3.1 Young inequality and Holder-Mccarthy inequality

3.1.1 Young inequality and generalized operator means

Theorem Y(Young inequality)

Let A and B be positive invertible operators on a Hilbert space H. Then the following inequality holds for $0 \leq \lambda \leq 1$.

$$(1-\lambda)A + \lambda B \geq A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2} \geq [(1-\lambda)A^{-1} + \lambda B^{-1}]^{-1}.$$

Proof:
Consider $f(x) = \lambda x + 1 - \lambda - x^{\lambda}$ for positive number x and $\lambda \in [0,1]$.

$$\begin{aligned} f'(x) &= \lambda - \lambda x^{\lambda-1},\ x \geq 0 \ and\ \lambda \in [0,1] \\ &= \lambda(1 - x^{\lambda-1}) \\ &= \lambda\left(1 - \frac{1}{x^{1-\lambda}}\right) \\ &= \lambda\left(1 - \left(\frac{1}{x}\right)^{1-\lambda}\right) \end{aligned}$$

$\Rightarrow\ f'(x) < 0$ for $0 < x < 1$.
$f'(x) = 0$ for $x = 1$.
$f'(x) > 0$ for $x > 1$.
Also $f(0^{+}) = 1 - \lambda > 0, f(1) = 0$.
Hence $f(x)$ is a non negative function
$\therefore$ for any positive operator T and $\lambda \in [0,1]$

$$f(T) = \lambda T + (1-\lambda) - T^{\lambda} \geq 0.$$

$\Rightarrow \lambda T + (1-\lambda) \geq T^{\lambda}\ \forall\ \lambda \in [0,1]$ (1)
If T is a positive operator then T^{-1} is also a positive operator.
Hence by (1)
$\lambda T^{-1} + (1-\lambda) \geq T^{-\lambda}$ (2)
$\Rightarrow (\lambda T^{-1} + (1-\lambda))^{-1} \leq T^{\lambda}$ (3)
From (1) and (3),
$\lambda T + 1 - \lambda \geq T^{\lambda} \geq (\lambda T^{-1} + 1 - \lambda)^{-1}$ (4)
Since A and B are positive invertible operator, $A^{-1/2}BA^{-1/2}$ is also a positive invertible operator.
Putting $T = A^{-1/2}BA^{-1/2}$ in (4) we get,
$\lambda(A^{-1/2}BA^{-1/2}) + 1 - \lambda \geq (A^{-1/2}BA^{-1/2})^{\lambda} \geq (\lambda A^{1/2}B^{-1}A^{1/2} + 1 - \lambda)^{-1}$ (5)

Multiplying (5) by $A^{1/2}$ on both sides, we get
$\lambda B + (1-\lambda)A \geq A^{1/2}(A^{1/2}BA^{-1/2})^{\lambda}A^{1/2} \geq A^{1/2}(\lambda A^{1/2}B^{-1}A^{1/2} + 1 - \lambda)^{-1}A^{1/2}$

$$\begin{aligned}(1-\lambda)A + \lambda B &\leq A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2} \\ &\geq A^{1/2}[A^{1/2}(\lambda B^{-1} + (1-\lambda)A^{-1})A^{1/2}]^{-1}A^{1/2} \\ &= A^{1/2}(A^{-1/2}(\lambda B^{-1} + (1-\lambda)A^{-1})A^{-1/2})A^{1/2} \\ &= [\lambda B^{-1} + (1-\lambda)A^{-1}]^{-1}\end{aligned}$$

Hence,
$(1-\lambda) + \lambda B \geq A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2} \geq [\lambda B^{-1} + (1-\lambda)A^{-1}]^{-1}$
Hence the theorem.

Theorem 1:

Let T be a positive operator on a Hilbert space H. Then the following hold.

(i) If $1 \geq \lambda \geq 0$ then $\lambda T + (1-\lambda) \geq T^{\lambda}$

(ii) If $\lambda > 0$ then $\lambda T + (1-\lambda) \leq T^{\lambda}$

(iii) If $\lambda < 0$ then $\lambda T + (1-\lambda) \leq T^{\lambda}$

In addition (i), (ii) and (iii) are mutually equivalent
Proof:

$$\begin{aligned}\text{Consider} f(x) &= \lambda x + 1 - \lambda - x^{\lambda} \text{ for } x > 0 \\ f'(x) &= \lambda - \lambda x^{\lambda-1} \\ &= \lambda\left(1 - \frac{1}{x^{1-\lambda}}\right).\end{aligned}$$

For $\lambda \in [0,1],\ f(0^+) = 1 - \lambda > 0$
$f(1) = 0$
$f'(x) < 0$ for $0 < x < 1$
$f'(1) = 0$
$f'(x) > 0$ for $x > 1$ (1)
For $\lambda > 1,\ f(0^+) = 1 - \lambda < 0$
$f(1) = 0$
$f'(x) > 0$ for $0 < x < 1$
$f'(1) = 0$
$f'(x) < 0$ for $x > 1$ (2)
For $\lambda < 0,\ f(0^+) < 0$
$f'(x) > 0,\ f(1) = 0,\ f'(x) < 0$ (3)
(1),(2),(3) impies,
$f(x) \geq 0\ \forall x > 0$ and $\lambda \in [0,1]$

i.e $1 \geq \lambda \geq 0$ (4)
$f(x) \leq 0 \; \forall \; x > 0$ and $\lambda > 1$ (5)
and $f(x) \leq 0 \; \forall \; x > 0$ and $\lambda < 0$ (6)
$\therefore$ If T is a positive invertible operator on a Hilbert space then from (4), (5) and (6), we get
$f(T) = \lambda T + (1-\lambda) - T^{\lambda} \geq 0$ for $1 \geq \lambda \geq 0$ (7)
$f(T) = \lambda T + (1-\lambda) - T^{\lambda} \leq 0$ for $\lambda > 1$ (8)
and $f(T) = \lambda T + (1-\lambda) - T^{\lambda} \leq 0$ for $\lambda < 0$ (9)
Hence
$\lambda T + (1-\lambda) \geq T^{\lambda}$ for $1 \geq \lambda \geq 0$
$\lambda T + (1-\lambda) \leq T^{\lambda}$ for $\lambda > 1$
and $\lambda T + (1-\lambda) \leq T^{\lambda}$ for $\lambda \leq 0$
Hence (i), (ii) and (iii) hold.
To prove that (i),(ii) and (iii) are mutually equivalent
To prove that (i) $\iff$ (ii):
Assum that $\lambda > 1$ then $\frac{1}{\lambda}$.
$\therefore$ by(1), $\left(\frac{1}{\lambda}\right)^{T} + \left(1 - \frac{1}{\lambda}\right) \geq T^{1/\lambda}$
$T + (\lambda - 1) \geq \lambda T^{1/\lambda}$
Put $S = T^{1/\lambda}$, then $S^{\lambda} + (\lambda - 1) \geq \lambda S$
$\Rightarrow S^{\lambda} \geq \lambda S + 1 - \lambda$ for $\lambda > 1$
Hence (i)$\Rightarrow$ (ii).
Similarly, (ii)$\Rightarrow$ (i)
Hence (i) $\iff$ (ii).
To prove that (ii) $\iff$ (iii)
Consider (ii),
$\lambda T + (1-\lambda) \leq T^{\lambda}$ for $\lambda > 1$.
Mutiplying this inequality by T^{-1} we get,
$\lambda + (1-\lambda)T^{-1} \leq T^{\lambda-1}$ for any $\lambda > 1$
Put $\mu = 1 - \lambda < 0$ and $S = T^{-1}$
$\Rightarrow \; (1-\mu) + \mu S \leq S^{1-\lambda} = S^{\mu}$
i.e $\mu S + (1-\mu) \leq S^{\mu}$ for $\mu < 0$
Hence (ii)$\Rightarrow$ (iii)
Similarly (iii)$\Rightarrow$ (ii).
Hence (ii) $\iff$ (iii)
Hence (i) $\iff$ (ii) $\iff$ (iii).
Hence the theorem.

Theorem 2:

Let A and B be the positive invertible operator on a Hilbert space H then the following hold and are mutually eqivalent.

(i) If $1 \geq \lambda \geq 0$ then $(1-\lambda)A + \lambda B \geq A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}$

(ii) If $\lambda > 1$ then $(1-\lambda)A + \lambda B \leq A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$

(iii) If $\lambda < 0$ then $(1-\lambda)A + \lambda B \leq A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$

Proof:
In the theorem,
"Let T be a positive operator on a Hilbert space H. Then the following hold.

(i) If $1 \geq \lambda \geq 0$ then $\lambda T + (1-\lambda) \geq T^{\lambda}$

(ii) If $\lambda > 1$ then $\lambda T + (1-\lambda) \leq T^{\lambda}$

(iii) If $\lambda \leq 0$ then $\lambda T + (1-\lambda) \leq T^{\lambda}$

In addition (i), (ii) and (iii) are mutually equivalent
Put $T = A^{-1/2}BA^{-1/2}$
Then we get,
(i) If $1 \geq \lambda \geq 0$
$\lambda(A^{-1/2}BA^{-1/2}) + 1 - \lambda \geq (A^{-1/2}BA^{-1/2})^{\lambda}$
Pre multiplying and post mutiplying by $A^{1/2}$ we get
$\lambda B + (1-\lambda)A \geq A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$
Similarly, we get
(ii) If $\lambda > 1$ then
$\lambda B + (1-\lambda)A \leq A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$
(iii) If $\lambda < 0$ then
$\lambda B + (1-\lambda)A \leq A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$
and (i), (ii) and (iii) are mutually equivalent.

3.1.2 Hölder-McCarthy inequality

Theorem: H-M(Hölder-McCarthy inequality)

Let A be a positive linear operator on a Hilbert space H. Then the following properties (i), (ii) and (iii) are hold.

(i) $\langle A^{\lambda}x, x\rangle \geq \langle Ax, x\rangle^{\lambda}$ for any $\lambda > 1$ and any unit vector x

(ii) $\langle A^{\lambda}x, x\rangle \leq \langle Ax, x\rangle^{\lambda}$ for any $\lambda \in [0,1]$ and any unit vector x

(iii) If A is invertible then $\langle A^{\lambda}x, x\rangle \geq \langle Ax, x\rangle^{\lambda}$ for any $\lambda < 0$ and any unit vector x

Moreover (i),(ii) and (iii) are equivalent to the following (i)', (ii)'and (iii)' respectively.

(i)' $\langle A^{\lambda}x, x\rangle \geq \langle Ax, x\rangle^{\lambda}\|x\|^{2(1-\lambda)}$ for any $\lambda > 1$ and any vector x

(ii)' $\langle A^{\lambda}x, x\rangle \leq \langle Ax, x\rangle^{\lambda}\|x\|^{2(1-\lambda)}$ for any $\lambda \in [0,1]$ and any vector x

(iii)' If A is invertible then $\left\langle A^{\lambda}x, x\right\rangle \geq \langle Ax, x\rangle^{\lambda} \|x\|^{2(1-\lambda)}$ for any $\lambda < 0$ and any vector x

Proof:
To prove (ii):
i.e $\left\langle A^{\lambda}x, x\right\rangle \leq \langle Ax, x\rangle^{\lambda} \|x\|^{2(1-\lambda)}$ for any $\lambda \in [0,1]$ and unit vector x
To prove this, it is sufficient to prove that if (ii) holds for some $\alpha, \beta \in [0,1]$ then (ii) holds for $\frac{\alpha+\beta}{2} \in [0,1]$, by the continuity of the operator.
Hence assume that,
$\langle A^{\alpha}x, x\rangle \leq \langle Ax, x\rangle^{\alpha}$ (1)
and $\left\langle A^{\beta}x, x\right\rangle \leq \langle Ax, x\rangle^{\beta}$ (2)
for any unit vector x and for some $\alpha, \beta \in [0,1]$
For any unit vector x consider

$$\begin{aligned}
\left|\left\langle A^{\frac{\alpha+\beta}{2}}x, x\right\rangle\right|^2 &= |\left\langle A^{\alpha/2}x, A^{\beta/2}x\right\rangle|^2 \; [\because A \geq 0] \\
&\leq \|A^{\alpha/2}x\|^2 \|A^{\beta/2}x\|^2 \\
&= \left\langle A^{\alpha/2}x, A^{\alpha/2}x\right\rangle \left\langle A^{\beta/2}x, A^{\beta/2}x\right\rangle \\
&= \langle A^{\alpha}x, x\rangle \left\langle A^{\beta}x, x\right\rangle \\
&= \langle Ax, x\rangle^{\alpha} \langle Ax, x\rangle^{\beta} \quad [by\ (1)\&(2)] \\
&= \langle Ax, x\rangle^{\alpha+\beta} \\
\left\langle A^{\frac{\alpha+\beta}{2}}x, x\right\rangle &\leq \langle Ax, x\rangle^{\frac{\alpha+\beta}{2}}
\end{aligned}$$

Hence (ii) holds for $\frac{\alpha+\beta}{2} \in [0,1]$
Hence for any $\lambda \in [0,1]$
$\left\langle A^{\lambda}x, x\right\rangle \leq \langle Ax, x\rangle^{\lambda}$ for any unit vector x (3)
Hence (ii) is true.
To prove(i):
Let $\lambda > 1$
$\Rightarrow \frac{1}{\lambda} \in [0,1]$
$\because$ by(ii), i.e (3) for any unit vector x

$$\begin{aligned}
\left\langle A^{1/\lambda}x, x\right\rangle &\leq \langle Ax, x\rangle^{1/\lambda} \\
\therefore \langle Ax, x\rangle &= \left\langle (A^{\lambda})^{1/\lambda}x, x\right\rangle \\
&\leq \left\langle A^{\lambda}x, x\right\rangle^{1/\lambda} \\
\Rightarrow \langle Ax, x\rangle^{\lambda} &\leq \left\langle A^{\lambda}x, x\right\rangle
\end{aligned}$$

i.e $\left\langle A^{\lambda}x, x\right\rangle \geq \langle Ax, x\rangle^{\lambda}$ (4)
for any unit vector x and $\lambda > 1$
Hence (i) holds.
(iii)Assume that A^{-1} exists

case (i): $\lambda = -1$
Then for any unit vector x

$$\begin{aligned} 1 = \|x\|^4 &= |\left\langle A^{-1/2}A^{1/2}x, x\right\rangle|^2 \\ &= |\left\langle A^{1/2}x, A^{-1/2}x\right\rangle|^2 \\ &\leq \|A^{1/2}x\|^2\|A^{-1/2}x\|^2 \\ &= \langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle \end{aligned}$$

$$\Rightarrow \langle A^{-1}x, x\rangle \geq \langle Ax, x\rangle^{-1} \tag{5}$$

for any unit vector x
case (ii):$\lambda < -1$
For any unit vector x

$$\begin{aligned} \left\langle A^{\lambda}x, x\right\rangle &= \left\langle A^{-|\lambda|}x, x\right\rangle \\ &= \left\langle (A^{-1})^{|\lambda|}x, x\right\rangle \quad where|\lambda| > 1 \\ &\geq \left\langle A^{-1}x, x\right\rangle^{|\lambda|} \quad by(4)\ i.e(i) \\ &\geq (\langle Ax, x\rangle^{-1})^{\lambda} \quad by(5) \\ &= \langle Ax, x\rangle^{-|\lambda|} \\ &= \langle Ax, x\rangle^{\lambda} \end{aligned}$$

Hence $\left\langle A^{\lambda}x, x\right\rangle \geq \langle Ax, x\rangle^{\lambda}$ for any $\lambda < -1$ (6)
Case(iii): Let $-1 \leq \lambda \leq 0$
Then for any unit vector x

$$\begin{aligned} \left\langle A^{\lambda}x, x\right\rangle &= \left\langle A^{-|\lambda|}x, x\right\rangle \\ &= \left\langle (A^{|\lambda|})^{-1}x, x\right\rangle \quad where|\lambda| \in [0, 1] \\ &\geq \left\langle A^{|\lambda|}x, x\right\rangle^{-1} \quad by(5) \\ &\geq (\langle Ax, x\rangle^{|\lambda|})^{-1} \quad by(6)i.e(ii) \\ &= \langle Ax, x\rangle^{-|\lambda|} \\ &= \langle Ax, x\rangle^{\lambda} \end{aligned}$$

Hence $\left\langle A^{\lambda}x, x\right\rangle \geq \langle Ax, x\rangle^{\lambda}$ for any $-1 \leq \lambda \leq 0$ (7)
(5),(6) and (7) implies,
$\left\langle A^{\lambda}x, x\right\rangle \geq \langle Ax, x\rangle^{\lambda}$ for any $\lambda < 0$ and any unit vector x
Hence (iii) holds.
Hence (i), (ii) and (iii) holds.
To prove that (i) $\iff$ (i)':
Consider (i),
$\left\langle A^{\lambda}x, x\right\rangle \geq \langle Ax, x\rangle^{\lambda}$ for any $\lambda > 1$ and any unit vector x

Replacing x by $\frac{x}{\|x\|}$ we get

$\left\langle A^{\lambda}\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle \geq \left\langle A\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle^{\lambda}$

for any $\lambda > 1$ and any vector x

$\iff \quad \frac{1}{\|x\|^2}\left\langle A^{\lambda}x, x\right\rangle \geq \frac{1}{\|x\|^{2\lambda}}\langle Ax, x\rangle^{\lambda}$

$\iff \quad \left\langle A^{\lambda}x, x\right\rangle \geq \frac{\|x\|^2}{\|x\|^{2\lambda}}\langle Ax, x\rangle^{\lambda}$

for any $\lambda > 1$ and any vector x

$\iff \left\langle A^{\lambda}x, x\right\rangle \geq \langle Ax, x\rangle^{\lambda}\|x\|^{2(1-\lambda)}$

for any $\lambda > 1$ and any vector x.

Hence (i) $\Rightarrow$ (i)'

Similarly, replacing x by $\frac{x}{\|x\|}$ in (ii)and (iii), we get (ii)' and (iii)'

Hence (i), (ii), and (iii) are equivalent to (i)', (ii)' and (iii)'

3.1.3 Hölder-McCarthy and Young inequalities are equivalent for Hilbert space operators

Theorem 1:

For a positive linear operator A on a Hilbert space H and $\lambda \in [0, 1]$. We give an elementary proof of the equivalence of the following two inequalities

(1) **H0lder- McCarrthy inequality:**
$\langle Ax, x\rangle^{\lambda} \geq \left\langle A^{\lambda}x, x\right\rangle$ for all unit vectors $x \in H$

(2) **Young inequality:**
$\lambda A + I - \lambda \geq A^{\lambda}$

Proof:

(1)$\Rightarrow$(2)

Assume (1), Hölder- McCarthy inequality

Consider, $f(x) = \lambda x + 1 - \lambda - x^{\lambda}$ for positive numbers x and $\lambda \in [0, 1]$

$$\begin{aligned} f'(x) &= \lambda - \lambda x^{\lambda-1},\ x > 0 \ and\ \lambda \in [0,1] \\ &= \lambda\left(1 - \frac{1}{x^{1-\lambda}}\right) \\ &= \lambda\left(1 - \left(\frac{1}{x}\right)^{1-\lambda}\right) \end{aligned}$$

$\Rightarrow f'(x) < 0$ for $0 < x < 1$

$f'(x) = 0$ for $x = 1$

$f'(x) > 0$ for $x > 1$

Also $f(0^+) = 1 - \lambda > 0$, f(1)=0

Hence f(x) is a nonnegative convex function with minimum value f(1)=0.
So we have
$\lambda a + 1 - \lambda \geq a^{\lambda}$ (1)
for positive a and $\lambda \in [0,1]$
Replacing a by $\langle Ax, x\rangle \geq 0$ for $\|x\| = 1$& $\lambda \in [0,1]$ in (1), we get

$$\begin{aligned}
\lambda \langle Ax, x\rangle + 1 - \lambda &\geq \langle Ax, x\rangle^{\lambda} \\
\langle \lambda Ax, x\rangle + 1 - \lambda &\geq \left\langle A^{\lambda}x, x\right\rangle \ by\ (1) \\
\lambda A + 1 - \lambda &\geq A^{\lambda}\ [\because \|x\| = 1]
\end{aligned}$$

Hence (1)$\Rightarrow$ (2) is proved.
(2)$\Rightarrow$ (1)
We may assume $\lambda \in [0,1]$
i.e $\quad \lambda A + 1 - \lambda \geq A^{\lambda}$
Replace A by $k^{1/\lambda}A$ for a positive number k, then
$\lambda k^{1/\lambda}A + 1 - \lambda \geq (k^{1/\lambda})^{\lambda}\ \forall\ x > 0$
$\Rightarrow \lambda k^{1/\lambda}\langle Ax, x\rangle + 1 - \lambda \geq (k^{1/\lambda})^{\lambda}\left\langle A^{\lambda}x, x\right\rangle \forall\ x > 0$ with $\|x\| = 1$
$\lambda k^{1/\lambda}\langle Ax, x\rangle + 1 - \lambda \geq k\left\langle A^{\lambda}x, x\right\rangle$ (2)
Put $k\langle Ax, x\rangle^{-\lambda}$ in(2) if $\langle Ax, x\rangle \neq 0$ then

$$\begin{aligned}
\lambda \langle Ax, x\rangle^{-1}\langle Ax, x\rangle + 1 - \lambda &\geq \langle Ax, x\rangle^{-\lambda}\left\langle A^{\lambda}x, x\right\rangle \\
\Rightarrow \lambda + 1 - \lambda &\geq \langle Ax, x\rangle^{-\lambda}\left\langle A^{\lambda}x, x\right\rangle \\
\Rightarrow 1 &\geq \langle Ax, x\rangle^{-\lambda}\left\langle A^{\lambda}x, x\right\rangle \\
\Rightarrow \langle Ax, x\rangle^{\lambda} &\geq \left\langle A^{\lambda}x, x\right\rangle
\end{aligned}$$

for $\|x\| = 1$.
If $\langle Ax, x\rangle = 0$ then $A^{1/2}x = 0$
So $A^{\lambda}x = 0$ for $\lambda \in [0,1]$, by induction and continuity of A.
Hence (2)$\Rightarrow$(1) is proved.

3.2 Lowner Heinz inequality and Furuta inequality

3.2.1 Simplified proofs three order preserving operator inequalities

Theorem L-H:(Lowner Heinz inequality)

$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\beta}$ for any $\alpha \in [0,1]$
Proof:
Case(i): $A \geq B > 0$
Let $A^{\alpha} \geq B^{\alpha}$ and $A^{\beta} \geq B^{\beta}$ for some $\alpha, \beta \in [0,1]$

It is sufficient to prove that $A^{\frac{\alpha+\beta}{2}} \geq B^{\frac{\alpha+\beta}{2}}$ by the continuity of an operator.
i.e To prove $A^{\frac{\alpha+\beta}{4}} A^{\frac{\alpha+\beta}{4}} \geq B^{\frac{\alpha+\beta}{2}}$
Pre and post multiply by $A^{-\frac{\alpha+\beta}{4}}$

$$\begin{aligned} A^{-\frac{\alpha+\beta}{4}} A^{\frac{\alpha+\beta}{4}} A^{\frac{\alpha+\beta}{4}} A^{-\frac{\alpha+\beta}{4}} &\geq A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}} \\ I &\geq A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}} \\ 0 &\leq A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}} \leq I \end{aligned}$$

By continuity to prove, $\|A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}}\| \leq 1$
Consider,

$$\begin{aligned} \|A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}}\| &= r(A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}}) \\ &= r[A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}} A^{\frac{\beta-\alpha}{4}} A^{\frac{\alpha-\beta}{4}}] \ [\because\ A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}}\ \textit{is positive}] \\ &= r(A^{\frac{\alpha-\beta}{4}} A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}} A^{\frac{\beta-\alpha}{4}}) \ [\because\ r(ST) = r(TS),\ S \geq 0\ \textit{and}\ T \geq 0] \\ &= r(A^{-\frac{\beta}{2}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha}{2}}) \\ &= r(A^{-\frac{\beta}{2}} B^{\frac{\beta}{2}} B^{\frac{\alpha}{2}} A^{-\frac{\alpha}{2}}) \\ &= \|(A^{-\frac{\beta}{2}} B^{\frac{\beta}{2}})(B^{\frac{\alpha}{2}} A^{-\frac{\alpha}{2}})\| \\ &\leq \|(A^{-\frac{\beta}{2}} B^{\frac{\beta}{2}})\| \|(B^{\frac{\alpha}{2}} A^{-\frac{\alpha}{2}})\| \\ &\leq 1 \\ \Rightarrow \|A^{-\frac{\alpha+\beta}{4}} B^{\frac{\alpha+\beta}{2}} A^{-\frac{\alpha+\beta}{4}}\| &\leq 1 \\ \Rightarrow A^{\frac{\alpha+\beta}{2}} &\geq B^{\frac{\alpha+\beta}{2}} \end{aligned}$$

$\therefore A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$
Hence proved.
Case(ii): In the general case $A \geq B \geq 0$
The condition $A \geq B \geq 0$ ensures $A + \epsilon \geq B + \epsilon \geq \epsilon$ for any $\epsilon > 0$
Then $A_1 = A + \epsilon$ and $B_1 = B + \epsilon$ are both invertible and $A_1 \geq B_1 \geq 0$
So that $A_1^\alpha \geq B_1^\alpha$ for any $\alpha \in [0,1]$ by case (i)
Let $\epsilon \to 0$ then we have
$A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$
Hence proved.

Theorem F:(Furuta inequality)

If $A \geq B \geq 0$ then for each $r \geq 0$.

(i) $(B^{r/2} A^p B^{r/2})^{1/q} \geq (B^{r/2} B^p B^{r/2})^{1/q}$ and

(ii) $(A^{r/2} A^p A^{r/2})^{1/q} \geq (A^{r/2} B^p A^{r/2})^{1/q}$
holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$

Proof:
For proving this theorem, we need the follwing lemma
Lemma:A
Let X be a positive invertible operator and Y be an invertible operator. For any real number λ

$$(YXY^*)^\lambda = YX^{1/2}(X^{1/2}Y^*YX^{1/2})^{\lambda-1}X^{1/2}Y^*$$

Proof: for lemma
Let $YX^{1/2} = UH$ be the polar decomposition of $YX^{1/2}$, where U is unitary and $H = |YX^{1/2}|$ [where $|T|$ denotes $(T*T)^{1/2}$]
Then

$$\begin{aligned}
(YXY^*)^\lambda &= (YX^{1/2}X^{1/2}Y^*)^\lambda \\
&= (UHH^*U^*)^\lambda [\because YX^{1/2} = UH,\ X^{1/2}Y^* = H^*U^*] \\
&= (UH^2U^*)^\lambda \\
&= UH^{2\lambda}U^* \\
&= YX^{1/2}H^{-1}H^{2\lambda}H^{-1}X^{1/2}Y^* [\because X > 0, U^* = (YX^{1/2}H^{-1})^* = H^{-1}X^{1/2}Y^*] \\
&= YX^{1/2}H^{2(\lambda-1)}X^{1/2}Y^* \\
&= YX^{1/2}(H^2)^{\lambda-1}X^{1/2}Y^* \\
&= YX^{1/2}(X^{1/2}Y^*YX^{1/2})^{\lambda-1}X^{1/2}Y^* [\because\ H^2 = (YX^{1/2})^*(YX^{1/2}) = X^{1/2}Y^*YX^{1/2}]
\end{aligned}$$

Hence the lemma.
Proof of the Theorem:
Let $A \geq B \geq 0,\ r \geq 0$
To prove (ii): i.e $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$ holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$
case(i): $0 \leq p \leq 1$
Since $0 \leq p \leq 1$, by Lowner- Heinz inequality,
$A \geq B \geq 0$
$\Rightarrow A^p \geq B^p$
$\Rightarrow (A^{r/2}A^pA^{r/2}) \geq (A^{r/2}B^pA^{r/2})$
Hence again by Lowner- Heinz inequality, $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$
for $q \geq 1$ and $(1+r)q \geq p+r$
Case (ii):
Consider $p \geq 1$ and $q > \frac{p+r}{1+r}$ and $r \geq 0$
then $q > \frac{p+r}{1+r} \geq 1 \Rightarrow \frac{1}{q} < \frac{1+r}{p+r} < 1$ [$\because p \geq 1, p+r \geq 1+r$]
if $(A^{r/2}A^pA^{r/2}) \geq (A^{r/2}B^pA^{r/2})^{1/q}$
Then by Lowner- Heienz inquality, $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$
Therefore it is enough to show that
$(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$
for $q = \frac{p+r}{1+r}, p \geq 1$ & $r \geq 0$
i.e $(A^{p+r})^{\frac{1+r}{p+r}} \geq (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}$

i.e $A^{1+r} \geq (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}$ (1)
for $p \geq 1$ and $r \geq 0$.
Without loss of generality, assume that A and B are invertible.
If $r \in [0,1]$ then $A \geq B \geq 0 \Rightarrow A^r \geq B^r$ [by L-H]
Therefore

$$\begin{aligned}
(A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}} &= A^{r/2}B^{p/2}(B^{p/2}A^rB^{p/2})^{\frac{1+r}{p+r}-1}B^{p/2}A^{r/2} \\
&= A^{r/2}B^{p/2}(B^{p/2}A^rB^{p/2})^{-\frac{p-1}{p+r}}B^{p/2}A^{r/2} \\
&= A^{r/2}B^{p/2}(B^{-p/2}A^{-r}B^{-p/2})^{\frac{p-1}{p+r}}B^{p/2}A^{r/2} \\
&\leq A^{r/2}B^{p/2}(B^{-p/2}B^{-r}B^{-p/2})^{\frac{p-1}{p+r}}B^{p/2}A^{r/2} \\
&= A^{r/2}(B^{-(r+p)})^{\frac{p-1}{p+r}+p}A^{r/2} \\
&= A^{r/2}(B^{-p+1+p})A^{r/2} \\
&= A^{r/2}BA^{r/2} \\
&\leq A^{r/2}AA^{r/2} \\
&= A^{1+r}
\end{aligned}$$

Hence $A \geq B \geq 0$ implies,
$A^{1+r} \geq (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}$ for $r \in [0,1]$ (2)
$p \geq 1$ and $q = \frac{1+r}{p+r}$
Put $A_1 = A^{1+r}$, $B_1 = (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}$ in (2) then $A_1 \geq B_1 \geq 0$.
Repeating (2) for $A_1 \geq B_1 \geq 0$,
$A_1^{1+r} \geq (A_1^{r/2}B_1^pA_1^{r/2})^{\frac{1+r}{p+r}}$ for $r_1 \in [0,1]\&p_1 \geq 1$.
Put $p_1 = \frac{p+r}{1+r} > 1$ and $r_1 = r$

$$\begin{aligned}
(A^{1+r})^2 &\geq \{(A^{r+1})^{1/2}[(A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}]^{\frac{p+r}{1+r}}(A^{r+1})^{1/2}\}^{\frac{2(1+r)}{p+2r+1}} \\
A^{2(1+r)} &\geq \{A^{(r+1)/2}A^{r/2}B^pA^{r/2}A^{(r+1)/2}\}^{\frac{2(1+r)}{p+2r+1}} \\
&= \{A^{r+1/2}B^pA^{r+1/2}\}^{\frac{2(1+r)}{p+2r+1}}
\end{aligned}$$

for $p \geq 1$ and $r \in [0,1]$.
Put $s/2 = r + 1/2$
$\Rightarrow s = 2r + 1$
$\therefore 2r + 2 = s + 1$
Substitute in (3)
$A^{s+1} \geq (A^{s/2}B^pA^{s/2})^{\frac{s+1}{s+p}}$ for $p \geq 1$ and $s = 2r + 1 \in [1,3]$ (4)
Therefore From (2)and (4),
$A^{1+r} \geq (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \in [0,3]$
Repeating this process,
(1) holds for any $r \geq 0$ and $p \geq 1$

Therefore $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$ holds $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$
Hence (ii) is proved.
To prove (i):
If $A \geq B > 0$ then $B^{-1} \geq A^{-1} > 0$.
Hence by (ii) for each $r \geq 0$
$((B^{-1})^{r/2}(B^{-1})^p(B^{-1})^{r/2})^{1/q} \geq ((B^{-1})^{r/2}(A^{-1})^p(B^{-1})^{r/2})^{1/q}$
$B^{-\frac{(p+r)}{q}} \geq (B^{-r/2}A^{-p}B^{-r/2})^{1/q}$
holds for each p and q such that $p \geq 0$ and $q \geq 1$ and $(1+r)q \geq p+r$.
Taking inverse,

$$\begin{aligned} (B^{-r/2}A^{-p}B^{-r/2})^{-1/q} &\geq B^{\frac{(p+r)}{q}} \\ (B^{r/2}A^pB^{r/2})^{1/q} &\geq B^{\frac{(p+r)}{q}} \\ (B^{r/2}A^pB^{r/2})^{1/q} &\geq (B^{r/2}B^pB^{r/2})^{1/q} \end{aligned}$$

for each $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+r)q \geq p+r$
Hence (i) is proved.

Theorem F1:

If $A \geq B \geq 0$ then the following inequalities hold:

(i) $(B^{r/2}A^pB^{r/2})^{\frac{1+r}{p+r}} \geq B^{1+r}$

(ii) $A^{1+r} \geq (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$

Proof:
Consider the Furuta inequality, If $A \geq B \geq 0$ then for each $r \geq 0$.

(i) $(B^{r/2}A^pB^{r/2})^{1/q} \geq (B^{r/2}B^pB^{r/2})^{1/q}$ (1)

(ii) $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$ (2)
holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$

Let $q = \frac{p+r}{1+r}$ then $q \geq 1$ if $p \geq 1$ and $r \geq 0$
$\therefore$ by (1), $(B^{r/2}A^pB^{r/2})^{\frac{1+r}{p+r}} \geq (B^{p+r})^{\frac{1+r}{p+r}}$
$\Rightarrow (B^{r/2}A^pB^{r/2})^{\frac{1+r}{p+r}} \geq B^{1+r}$
and by (2),
$A^{1+r} \geq (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$
Hence the theorem.

Theorem F'

If $A \geq C \geq B \geq 0$ then for each $r \geq 0$
$(C^{r/2}A^pC^{r/2})^{1/q} \geq (C^{r/2}C^pC^{r/2})^{1/q} \geq (C^{r/2}B^pC^{r/2})^{1/q}$
holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$
Proof:
Consider the Furuta inequality, If $A \geq B \geq 0$ then for each $r \geq 0$.

(i) $(B^{r/2}A^pB^{r/2})^{1/q} \geq (B^{r/2}B^pB^{r/2})^{1/q}$

(ii) $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$
holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$

Since $A \geq C$ by (i)
$$(C^{r/2}A^pC^{r/2})^{1/q} \geq (C^{r/2}C^pC^{r/2})^{1/q} \quad (1)$$
Since $C \geq B$ by (ii)
$$(C^{r/2}C^pC^{r/2})^{1/q} \geq (C^{r/2}B^pC^{r/2})^{1/q} \quad (2)$$
By (1) and (2), for each $r \geq 0$,
$(C^{r/2}A^pC^{r/2})^{1/q} \geq (C^{r/2}C^pC^{r/2})^{1/q} \geq (C^{r/2}B^pC^{r/2})^{1/q}$
holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$
Similarly taking B=C in (1), we get (i) and taking A=C in(2), we get (ii)
Hence Furuta inequality and (1) and (2) are equivalent.

Theorem F"

If $A \geq C \geq B \geq 0$ holds iff
$(C^{r/2}A^pC^{r/2})^{1/q} \geq (C^{r/2}C^pC^{r/2})^{1/q} \geq (C^{r/2}B^pC^{r/2})^{1/q}$
holds for all $r \geq 0,\ p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$
Proof:
By Theorem F',
$$A \geq C \geq B \geq 0 \ \Rightarrow (C^{r/2}A^pC^{r/2})^{1/q} \geq (C^{r/2}C^pC^{r/2})^{1/q} \geq (C^{r/2}B^pC^{r/2})^{1/q} \quad (1)$$
holds for all $r \geq 0,\ p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$
Conversely, assume (1),
Put $r = 0,\ p = q = 1$ then we get $A \geq C \geq B$
Hence the theorem.

Theorem G: (Generalized Furuta inequalitiy)

If $A \geq B \geq 0$ with $A > 0$ then for $t \in [0,1]$ and $p \geq 1$
$A^{1-t+r} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+t}}$
for $s \geq 1$ and $r \geq t$
Proof:
Assume that B is invertible.

Claim: If $A \geq B \geq 0$ with $A > 0$, then
$A \geq \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{\frac{1}{(p-t)s+t}}$ (1)
for $t \in [0,1]$ $p \geq 1$ and $s \geq 1$.
Let $A \geq B > 0$, $t \in [0,1]$
Suppose if $1 \leq s \leq 2, \quad p \geq 1$.
then $s-1, \ \frac{1}{(p-t)s+t} \in [0,1]$
Since $t \in [0,1]$ by Lowner- Heinz inequality $A \geq B > 0 \ \Rightarrow A^t \geq B^t$ (2)
Let $B_1 = \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{\frac{1}{(p-t)s+t}}$ and $A_1 = A$
By lemma,

$$(YXY^*)^{\lambda} = YX^{1/2}(X^{1/2}Y^*YX^{1/2})^{\lambda-1}X^{1/2}Y^*$$

where $X \geq 0$ we have
$(A^{-t/2}B^pA^{-t/2})^s = A^{-t/2}B^{p/2}(B^{p/2}A^{-t/2}A^{-t/2}B^{p/2})^{s-1}B^{p/2}A^{-t/2}$

$$\begin{aligned}
\therefore B_1 &= \{B^{t/2}(B^{p/2}A^{-t}B^{p/2})^{s-1}B^{p/2}\}^{\frac{1}{(p-t)s+t}} \\
&\leq \{B^{t/2}(B^{p/2}B^{-t}B^{p/2})^{s-1}B^{p/2}\}^{\frac{1}{(p-t)s+t}} \ [by(2)] \\
&= \{B^{p/2+(p-t)(s-1)+p/2}\}^{\frac{1}{(p-t)s+t}} \\
&= \{B^{p+(p-t)s-p-t}\}^{\frac{1}{(p-t)s+t}} \\
&= B \leq A = A_1
\end{aligned}$$

i.e $B_1 \leq A_1$ (3)
Hence (1) is proved for $1 \leq s \leq 2$.
Now since $A_1 \geq B_1 > 0$, replacing A by A_1 and B by B_1 in (1), we get ,

$$A_1 \geq \{A_1^{t_1/2}(A_1^{-t_1/2}B_1^{p_1}A_1^{-t_1/2})^{s_1}A_1^{t_1/2}\}^{\frac{1}{(p_1-t_1)s_1+t_1}} \quad (4)$$

for $1 \leq s_1 \leq 2, \ p_1 \geq$ and $t_1 \in [0,1]$
Put $t_1 = t \in [0,1]$ and $p_1 = (p-t)s+t \geq 1$ in (4)
then,

$$\begin{aligned}
A &\geq \{A^{t/2}(A^{-t/2}([A^{t/2}(A^{-t/2}B^pA^{-t/2})A^{t/2}]^{\frac{1}{(p-t)s+t}})^{(p-t)s+t}A^{-t/2})^{s_1}A^{t/2}\}^{\frac{1}{(p-t)ss_1+t}} \\
&= \{A^{t/2}(A^{-t/2}[A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}]A^{-t/2})^{s_1}\}^{\frac{1}{(p-t)ss_1+t}} \\
&= \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^{ss_1}A^{t/2}\}^{\frac{1}{(p-t)ss_1+t}} \text{ for } t \in [0,1], \ p \geq 1 \text{ and } 1 \leq ss_1 \leq 4 \quad (5)
\end{aligned}$$

Repeating this process, we get (1), for $t \in [0,1], \ p \geq 1$ and any $s \geq 1$
Put $A_2 = A$ and $B_2 = \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{\frac{1}{(p-t)s+t}}$ in (1) then $A_2 \geq B_2 \geq 0$
Therefore by Furuta inequality (ii) for $t \in [0,1]$ and $p \geq 1$ and $s \geq 1$

$$A_2^{1+r_2} \geq (A_2^{r_2/2}B_2^{p_2}A_2^{r_2/2})^{\frac{1+r_2}{p_2+r_2}} \text{ holds for } p_2 \geq 1 \text{ and } r_2 \geq 0 \quad (6)$$

Put $r_2 = r-t \geq 0$ and $p_2 = (p-t)s+t \geq 1$ in (6) then

$$\begin{aligned}
A^{1+r-t} &\geq (A^{\frac{r-t}{2}}A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}{}^{\frac{r-t}{2}}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+t}} \\
&= A^{r/2}(A^{-t/2}B^pA^{-t/2})A^{r/2}{}^{\frac{1+r-t}{(p-t)s+r}}
\end{aligned}$$

for $s \geq 1$ and $r \geq t$
Hence the Generalized Furuta inequality is proved.