lambda)$ such that

$$p(\lambda) - p(\lambda_0) = (\lambda - \lambda_0)g(\lambda),$$

$p(T) - p(\lambda_0 I) = (T - \lambda_0 I)g(T)$
$\Rightarrow p(T) - p(\lambda_0)I$ is not invertible. $[\because T - \lambda_0 I$ is not invertible$]$
$\Rightarrow p(\lambda_0) \in \sigma(p(T))$
$\Rightarrow p(\sigma(T)) \subset \sigma(p(T))$ (1)

### C

onversely,
Let $\lambda_0 \in \sigma(p(T))$
$\Rightarrow p(T) - \lambda_0 I$ is not invertible.
Let $\lambda_1, \lambda_2, \ldots \lambda_n$, be such that $p(\lambda_j) = \lambda_0$, for $\lambda = 1, 2, \ldots n$
$\Rightarrow p(\lambda) - \lambda_0 = \alpha(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$, for some scalar $\alpha$
$\Rightarrow p(T) - \lambda_0 = \alpha(T - \lambda_1)(T - \lambda_2)\ldots(T - \lambda_n)$,
If each of $T - \lambda_j I$ is invertible, then $p(T) - \lambda_0 I$ is also invertible.
Hence there exists k $\ni$ $T - \lambda_k I$ is not invertible. i.e $\lambda_k \in \sigma(T)$
$\Rightarrow p(\lambda_k) \in p(\sigma(T))$
$\Rightarrow \lambda_0 \in p(\sigma(T))$
$$\Rightarrow \sigma(p(T)) \subset p(\sigma(T)) \tag{2}$$

$$\text{From (1) and (2), } p(\sigma(T)) = \sigma(p(T))$$

## Example

Let T be defined as $T = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix}$

$$|T - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \ (2-\lambda)(1-\lambda) - 6 = 0$$
$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 6$$
$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$$
$$\Rightarrow T + I \text{ and } T - 4I \text{ not invertible}$$
$$\Rightarrow -1, \ 4 \in \sigma(T)$$
$$\Rightarrow \sigma(T) = \{4, -1\} \tag{1}$$
$$T^2 = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 10 & 3 \\ 18 & 7 \end{pmatrix}$$

$|T^2 - \lambda I| = \begin{vmatrix} 10-\lambda & 3 \\ 18 & 7-\lambda \end{vmatrix} = 0$
$\Rightarrow (10-\lambda)(7-\lambda) - 54 = 0$
$\Rightarrow 70 - 7\lambda - 10\lambda + \lambda^2 - 54 = 0$
$\Rightarrow \lambda^2 - 17\lambda + 16 = 0$
$\Rightarrow (\lambda - 1)(\lambda - 16) = 6$
$\Rightarrow \lambda = 1,\ 16$
$\Rightarrow T^2 - I$ and $T^2 - 16I$ are not invertible
$\Rightarrow \sigma(T^2) = \{1, 16\}$
Since $\sigma(T) = \{4, -1\}$, $\{\sigma(T)\}^2 = \{16, 1\}$
Hence $\sigma(T^2) = \{\sigma(T)\}^2$

## Theorem

Let $\sigma(T)$ be the spectrum of an invertible operator T. Then

$$\sigma(T^{-1}) = \{\sigma(T)\}^{-1}$$

## Proof:

Let $\sigma(T)$ be the spectrum of an invertible operator T.
Since T is invertible, $0 \neq \sigma(T)$.
Now,

$$\begin{aligned} T^{-1} - \lambda^{-1}I &= \lambda\lambda^{-1}T^{-1} - \lambda^{-1}TT^{-1} \\ &= \lambda\lambda^{-1}T^{-1} - T\lambda^{-1}T^{-1} \\ &= (\lambda - T)\lambda^{-1}T^{-1} \end{aligned}$$

$\Rightarrow \lambda - T$ is invertible iff $(T^{-1} - \lambda^{-1})$ is invertible.
Hence $\lambda \notin \sigma(T) \iff \lambda^{-1} \notin \sigma(T^{-1})$
Hence $\sigma(T^{-1}) = \{\sigma(T)\}^{-1}$

## Theorem

Let $\sigma(\mathrm{T})$ be the spectrum of an operator T. Then
$\sigma(\mathrm{T}^*) = \{\sigma(\mathrm{T})\}^* = \{\lambda^* / \ \lambda \in \sigma(\mathrm{T})\}$

## Proof:

Let $\sigma(\mathrm{T})$ be the spectrum of an invertible operator T.
If $\lambda \notin \sigma(\mathrm{T})$, then $\mathrm{T} - \lambda \mathrm{I}$ is invertible.
$\Rightarrow \mathrm{T}^* - \lambda^* \mathrm{I}$ is also invertible.
$\Rightarrow \lambda^* \notin \sigma(\mathrm{T}^*)$
$\Rightarrow \sigma(\mathrm{T}^*) \subset \{\sigma(\mathrm{T})\}^*$ (1)
Replacing T by $\mathrm{T}^*$ in (1)
$\sigma(\mathrm{T}^{**}) \subset (\sigma(\mathrm{T}^*))^*$
$\Rightarrow \sigma(\mathrm{T}) \subset (\sigma(\mathrm{T}^*))^2$
$\Rightarrow \{\sigma(\mathrm{T})\}^* \subset \sigma(\mathrm{T}^*)$ (2)
From (1)and (2),
$\sigma(\mathrm{T}^*) = \{\sigma(\mathrm{T})\}^*$

section 2.5 Numerical Range of an operator

## 2.5.1 Numerical range is convex set

**Definition**

The numerical range of W(T) of an operator T on a Hilbert space H is defined by

$$W(T) = \{< Tx, x > / \|x\| = 1\}$$

**Theorem** (Toeplitz-Hausdorff theorem)

The numerical range W(T) of an operator T is a convex set in the complex plane

## proof

Let T be an operator on a Hilbert space H.
Let $\xi = < Tx, x >$, $\eta = < Ty, y > \in W(T)$
where x and y are unit vectors in H.
To prove that W(T) is a convex, it is sufficient to prove that every point of the line segment joining $\xi$ and $\eta$ is in W(T)
If $\xi = \eta$, then the result is true.
If $\xi \neq \eta$, then there exist complex numbers $\alpha$ and $\beta$ such that
$\alpha\xi + \beta = 1$ and $\alpha\eta + \beta = 0$
Then it is sufficient to prove that the unit interval
$[0, 1] \subset W(\alpha T + \beta) = \alpha W(T) + \beta$
If $\alpha < Tx, x > +\beta = t$, then

$$\begin{aligned} \alpha < Tx, x > +\beta &= t(\alpha\xi + \beta) + (1 - t)(\alpha\eta + \beta) \\ &= \alpha t\xi + \beta t + \alpha\eta + \beta - \alpha t\eta - \beta t \\ &= \alpha(t\xi + (1 - t)\eta) + \beta \end{aligned}$$

$\therefore$ Without loss of generality, we can assume that $\xi = 1$ and $\eta = 0$

Since T can be written as $T = A + iB$,
where A and B are self adjoint operators and $< Tx, x >= 1$ and $< Ty, y >= 0$ are real, we get

$$< (A + iB)x, x >= 1 \text{ and } < (A + iB)y, y >= 0$$

$$\Rightarrow < Ax, x >= 1, < Bx, x >= 0, < Ay, y >= 0, < By, y >= 0$$

If x is replaced by $\lambda x$, where $|\lambda| = 1$, then

$$< T(\lambda x), \lambda x >= \lambda\bar{\lambda} < Tx, x >=< Tx, x >$$

Hence $< Tx, x >$ remains the same, but $< Bx, y >$ becomes $\lambda < Bx, y >$
Hence without loss of generality, we may assume that $< Bx, y >$ is purely imaginary.
Put $h(t) = tx + (1 - t)y$, where $t \in [0, 1]$.
If x and y were linearly dependent, then since they are unit vectors, $y = \mu x$, where $|\mu| = 1$
then $< Ty, y >=< T(\mu x), \mu x >= \mu\overline{\mu} < Tx, x >=< Tx, x >$
$\Rightarrow 1 - \xi = \eta = 0$, which is a contradiction.
Hence x and y are linearly independent.

Therefore $h(t) \neq 0$.

$$
\begin{aligned}
< Bh(t), h(t) > &= < B(tx + (1-t)y), tx + (1-t)y > \\
&= < tBx + (1-t)By, tx + (1-t)y > \\
&= t^2 < Bx, x > + t(1-t) < Bx, y > + (1-t)t < By, x > + \\
&= t(1-t)\{< Bx, y > + < y, Bx >\} \\
&= t(1-t)\{< Bx, y > + \overline{< Bx, y >}\} \\
&= t(1-t)2Re < Bx, y > = 0 \; [\because < Bx, y > \text{ is purely imagi}
\end{aligned}
$$

Hence
$< Th(t), h(t) > = < Ah(t), h(t) > + i < Bh(t), h(t) > = < Ah(t), h(t) >$
Hence $< Th(t), h(t) >$ is real for all t.

Hence the function,

$$f(t) = \left\langle T\frac{h(t)}{\|h(t)\|}, \frac{h(t)}{\|h(t)\|}\right\rangle \in W(T)$$

and f(t) is real-valued and continues on the closed interval [0, 1].
Hence f([0, 1]) is connected.
Since

$$\begin{aligned} f(0) &= \left\langle T\frac{h(0)}{\|h(0)\|}, \frac{h(0)}{\|h(0)\|}\right\rangle \\ &= < Ty, y > \\ &= \eta = 0 \quad \text{and} \\ f(1) &= \left\langle T\frac{h(1)}{\|h(1)\|}, \frac{h(1)}{\|h(1)\|}\right\rangle \\ &= < Tx, x > \\ &= \xi = 1 \end{aligned}$$

$0, 1 \in f([0, 1])$
$\Rightarrow [0, 1] \subset f([0, 1])$ [$\because$ f([0, 1])is connected]
Hence $[0, 1] \subset W(T)$ [$\because f([0, 1]) \subset W(T)$]
Hence W(T) is a convex set in the complex plane.

## Theorem

(i) If T is a two- by-two matrix with distinct eigenvalues $\alpha$ and $\beta$ and corresponding normalized eigenvectors x and y, then W(T) is a closed elliptied disc with foci at $\alpha$ and $\beta$; if $\gamma = | < x, y > |$ and $\delta = \sqrt{1-\gamma^2}$, then the minor axis and the major axis can be expressed respectively as follows

the minor axis $= \dfrac{\gamma|\alpha-\beta|}{\delta}$

and the major axis $= \dfrac{|\alpha-\beta|}{\delta}$

(ii) If T has only one eigenvalue $\alpha$, then W(T) is the disc with center $\alpha$ and radius $\frac{1}{2}\|T-\alpha\|$