## 2.6 Relations among several classes of Non-normal operator

### 2.6.1 Paranormal operator

**Definition 1:**

An operator T on a Hilbert space H is said to be a paranormal operator if $\|T^2x\| \geq \|Tx\|^2$ for any unit vector $x \in H$.

**Definition 2:**

An operator T on a Hilbert space H is said to be a subnormal operator if T has a normal extension N that is , there exists a normal operator N on a larger Hilbert space $K \supset H$ such that $Tx = Nx$ for all $x \in H$.

**Theorem 1**

If T is a paranormal operator, then the following inequalities hold for any vector $x \in H$

$(p-1)$ $\|T\| \geq \dots \dfrac{\|T^{n+1}x\|}{\|T^n x\|} \geq \dots \dfrac{\|T^4x\|}{\|T^3x\|} \geq \dfrac{\|T^3x\|}{\|T^2x\|} \geq \dfrac{\|T^2x\|}{\|Tx\|} \geq \dfrac{\|Tx\|}{\|x\|}$

If T is an invertible paranormal operator then the following inequalities hold for any vector $x \in H$

$(p-2)$ $\dfrac{\|Tx\|}{\|T^{-1}x\|} \geq \dfrac{\|T^{-1}x\|}{\|T^{-2}x\|} \geq \dfrac{\|T^{-2}x\|}{\|T^{-3}x\|} \geq \dots \dfrac{\|T^{-n+1}x\|}{\|T^{-n}x\|} \geq \dots \dfrac{1}{\|T^{-1}\|}$

**Proof:**

Since T is paranormal,

$$\|T^2x\| \geq \|Tx\|^2 \text{ for any } x \in H \text{ with } \|x\| = 1 \qquad (1)$$

Replacing $x$ by $\dfrac{x}{\|x\|}$ in (1),

$$\left\|T^2\left(\frac{x}{\|x\|}\right)\right\| \geq \left\|T\left(\frac{x}{\|x\|}\right)\right\|^2$$

$$\Rightarrow \frac{1}{\|x\|}\|T^2x\| \geq \frac{1}{\|x\|^2}\|Tx\|^2$$

$$\Rightarrow \frac{\|T^2x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|x\|} \text{ for any } x \in H. \tag{2}$$

Replacing $x$ by $Tx$ in (2),

$$\Rightarrow \frac{\|T^3x\|}{\|T^2x\|} \geq \frac{\|T^2x\|}{\|Tx\|}$$

Repeating this, we get,

$$\Rightarrow \frac{\|T^{n+1}x\|}{\|T^nx\|} \geq \frac{\|T^nx\|}{\|T^{n-1}x\|} \ \forall \ n = 1, 2, 3, \ldots$$

Also $\|T^{n+1}x\| \leq \|T\|\|T^nx\|, \ \forall n = 1, 2, \ldots$

Hence we get,

$$\|T\| \geq \ldots \geq \frac{\|T^{n+1}x\|}{\|T^nx\|} \geq \ldots \frac{\|T^4x\|}{\|T^3x\|} \geq \frac{\|T^3x\|}{\|T^2x\|} \geq \frac{\|T^2x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|x\|}.$$

Let T be an invertible paranormal operators

Replacing $x$ by $T^{-2}x$ in (2), we get

$$\frac{\|x\|}{\|T^{-1}x\|} \geq \frac{\|T^{-1}x\|}{\|T^{-2}x\|} \tag{3}$$

Replacing $x$ by $T^{-1}x$ in (3)

$$\frac{\|T^{-1}x\|}{\|T^{-2}x\|} \geq \frac{\|T^{-2}x\|}{\|T^{-3}x\|} \tag{4}$$

Repeating

$$\frac{\|T^{-2}x\|}{\|T^{-3}x\|} \geq \frac{\|T^{-3}x\|}{\|T^{-4}x\|} \ldots$$

$$\frac{\|T^{-n+1}x\|}{\|T^{-n}x\|} \geq \frac{\|T^{-n}x\|}{\|T^{-(n+1)}x\|} \ \forall \ n = 1, 2, \ldots$$

Also

$$\|T^{-(n+1)}x\| = \|T^{-1}T^{-n}x\| \leq \|T^{-1}\|\|T^{-n}x\|$$

$$\Rightarrow \frac{\|T^{-n}x\|}{\|T^{-(n+1)}x\|} \geq \frac{1}{\|T^{-1}\|}, \forall \ n = 1, 2, \ldots$$

Combining these inequalities, we get,

$$\frac{\|x\|}{\|T^{-1}x\|} \geq \frac{\|T^{-1}x\|}{\|T^{-2}x\|} \geq \frac{\|T^{-2}x\|}{\|T^{-3}x\|} \geq \ldots \geq \frac{\|T^{-n+1}x\|}{\|T^{-n}x\|} \geq \ldots \geq \frac{1}{\|T^{-1}\|}$$

Hence the theorem.

**Theorem 2**

If T is a paranormal operator, then the following properties hold:

(i) $T^n$ is also paranormal for any natural number n.

(ii) T is normaloid operator. i.e $\|T\| = r(T)$.

(iii) If T is an invertible paranormal operator then so is $T^{-1}$

**Proof:**
(i) From theorem 1,
If T is a paranormal operator, then

$$\|T\| \geq \ldots \geq \frac{\|T^{n+1}x\|}{\|T^n x\|} \geq \ldots \frac{\|T^4 x\|}{\|T^3 x\|} \geq \frac{\|T^3 x\|}{\|T^2 x\|} \geq \frac{\|T^2 x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|x\|} \tag{1}$$

Applying (1) repeatedly, we get

$$\begin{aligned} \frac{\|T^{2n}x\|}{\|T^n x\|} &= \frac{\|T^{2n}x\|}{\|T^{2n-1}x\|}\frac{\|T^{2n-1}x\|}{\|T^{2n-2}x\|}\cdots\frac{\|T^{n+1}x\|}{\|T^n x\|} \\ &\geq \frac{\|T^n x\|}{\|T^{n-1}x\|}\frac{\|T^{n-1}x\|}{\|T^{n-2}x\|}\cdots\frac{\|Tx\|}{\|x\|} \\ &= \frac{\|T^n x\|}{\|x\|} \end{aligned}$$

$$\text{i.e } \frac{\|T^{2n}x\|}{\|T^n x\|} \geq \frac{\|T^n x\|}{\|x\|} \tag{2}$$

$\Rightarrow \|T^{2n}x\| \geq \|T^n x\|^2$ for any $x \in H$, with $\|x\| = 1$
$\Rightarrow$ $T^n$ is paranormal for any natural number n
Hence (i) is proved.
(ii) From (1), we get
$\frac{\|T^n x\|}{\|Tx\|} = \frac{\|T^n x\|}{\|T^{n-1}x\|}\frac{\|T^{-2}x\|}{\|T^{n-3}x\|}\cdots\frac{\|T^3 x\|}{\|T^2 x\|}\frac{\|T^2 x\|}{\|Tx\|} \geq \left(\frac{\|Tx\|}{\|x\|}\right)^{n-1}$ for any $x \in H$.
$\|T^n x\| \geq \|Tx\|^n$ for any $x \in H$. with $\|x\| = 1$
$\|T^n\| \geq \|T\|^n$ for all $n \in I$
$\|T^n\| = \|T\|^n$, for all $n \in I$ $[\because \|T^n\| \leq \|T\|\|T^{n-1}\| \ldots \leq \|T\|^n,\ n \in I]$
Hence $\|T\| = \|T^n\|^{1/n},\ \forall\ n \in I$.
$\Rightarrow \|T\| = \lim_{n\to\infty} \|T^n\|^{1/n} = r(T)$
Hence T is normaloid.
Hence (ii) is proved.
(iii)From $(p-2)$ of theorem 2,

$$\frac{\|x\|}{\|T^{-1}x\|} \geq \frac{\|T^{-1}x\|}{\|T^{-2}x\|}$$
$\|T^{-2}x\| \geq \|T^{-1}x\|^2, \; \forall x \in H$ with $\|x\| = 1$
$\Rightarrow T^{-1}$ is paranormal.
Hence (iii) is proved.
Hence the theorem.

## 2.6.2 Implication relations among several classes of non-normal opertor.

**Theorem 1:**

The following inclusion relation hold:
Self adjoint $\subseteq$ Normal$\subseteq$ Quasinormal$\subseteq$ Subnormal $\subseteq$ Hyponormal $\subseteq$ Paranormal $\subseteq$ Normaloid $\subseteq$ Spectraloid.
**Proof:**
Let T be a Self-adjoint operator
$\Rightarrow T = T^*$
Hence $T^*T = TT = TT^*$
Hence T is normal.
Therefore Self-adjoint $\subseteq$ Normal. (i)
Let T be a normal operator.
$\Rightarrow T^*T = TT^*$
Hence $T(T^*T) = (TT^*)T = (T^*T)T$
Hence T is quasi normal.
Therefore Normal $\subseteq$ Quasinormal. (ii)
Let T be a Quasi normal operator.
Let $T = U|T|$ be the polar decomposition of T.
Then $U|T| = |T|U$ (1)
By the theorem,
" Let $T = U|T|$ be the polar decomposition of an operator T. Then

$T = U|T|$ is quasinormal iff $U|T| = |T|U$"

$$\begin{aligned} \therefore T^*T - TT^* &= |T|^2 - U|T||T|U^* \\ &= |T|^2 - |T|UU^*|T| \quad [Using(1)] \\ &= |T|(I - UU^*)|T| \geq 0 \end{aligned}$$

Hence T is hyponormal
Define $[T] = T^*T - TT^*$ then $[T] \geq 0$
Since T is quasi normal,

$$\begin{aligned} [T]T &= (T^*T - TT^*)T \\ &= T^*TT - TT^*T \\ &= (T^*T)T - T(T^*T) = 0 \end{aligned}$$

$\therefore ([T]T)^* = 0$
$\Rightarrow T^*[T] = 0$
$\Rightarrow T^*[T] = 0$ ( $\because$ $[T^*] = (T^*T - TT^*)^* = T^*T - TT^* = [T]$)
$[T]^2T = [T][T]T = 0$
Similarity $T^*[T]^2 = 0 \ldots$
$[T^n]T = T^*[T^n] = 0$ for all natural number n.
Let S be the square root of $[T]$. Since the square root is approximated uniformly by polynomials of $[T]$ without constant terms,
$ST = T^*S$ (2)
Define N= $\begin{pmatrix} T & S \\ 0 & T^* \end{pmatrix}$, where S=$[T]^{1/2}$
then

$$\begin{aligned} NN^* - N^*N &= \begin{pmatrix} T & S \\ 0 & T^* \end{pmatrix}\begin{pmatrix} T^* & 0 \\ S^* & T \end{pmatrix} - \begin{pmatrix} T^* & 0 \\ S^* & T \end{pmatrix}\begin{pmatrix} T & S \\ 0 & T^* \end{pmatrix} \\ &= \begin{pmatrix} TT^* + SS^* & ST \\ T^*S^* & T^*T \end{pmatrix} - \begin{pmatrix} T^*T & T^*S \\ S^*T & S^*S + TT^* \end{pmatrix} \\ &= \begin{pmatrix} TT^* + SS^* - T^*T & ST - T^*S \\ T^*S^* - S^*T & T^*T - S^*S - TT^* \end{pmatrix} \end{aligned}$$

$TT^* + SS^* - T^*T = SS^* - (T^*T - TT^*)$
$= [T] - [T] = 0$

$ST - T^*S = 0$ by (2)
$T^*S^* - S^*T = 0$ by (2)

$$\begin{aligned} T^*T - S^*S - TT^* &= T^*T - TT^* - S^*S \\ &= [T] - [T] = 0 \end{aligned}$$

Hence $NN^* - N^*N = 0$
Hence N is normal.
Hence T has a normal extension N.
Hence T is subnormal.
Quasinormal $\subseteq$ Subnormal (iii)
Let T be a subnormal operator.
Then T has a normal extension N, which can be expressed as
N=$\begin{pmatrix} T & X \\ 0 & Y \end{pmatrix}$ on a larger Hilbert space $K \supset H$
Since N is normal,
$N^*N = NN^*$
$\begin{pmatrix} T^* & 0 \\ X^* & Y^* \end{pmatrix} \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix} = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix} \begin{pmatrix} T^* & 0 \\ X^* & Y^* \end{pmatrix}$
$\Rightarrow \begin{pmatrix} T^*T & T^*X \\ X^*T & X^*X + Y^*Y \end{pmatrix} = \begin{pmatrix} TT^* + XX^* & XY^* \\ YX^* & YY^* \end{pmatrix}$
$\Rightarrow T^*T = TT^* + XX^*$ (3)
$T^*X = XY^*$ (4)
$X^*T = YX^*$ (5)
$X^*X + Y^*Y = YY^*$ (6)
Since $XX^* \geq 0$, (3)$\Rightarrow T^*T \geq TT^*$
$\Rightarrow T$ is hyponormal.
$\therefore$ Subnormal$\subseteq$ hyponormal (iv)
Let T be hyponormal
$\Rightarrow \|Tx\| \geq \|T^*x\|, \ \forall x \in H.$
$\Rightarrow \|TTx\| \geq \|T^*Tx\|, \ \forall x \in H.$ (7)

Consider,

$$\begin{aligned}\|Tx\|^2 &= <Tx, Tx> \\ &= <T^*Tx, x> \\ &\leq \|T^*Tx\|\|x\| \\ &\leq \|T^2x\|\|x\|\end{aligned}$$

$\Rightarrow \|T^2x\| \geq \|Tx\|^2 \ \forall x \in H$ with $\|x\| = 1$
Hence T is paranormal.
Therefore Hyponormal $\subseteq$ paranormal (v)
Let T be a paranormal operator, then T is normaloid.
( by Theorem,
"If T is a paranormal operator, then the following properties hold:

(i) $T^n$ is also paranormal for any natural number n.

(ii) T is normaloid operator. i.e $\|T\| = r(T)$.

(iii) If T is an invertible paranormal operator then so is $T^{-1}$"

)
Hence paranormal$\subseteq$ Normaloid (vi)
Let T be a normaloid operator
$\Rightarrow \|T\| = r(T)$ (8)
In general, $\|T\| \geq w(T) \geq r(T) = \|T\|$
$\Rightarrow w(T) = r(T)$
Hence T is spectraloid.
Hence Normaloid $\subseteq$ Spectraloid (viii)
Hence the theorem.

## 2.7 Characterization of Convexoid operators and Related Examples

### 2.7.1 Characterization of Convexoid operators

**Definitions**

(1) An operator T is said to be a **convexoid operator** if

$$\overline{W(T)} = co\sigma(T)$$

where co$\sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of T.
(2) An operator T is said to be a **condition $G_1$ operator** if

$$\|(T-\mu)^{-1}\| = \frac{1}{d(\mu, \sigma(T))} \text{for all} \mu \notin \sigma(T),$$

where $\sigma(T)$ means the spectrum $\sigma(T)$ of T.
(3) An operator T is said to be a **transaloid operator** if

$$T - \mu \ \text{ is normaloid for any } \mu \in C$$

**Lemma 1:**

If X is any bounded closed set in the complex plane, then

(i) co X={the intersection of all circles which contain the set X}
$= \cap_\mu \{\lambda / |\lambda - \mu| \leq \sup_{x \in X} |x - \mu|\}$

(ii) co X={the intersection of all closed half planes which contain the set X}
$= \cap_\theta \{\lambda / Re\lambda e^{i\theta} \geq \inf_{x \in X} Rese^{i\theta}\}$

**Theorem 1:**

An operator T is convexoid if and only if $T - \lambda$ is spectraloid for all complex numbers $\lambda$.
**Proof:**
Since $\overline{W(T)}$ is convex,
$\overline{W(T)} = \cap_\mu \{\lambda / |\lambda - \mu| \leq w(T - \mu)\}$ (1)
By lemma,

(i) co X={the intersection of all circles which contain the set X}
$= \cap_{\mu}\{\lambda/|\lambda-\mu| \le \sup_{x\in X}|x-\mu|\}$

(ii) co X={the intersection of all closed half planes which contain the set X}
$= \cap_{\theta}\{\lambda/Re\lambda e^{i\theta} \ge \inf_{x\in X} Rese^{i\theta}\}$ (2)

Now, T is convexoid $\iff \overline{W(T)} = co\sigma(T)$ [by Definition]
$\iff \cap_{\mu}\{\lambda/|\lambda-\mu| \le w(T-\mu)|\}$
$= \cap_{\mu}\{\lambda/|\lambda-\mu| \le r(T-\mu)|\}$
$\iff w(T-\lambda) = r(T-\lambda),\ \forall\lambda \in C$ [$\because \sigma(T)-\lambda = \sigma(T-\lambda), co\sigma(T)-\lambda = co\sigma(T-\lambda)$]
Hence $T-\lambda$ is spectraloid for all $\lambda$.

**Theorem 2**

An operator T is convexoid iff $\|(T-\mu)^{-1}\| \le \dfrac{1}{d(\mu, co\sigma(T))},\ \forall\mu \notin co\sigma(T)$

**Proof:**
Let T be convexoid operator.
$\Rightarrow \overline{W(T)} = co\sigma(T)$
By the result,
$\|(T-\mu)^{-1}\| \le \dfrac{1}{d(\mu, \overline{W(T)})},\ \forall\mu \notin \overline{W(T)}$"
$\|(T-\mu)^{-1}\| \le \dfrac{1}{d(\mu, co\sigma(T))},\ \forall\mu \notin co\sigma(T)$
Conversely, assume that $\|(T-\mu)^{-1}\| \le \dfrac{1}{d(\mu, co\sigma(T))},\ \forall\mu \notin co\sigma(T)$
$\Rightarrow \|(T-\mu)x\| \ge d(\mu, co\sigma(T)), \forall\mu \notin co\sigma(T)\ and\ \|x\| = 1$
$\Rightarrow \|Tx-\mu x\|^2 \ge d(\mu, co\sigma(T))^2, \forall\mu \notin co\sigma(T)\ and\ \|x\| = 1$
$\Rightarrow \|Tx\|^2 - 2Re < Tx, x > \overline{\mu} + |\mu|^2 \ge \inf_{s\in co\sigma(T)}(|s|^2 - 2Res\overline{\mu} + |\mu|^2)$ (1)
Take $\mu = |\mu|e^{-i(\theta+\pi)}$ in (1)
$\|Tx\|^2 - 2Re < Tx, x > e^{i(\theta+\pi)} + |\mu|^2 \ge \inf_{s\in co\sigma(T)}(|s|^2 - 2Res|\mu|e^{i(\theta+\pi)} + |\mu|^2)$
(2)
(2)$\div\ |\mu|$,
$\dfrac{\|Tx\|^2}{|\mu|} - 2Re < Tx, x > e^{i(\theta+\pi)} + |\mu| \ge \inf_{s\in co\sigma(T)}\left(\dfrac{|s|^2}{|\mu|} - 2Res|\mu|e^{i(\theta+\pi)} + |\mu|\right)$

$\frac{\|Tx\|^2}{|\mu|} - 2Re < Tx, x > e^{i(\theta+\pi)} \geq \inf_{s \in co\sigma(T)} \left( \frac{|s|^2}{|\mu|} - 2Res|\mu|e^{i(\theta+\pi)} \right)$
$-2Re < Tx, x > |\mu|e^{i(\theta+\pi)} \geq \inf_{s \in co\sigma(T)} \left( -2Rese^{i(\theta+\pi)} \right) \; as \; |\mu| \to \infty$
$Re < Tx, x > |\mu|e^{i(\theta+\pi)} \geq \inf_{s \in co\sigma(T)} Res|\mu|e^{i(\theta+\pi)}$ for $\|x\| = 1$ (3)
$\Rightarrow \; If \lambda \in \overline{W(T)}$, then $\lambda =< Tx, x >$ for some $x \in H$ with $\|x\| = 1$
$\therefore$(3)$\Rightarrow Re\lambda e^{i\theta} \geq inf_{s \in co\sigma(T)} Rese^{i\theta}$ for all $\theta$.
$\Rightarrow \lambda \in \cap_\theta \{\lambda / Re\lambda e^{i\theta} \geq \inf_{s \in co\sigma(T)} Rese^{i\theta}\}$
$\Rightarrow \lambda \in co\sigma(T)$
Hence $\overline{W(T)} \subseteq co\sigma(T)$
Since $\overline{W(T)} \supseteq co\sigma(T)$ is always true.
$\overline{W}(T) = co\sigma(T)$
Hence T is convexoid.
Hence the theorem.

**Theorem 3:**

Let T be a hyponormal operator. Then the following properties hold:

(i) $T - \mu$ is also a hyponormal for any $\mu \in C$.

(ii) $T$ is a transaloid operator.

(iii) $T^{-1}$ ia also a hyponormal operator if $T^{-1}$ exists.

(iv) $T$ is a condition $G_1$ operator.

**Proof:**
Let T be a hyponormal operator.
$\Rightarrow T^*T \geq TT^*$ (1)
(i) let $\mu \in C$

$$\begin{aligned} (T-\mu)^*(T-\mu) - (T-\mu)(T-\mu)^* &= (T^* - \overline{\mu})(T-\mu) - (T-\mu)(T^* - \overline{\mu}) \\ &= (T^*T - \overline{\mu}T - \mu T^* + |\mu|^2) - (TT^* - \mu T^* - \overline{\mu} \\ &= T^*T - TT^* \geq 0 \end{aligned}$$

Hence $T - \mu$ is hyponormal for all $\mu \in C$.
(ii) By (i), $T - \mu$ is hyponormal for all $\mu \in C$

$\Rightarrow T-\mu$ is normaloid for all $\mu \in C$ [$\because$ hyponormal operators are normaloid.]
$\Rightarrow$ T is transaloid operator.
(iii) By (i), $T^*T \geq TT^*$
$\Rightarrow T^{-1}T^*TT^{*-1} \geq I$
$\Rightarrow (T^{-1}T^*TT^{*-1})^{-1} \leq I^{-1}$
$\Rightarrow T^*T^{-1}T^{*-1}T \leq I$
or $I \geq T^*T^{-1}T^{*-1}T$
i.e $T^{*-1}T^{-1} \geq T^{-1}T^{*-1}$
i.e $(T^{-1})^*T^{-1} \geq T^{-1}(T^{-1})^*$
$\Rightarrow T^{-1}$ is also hyponormal, if $T^{-1}$ exists.
(iv) Consider

$$\begin{aligned} \frac{1}{d(\mu,\sigma(T))} &= \sup \frac{1}{|\sigma(T)-\mu|} for\ any\ \mu \notin \sigma(T) \\ &= \sup \frac{1}{|\sigma(T-\mu)|} [\text{by spectral mapping theorem}] \\ &= \sup |\sigma(T-\mu)^{-1}| [\because \sigma(T^{-1}) = \{\sigma(T)\}^{-1}] \\ &= \sigma(T-\mu)^{-1} \end{aligned}$$

i.e $\dfrac{1}{d(\mu,\sigma(T))} = \sigma(T-\mu)^{-1}$ (1)
By (i), $T-\mu$ is hyponormal for any $\mu \in C$.
By (iii), $(T-\mu)^{-1}$ is hyponormal for any $\mu \in C$.
Since a hyponormal operator is a normaloid, $r(T-\mu)^{-1} = \|(T-\mu)^{-1}\|$ (2)
From (1) and(2)
$\|(T-\mu)^{-1}\| = \dfrac{1}{d(\mu,\sigma(T))}\ \forall \mu \notin \sigma(T)$.
Hence T is a condition $G_1$ operator.
Hence the theorem.

**Theorem 4:**

The following properties hold:

(i) If T is a transaloid operator, then T is a convexoid operator

(ii) If T is a condition $G_1$ operator, then T is a convexoid operator

**Proof:**
(i)Let T be a transaloid operator, then by definition
$T-\mu$ is normaloid for any $\mu \in C$
Since a normaloid operator is also a spectraloid operator, $T-\mu$ is spectraloid for any $\mu \in C$
Hence T is convexoid.
By the theorem, " An operator T is convexoid iff $T-\lambda$ is spectraloid for all complex number $\lambda$"
Hence (i) is proved.
(ii) Let T be a condition $G_1$ operator
$\Rightarrow \|(T-\mu)^{-1}\| = \dfrac{1}{d(\mu,\sigma(T))}, \forall\ \mu \notin \sigma(T)$
$\because\ co\sigma(T) \supseteq \sigma(T),\ d(\mu,\sigma(T)) \geq d(\mu, co\sigma(T))$
$\dfrac{1}{d(\mu,\sigma(T))} \leq \dfrac{1}{d(\mu, co\sigma(T))}$
$\Rightarrow \|(T-\mu)^{-1}\| \leq \dfrac{1}{d(\mu, co\sigma(T))},\ \forall \mu \notin \sigma(T)$
Hence T is convexoid.
By theorem," An operator T is convexoid iff $\|(T-\mu)^{-1}\| \leq \dfrac{1}{d(\mu, co\sigma(T))},\ \forall \mu \notin co\sigma(T)$"
Hence (ii)is proved.
Hence the theorem.

**Corollary 5:**

(i) Every hyponormal operator is convexoid

(ii) Every normal operator is convexoid

**Proof:**
(i)Let T be hyponormal operator, then T is transaloid.
Then T is a convexoid operators.
By theorem,"Let T be a hyponormal operator. Then the following properties hold:

(i) $T-\mu$ is also a hyponormal for any $\mu \in C$.

(ii) $T$ is a transaloid operator.

(iii) $T^{-1}$ ia also a hyponormal operator if $T^{-1}$ exists.

(iv) $T$ is a condition $G_1$ operator.

and " If T is a transaloid operator, then T is a convexoid operator"
Hence a hyponormal is convexoid.
(ii) Since every normal operator is a hyponormal operator and every hyponormal operator is convexoid, every normal operator is convexoid.

**Theorem 6:**

An operator T is convexoid iff $Re\Sigma(e^{i\theta}T) = \Sigma(Re(e^{i\theta}T))$ for any $0 \leq \theta \leq 2\pi$, where $\Sigma(S)$ denotes $co\sigma(S)$.
**Proof:**
Let T be a convexoid operator.
$\Rightarrow \overline{W(T)} = co\sigma(T) = \Sigma(T)$
$\Rightarrow \overline{W(Re(e^{i\theta}T))} = \Sigma(Re(e^{i\theta}T))$ (1)
But $\overline{W(Re(e^{i\theta}T))} = Re\overline{W}(e^{i\theta}T) = Re\Sigma(e^{i\theta}T)$ (2)
From (1) and (2),
$Re\Sigma(e^{\theta}T) = \Sigma(Re(e^{i\theta}T))$ for any $0 \leq \theta \leq 2\pi$, where $\Sigma(S)$ denotes $co\sigma(S)$.
Conversely, assume that $Re\Sigma(e^{i\theta}T) = \Sigma(Re(e^{i\theta}T))$ then $Re\{\Sigma(e^{i\theta}T)\} = \Sigma\{(Re(e^{i\theta}T))\}$