25-11-19
(Mon)

Fluid Dynamics

Subject code :- 18MMA42C

Unit I :-

Kinematics of Fluids :- Methods of describing fluid motion; Lagrangian method - Eulerian method - Translation, Rotation and rate of deformation - Stream lines, path lines and streak lines - The Material derivatives and acceleration - Vorticity.

Fundamental Equations of the flow of Viscous Compressible Fluids :-

The equation of continuity - Conservation of mass - The equation of mass - Conservation of momentum, The equation of energy - Conservation of energy.

Chapter 3 - Section 3·1, 3·1a, 3·1b, 3·2, 3·3a, 3·3b, 3·3c; 3·4, 3·5

chapter 5 - Section 5·1 to 5·3

unit II :-

One dimensional Inviscid Incompressible Flow :- The equation of continuity - Stream flow tube; equation of motion - Euler's equation - The Bernoulli's equation - Flow from a tank through a small orifice - Trajectory of a free jet - The momentum theorem.

Two and Three dimensional Inviscid Incompressible flow :- Equation of continuity - Eulerian equation of motion - Circulation theorem (Kelvin's) - Velocity potential - Irrotational flow - Integration of the equations of motion - Bernoulli's equation - The momentum theorem - The moment of momentum theorem

chapter 6 - Section 6·1 to 6·3, 6·4a, 6·4b, 6·6 chapter 7 - Section 7.1, 7.2, 7.3a, 7.3b, 7.3c, 7.4, 7.5, 7.5a, 7.5b, 7.6, 7.7

Unit III :-

Laplace equation - Boundary conditions - Stream function in two dimensional motion. The flow net - Stream function in three dimensional motion - two dimensional flow examples - Rectilinear flow - Source and sink - Radial flow - Vortex flow - Doublet - Three dimensional axially symmetric flow - uniform flow - Radial flow - Radial flow (source or sink) - Doublet

chapter 7 - Section 7.8a, 7.8b, 7.9 to 7.11, 7.12a, 7.12b, 7.12c, 7.12d, 7.13a, 7.13b, 7.13c

Unit IV :- Laminar flow of Viscous Incompressible Fluids :-

Similarity of flows - The Reynolds number - Flow between parallel flat plates - Coutte flow - Plane Poiseuille flow - Steady flow in pipes - Flow through a pipe - The Hagen Poiseuille flow - Flow between coaxial cylinders

chapter 8 - Section 8.1, 8.3, 8.3a, 8.3b, 8.4a, 8.4b

Unit V :- Boundary layer Theory :-

Properties of Navier-Stokes Equations - Boundary layer concept - The Boundary layer equations in two dimensional flow - The Boundary layer along a flat plate - The Blasius solution - Shearing stress and Boundary layer thickness - Boundary layer on a surface with pressure gradient - Momentum integral theorem for the boundary layer - The Von Karman Integral relation.

chapter 9- Section 9.1, 9.2, 9.3a, 9.3b, 9.4, 9.5a

Textbook :-

Foundation of Fluid Mechanics - S.W Yuan, Prentice Hall of India Private Limited.

26-11-19 (Tue)

Unit - I

Basic definitions

Fluid :-

A fluid is any substance which is capable of flow.

Examples :- Water, air, oil etc.

Incompressible fluids :-

A fluid in which density of fluid does not change while change in external force or pressure is known as incompressible fluid.

Example :- All liquids

Compressible fluid :-

A fluid in which density of fluid changes while changes in external forces or pressure is known as compressible fluid

Example :- All gases.

2M (S) **Steady flow :-**

If the flow parameters do not vary with time (ie) independent of time) then the flow is called steady.

Example :- Pipe flow

2M (S) **Unsteady flow :-**

If the flow parameters (velocity, pressure...) vary with time then the flow is called unsteady.

Example :- Heating oil

Uniform flow :-

A flow is said to be uniform when the velocity of flow does not change either in magnitude or in direction in a flowing fluid at any point.

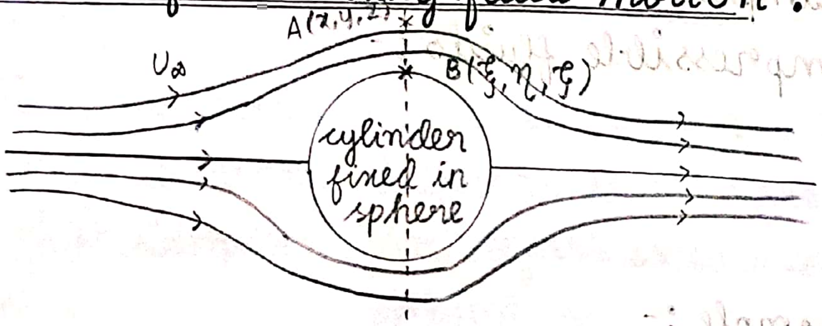
Example :- The flow of liquid under pressure through long pipeline with constant diameter

Non-uniform flow :-

A flow is said to be non-uniform when there is a change in velocity of the flow at different points in a flowing fluid.

Example :- natural sources

Methods of describing fluid motion :-



To determine the position of every particle of the fluid for every instant of time. The velocity and acceleration of a fluid particle can be found from its change in position as time goes on.

The study of motion of fluid particle by two methods

i) Lagrangian method

ii) Eulerian method

5M(S) 2M(S) **Lagrangian method :-**

In the Lagrangian representation, a rectangular system of co-ordinates (x, y, z) is used. A particle of a fluid is given a fixed identity by specifying the initial position ξ at a given time $t = t_0$. At a later time $t = t_1$ the same particle is at a position $\bar{r}$. Then $\bar{r} = \bar{F}(\vec{\xi}, t)$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\bar{\xi} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ (or)

$$x\hat{i} + y\hat{j} + z\hat{k} = \bar{F}(x_0\hat{i} + y_0\hat{j} + z_0\hat{k}, t)$$

$$\left.\begin{array}{l} x = F_1(x_0, y_0, z_0, t) \\ y = F_2(x_0, y_0, z_0, t) \\ z = F_3(x_0, y_0, z_0, t) \end{array}\right\} \text{—②}$$

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The velocity and acceleration of a fluid particle ξ can be obtained from the material derivatives of the position of the particle

$$u = \left(\frac{dx}{dt}\right)_\xi \quad v = \left(\frac{dy}{dt}\right)_\xi \quad w = \left(\frac{dz}{dt}\right)_\xi$$

$$\left.\begin{array}{l} u = \left(\frac{dx}{dt}\right)_\xi = \left(\frac{\partial x}{\partial t}\right)_\xi \\ v = \left(\frac{dy}{dt}\right)_\xi = \left(\frac{\partial y}{\partial t}\right)_\xi \\ w = \left(\frac{dz}{dt}\right)_\xi = \left(\frac{\partial z}{\partial t}\right)_\xi \end{array}\right\} \text{—③} \qquad \left.\begin{array}{l} a_x = \left(\frac{\partial^2 x}{\partial t^2}\right)_\xi \\ a_y = \left(\frac{\partial^2 y}{\partial t^2}\right)_\xi \\ a_z = \left(\frac{\partial^2 z}{\partial t^2}\right)_\xi \end{array}\right\} \text{—④}$$

and for the acceleration in x, y, z direction are a_x, a_y and a_z respectively.

Thus ~~in~~ Lagrangian method describ~~ing~~es the fluid motion in terms of fluid particles.

2M(S) 5M(S)

Eulerian method :-

In this method the individual fluid particles are not identified. Instead a fixed position in space is chosen, and the velocity of particles as function of time at this position. The velocity of particles at any point in the space can be written as

$$\bar{q} = \bar{f}(\bar{r}, t)$$

where $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$, $\bar{r}$ is the position vector

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(or) $u\hat{i} + v\hat{j} + w\hat{k} = \bar{f}(x\hat{i} + y\hat{j} + z\hat{k}, t)$

$$\left.\begin{array}{l} u = f_1(x, y, z, t) \\ v = f_2(x, y, z, t) \end{array}\right\}$$

$w = f_3(x,y,z,t)\}$ — (5)

(S) ※ The [relation between the Eulerian and Lagrangian method] (3) and (5) ⇒

According to the Lagrangian method (S) ※ 5M(S)

$$\left.\begin{aligned} \frac{dx}{dt} &= u(x,y,z,t) \\ \frac{dy}{dt} &= v(x,y,z,t) \\ \frac{dz}{dt} &= w(x,y,z,t) \end{aligned}\right\} \text{—(6)}$$

the fluid particle is described in terms of rectangular components x, y, z and $\bar{q}$ be the velocity vector
$\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

Integrating (6) with respect to t and using the initial condition x_0, y_0, z_0

$$\left.\begin{aligned} x &= F_1(x_0, y_0, z_0, t) \\ y &= F_2(x_0, y_0, z_0, t) \\ z &= F_3(x_0, y_0, z_0, t) \end{aligned}\right\} \text{—(7)}$$

(2) and (7) are identical] 5M(S)] 5M(S)

Example :-

10M (S) + 5M(S)

1\. Assuming that the velocity components for a 2D flow system can be given in the Eulerian system by

$$\left.\begin{aligned} u &= A(x+y) + ct \\ v &= B(x+y) + Et \end{aligned}\right\} \text{—(1)}$$

where A, B, C, E are constants. Find the displacement of a fluid particle in the Lagrangian system.

Solution :-

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt}$$

∴ (1) becomes

$$\frac{dx}{dt} = Ax + Ay + ct$$

$$\Rightarrow \left(\frac{d}{dt} - A\right)x - Ay = ct$$

Let $\frac{d}{dt} = D$

$$(D-A)x - Ay = Ct \quad \text{—— (2)}$$

$$\frac{dy}{dt} - Bx - By = Et$$

$$-Bx + (D-B)y = Et \quad \text{—— (3)}$$

<u>To find y</u> :- Solving (2) and (3)

$$(2) \times B \Rightarrow B\cancel{(D-A)x} - ABy = BCt$$

$$(3) \times (D-A) \Rightarrow -B\cancel{(D-A)x} + (D-A)(D-B)y = (D-A)Et$$

$$[(D-A)(D-B) - AB]y = [(D-A)E + BC]t$$

$$[D^2 - DA - DB + \cancel{AB} - \cancel{AB}]y = DEt + BCt - AEt$$

$$[D^2 - (A+B)D]y = (BC-AE)t + \frac{d}{dt}(Et)$$

$$[D^2 - (A+B)D]y = (BC-AE)t + E$$

It is of the form $[f(D)]y = f(t)$

where $f(D) = D^2 - (A+B)D$

$f(t) = (BC-AE)t + E$

Let $A+B = P \quad BC-AE = Q$

$$[D^2 - PD]y = Qt + E$$

<u>Complementary function</u> (C.F) :-

Replace D by m in $f(D) = 0$

$$m^2 - Pm = 0$$

$$m(m-P) = 0$$

$$\Rightarrow m = 0 \text{ or } m = P$$

$$C.F = C_1 e^{0\cdot \cancel{x} t} + C_2 e^{P \cancel{x} t}$$

ie) $C.F = C_1 + C_2 e^{P\cancel{x}t}$

<u>Particular Integral</u> :- (P.I)

$$P.I_1 = \frac{1}{f(D)} E(e^{0t}) = E \frac{1}{D^2 - PD} e^{0t}$$

Replace D by 0

$$P.I_1 = E \frac{1}{0} e^{0t} = E \frac{t}{2D-P} e^{0t}$$

[multiply the Nr by t and differentiate the Dr w.r.to D]

$$= \frac{Et}{0-P}$$

$$P \cdot I_1 = \frac{-Et}{P}$$

$$P \cdot I_2 = \frac{1}{f(D)} Qt$$

$$= Q \frac{1}{D^2 - PD} t$$

$$= Q \frac{1}{-PD\left[\frac{D^{\cancel{2}}}{-P\cancel{D}} + 1\right]} t$$

$$= \frac{Q}{-PD}\left[1 - \frac{D}{P}\right]^{-1} t$$

$$= \frac{-Q}{P} \frac{1}{D}\left(1 + \frac{D}{P} + \frac{D^2}{P^2}\right) t$$

$$= \frac{-Q}{P}\left[\frac{1}{D} + \frac{1}{\cancel{D}}\left(\frac{\cancel{D}}{P}\right) + \frac{1}{\cancel{D}}\left(\frac{D^{\cancel{2}}}{P^2}\right)\right] t$$

$$= \frac{-Q}{P}\left[\frac{1}{D}(t) + \frac{1}{P}(t) + \frac{D}{P^2}(t)\right]$$

$$= \frac{-Q}{P}\left[\int t\,dt + \frac{t}{P} + \frac{1}{P^2}\right]$$

$$P \cdot I_2 = \frac{-Q}{P}\left[\frac{t^2}{2} + \frac{t}{P} + \frac{1}{P^2}\right]$$

$$\therefore \quad y(t) = C \cdot F + P \cdot I_1 + P \cdot I_2$$

$$y(t) = c_1 + c_2 e^{Pt} - \frac{Et}{P} - \frac{Q}{P}\left[\frac{t^2}{2} + \frac{t}{P} + \frac{1}{P^2}\right]$$

$$(3) \Rightarrow -Bx + (D - B)y = Et$$

$$-Bx = Et - Dy + By \qquad D = \frac{d}{dt}$$

$$x = -\frac{Et}{B} + \frac{Dy}{B} - \frac{\cancel{B}}{\cancel{B}} y$$

$$x(t) = -\frac{E}{B} t - y(t) + \frac{1}{B} \frac{d}{dt}[y(t)]$$

$$\frac{d}{dt}[y(t)] = c_2 P e^{Pt} - \frac{E}{P} - \frac{Q}{P}\left(t + \frac{1}{P}\right)$$

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$$x(t) = -\frac{Et}{B} - C_1 - C_2 e^{Pt} + \frac{Et}{P} + \frac{Q}{P}\left(\frac{t^2}{2} + \frac{t}{P} + \frac{1}{P^2}\right) + \frac{1}{B}\left[C_2 P e^{Pt} - \frac{E}{P} - \frac{Q}{P}\left(t + \frac{1}{P}\right)\right]$$

$$= -\frac{Et}{B} + \frac{Et}{P} - C_1 - C_2 e^{Pt} + \frac{Q}{P}\left(\frac{t^2}{2} + \frac{t}{P} + \frac{1}{P^2}\right) + \frac{1}{B}\left[C_2 P e^{Pt} - \frac{E}{P} - \frac{Q}{P}\left(t + \frac{1}{P}\right)\right]$$

The constants c_1 and c_2 are found from the initial conditions $x = x_0$, $y = y_0$ at $t = t_0 = 0$

$$x_0 = -C_1 - C_2 + \frac{Q}{P}\left(\frac{1}{P^2}\right) + \frac{1}{B}\left(C_2 P - \frac{E}{P} - \frac{Q}{P^2}\right)$$

$$y_0 = C_1 + C_2 - \frac{Q}{P}\left(\frac{1}{P^2}\right)$$

$$x_0 = -C_1 - C_2 + \frac{Q}{P^3} + \frac{1}{B} C_2 P - \frac{E}{PB} - \frac{Q}{BP^2}$$

$$C_1 + C_2 - \frac{1}{B} P C_2 = \frac{Q}{P^3} - \frac{E}{PB} - \frac{Q}{BP^2} - x_0$$

$$C_1 + \left(1 - \frac{P}{B}\right) C_2 = \frac{Q}{P^3} - \frac{E}{PB} - \frac{Q}{BP^2} - x_0 = R_2$$

$$C_1 + C_2 = y_0 + \frac{Q}{P^3} = R_1$$

$$\cancel{C_1} + C_2 = R_1$$

$$-\cancel{C_1} - \left(1 - \frac{P}{B}\right) C_2 = -R_2$$

$$\left(\cancel{1} - \cancel{1} + \frac{P}{B}\right) C_2 = R_1 - R_2$$

$$C_2 = (R_1 - R_2)\frac{B}{P}$$

$$C_1 + (R_1 - R_2)\frac{B}{P} = R_1$$

$$C_1 = R_1 - \frac{R_1 B}{P} + \frac{R_2 B}{P} = R_1\left(1 - \frac{B}{P}\right) + R_2\left(\frac{B}{P}\right)$$

$$= \frac{R_1 P - (R_1 - R_2) B}{P}$$

$$C_1 = \frac{R_1(P - B) + R_2 B}{P} \qquad C_2 = \frac{(R_1 - R_2) B}{P}$$

∴ The solution can be written in the form

Substituting c_1 and c_2 in $x(t)$ and $y(t)$,

$x = F_1(x_0, y_0, t)$

$y = F_2(x_0, y_0, t)$

∴ The Lagrangian system is the functions of the initial positions x_0, y_0 and the time t

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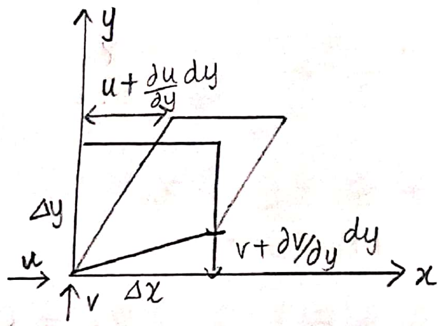
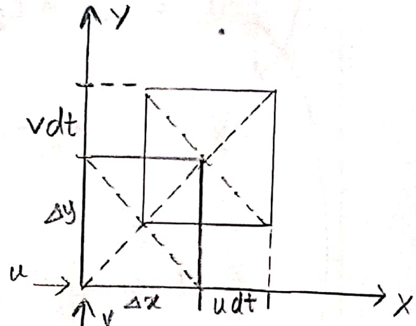
(10M) **Translation, Rotation and rate of deformation :-**

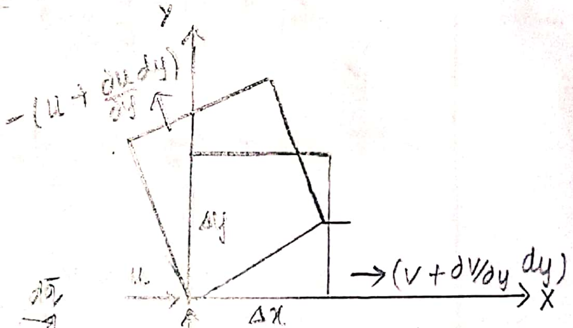
The movement of fluid element may consist of the translation, a rotation and a rate of deformation.



d) Rate of Angular deformation.

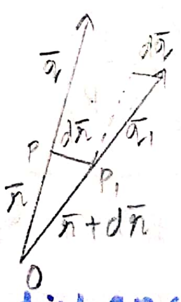


a) Translation

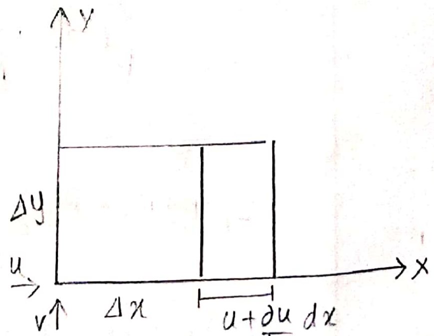


b) Rotation

Consider the motion of the fluid. The velocity at point $P(x, y, z)$ is $\bar{q}$ and at a point



$P_1(x, y, z)$, a small distance $d\bar{r}$ from P is $\bar{q}_1$. The velocity $\bar{q}_1$ can be written as



c) Rate of linear deformation

$\bar{q}_1 = u_1 \hat{i} + v_1 \hat{j} + w_1 \hat{k}$

$\bar{q}_1 = \bar{q} + d\bar{q}$

$u_1 \hat{i} + v_1 \hat{j} + w_1 \hat{k} = (u\hat{i} + v\hat{j} + w\hat{k}) + \left(\frac{\partial \bar{q}}{\partial x} dx + \frac{\partial \bar{q}}{\partial y} dy + \frac{\partial \bar{q}}{\partial z} dz \right)$

$$u_1\hat{i}+v_1\hat{j}+w_1\hat{k} = (u\hat{i}+v\hat{j}+w\hat{k}) + \left[\frac{\partial}{\partial x}(u\hat{i}+v\hat{j}+w\hat{k})dx + \frac{\partial}{\partial y}(u\hat{i}+v\hat{j}+w\hat{k})dy + \frac{\partial}{\partial z}(u\hat{i}+v\hat{j}+w\hat{k})dz\right]$$

$$= \hat{i}\left[u+\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz\right] + \hat{j}\left[v+\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz\right] + \hat{k}\left[w+\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz\right] \quad \text{—①}$$

We have to prove that

$$\bar{q}_1 = \underset{\text{translation}}{\bar{q}} + \underbrace{\frac{1}{2}(\bar{\Omega}\times d\bar{r})}_{\text{rotation}} + \underset{\text{rate of deformation}}{\bar{D}}$$

$$\bar{\Omega} = \nabla\times\bar{q} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ u & v & w\end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) - \hat{j}\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right) + \hat{k}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$= \hat{i}\underbrace{\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)}_{A} + \hat{j}\underbrace{\left(-\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)}_{B} + \hat{k}\underbrace{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}_{C}$$

$$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$d\bar{r} = dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}$$

$$\bar{\Omega}\times d\bar{r} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A & B & C\\ dx & dy & dz\end{vmatrix} = \hat{i}(B\,dz-C\,dy) + \hat{j}(C\,dx-A\,dz)+\hat{k}(A\,dy-B\,dx)$$

$$\frac{1}{2}(\bar{\Omega}\times d\bar{r}) = \frac{1}{2}\left\{\hat{i}\left[\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dz-\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dy\right] + \hat{j}\left[\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx-\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)dz\right] + \hat{k}\left[\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)dy-\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dx\right]\right.$$

① can be rearranged in the form,

$$\bar{q}_1 = \hat{i}\Big[u + \frac{\partial u}{\partial x}dx + \cancel{\frac{1}{2}\frac{\partial u}{\partial y}dy} + \cancel{\frac{1}{2}\frac{\partial u}{\partial z}dz} + \Big\{\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$-\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy\Big\} - \cancel{\frac{1}{2}\frac{\partial u}{\partial z}dz} + \frac{1}{2}\frac{\partial w}{\partial x}dz$$

$$+\frac{1}{2}\frac{\partial v}{\partial x}dy - \cancel{\frac{1}{2}\frac{\partial u}{\partial y}dy} + \frac{1}{2}\frac{\partial u}{\partial y}dy + \frac{1}{2}\frac{\partial u}{\partial z}dz\Big]$$

$$+\hat{j}\Big[v + \frac{\partial v}{\partial y}dy + \cancel{\frac{1}{2}\frac{\partial v}{\partial x}dx} + \cancel{\frac{1}{2}\frac{\partial v}{\partial z}dz} + \Big\{\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx$$

$$-\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dz\Big\} - \cancel{\frac{1}{2}\frac{\partial v}{\partial x}dx} + \frac{1}{2}\frac{\partial u}{\partial y}dx + \frac{1}{2}\frac{\partial w}{\partial y}dz$$

$$-\cancel{\frac{1}{2}\frac{\partial v}{\partial z}dz} + \frac{1}{2}\frac{\partial v}{\partial x}dx + \frac{1}{2}\frac{\partial v}{\partial z}dz\Big] +$$

$$\hat{k}\Big[w + \frac{\partial w}{\partial z}dz + \cancel{\frac{1}{2}\frac{\partial w}{\partial x}dx} + \cancel{\frac{1}{2}\frac{\partial w}{\partial y}dy} + \Big\{\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy$$

$$-\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dx\Big\} - \cancel{\frac{1}{2}\frac{\partial w}{\partial y}dy} + \frac{1}{2}\frac{\partial v}{\partial z}dy + \frac{1}{2}\frac{\partial u}{\partial z}dx$$

$$-\cancel{\frac{1}{2}\frac{\partial w}{\partial x}dx} + \frac{1}{2}\frac{\partial w}{\partial x}dx + \frac{1}{2}\frac{\partial w}{\partial y}dy\Big]$$

$$= u\hat{i} + v\hat{j} + w\hat{k} + \frac{1}{2}\Big\{\Big[\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy\Big]\hat{i}$$

$$+\hat{j}\Big[\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx - \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dz\Big]$$

$$+\hat{k}\Big[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy - \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dx\Big]\Big\}$$

$$+\hat{i}\Big[\frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)dz + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy\Big]$$

$$+\hat{j}\Big[\frac{\partial v}{\partial y}dy + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dz\Big]$$

$$+\hat{k}\Big[\frac{\partial w}{\partial z}dz + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dx\Big]$$

i.e) $\bar{q}_1 = \bar{q} + \frac{1}{2}(\bar{\Omega} \times d\bar{r}) + D$ — (2)

(2) represents [the most general form of the movement of fluid element]. The first

term $\bar{q}$ represents translation velocity vector. which indicates linear motion of fluid element without change of shape. Hence the velocity gradient are zero.

ie) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = \frac{\partial u}{\partial y} = \dots = 0$

The second term $\frac{1}{2}(\bar{\Omega} \times d\bar{r})$ represents the rigid rotation of the fluid element.

The term third represents D the rate of deformation (or) rate of strain term.

Problem :-

1. Given velocity field u, v, w (or) given

$u = cx + 2\omega_0 y + u_0$

$v = cy + v_0$

$w = -2cz + w_0$

where c, w_0, u_0, v_0 are constants

Determine the velocity components at the neighbouring point of (x_1, y_1, z_1) and determine the different type of motion involved.

Solution :-

Velocity at the point (x_1, y_1, z_1)

$\bar{q}_1 = \bar{q} + \frac{1}{2}(\bar{\Omega} \times d\bar{r}) + D$ — ①

Translational velocity :-

$\bar{q} = u\hat{i} + v\hat{j} + z\hat{k} = (cx + 2\omega_0 + u_0)\hat{i} + (cy + v_0)\hat{j} + (-2cz + w_0)\hat{k}$ — ②

$\bar{\Omega} = \nabla \times \bar{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$

$= \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(0 - 2\omega_0)$

$= -2\omega_0\hat{k}$

$$\bar{\Omega}\times d\bar{r} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & -2\omega_0\\ dx & dy & dz\end{vmatrix}$$

$$= \hat{i}\,(0+2\omega_0 dy) - \hat{j}\,(2\omega_0 dx) + \hat{k}\,(0)$$

$$= 2\omega_0 \hat{i}\, dy - 2\omega_0 \hat{j}\, dx$$

$$\frac{1}{2}(\bar{\Omega}\times d\bar{r}) = \omega_0 \hat{i}\, dy - \omega_0 \hat{j}\, dx$$

$$D = \hat{i}\left[\frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)dz + \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dy\right]$$

$$+ \hat{j}\left[\frac{\partial v}{\partial y}dy + \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)dz\right]$$

$$+ \hat{k}\left[\frac{\partial w}{\partial z}dz + \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)dx\right]$$

$$D = \hat{i}\left[c\,dx + \frac{1}{2}(0+0).dz + \frac{1}{2}(2\omega_0)dy\right]$$

$$+ \hat{j}\left[c\,dy + \frac{1}{2}(2\omega_0)\,dx + \frac{1}{2}(0+0)\,dz\right]$$

$$+ \hat{k}\left[-2c\,dz + \frac{1}{2}(0+0)\,dy + \frac{1}{2}(0+0)\,dx\right]$$

$$= \hat{i}\left[c\,dx + \omega_0 dy\right] + \hat{j}\left[c\,dy + \omega_0 dx\right]$$

$$+ \hat{k}\left[-2c\,dz\right]$$

$$\bar{q}_1 = \bar{q} + \frac{1}{2}(\bar{\Omega}\times d\bar{r}) + D$$

$$= (cx + 2\omega_0 y + u_0)\,\hat{i} + (cy+v_0)\,\hat{j} + (-2cz + w_0)\,\hat{k}$$

$$+ \frac{1}{2}\left[2\omega_0 \hat{i}\,dy - 2\omega_0 \hat{j}\,dx\right] + \hat{i}\left[c\,dx + \omega_0 dy\right]$$

$$+ \hat{j}\left[c\,dy + \omega_0\,dx\right] - 2\hat{k}\,c\,dz$$

$$= \left[cx + 2\omega_0 y + u_0 + \omega_0 dy + c\,dx + \omega_0 dy\right]\hat{i}$$

$$\left[cy + v_0 - \cancel{\omega_0 dx} + c\,dy + \cancel{\omega_0 dx}\right]\hat{j} + \left[-2cz + w_0 - 2c\,dz\right]\hat{k}$$

$$= (cx + 2\omega_0 y + 2\omega_0 dy + c\,dx + u_0)\,\hat{i} + \left[cy + v_0 + c\,dy\right]\hat{j}$$

$$+ (w_0 - 2cz - 2c\,dz)\,\hat{k}$$

03-12-19 (Tue)

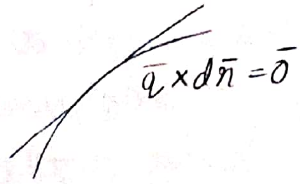
Stream lines, path lines and streak lines

2M(S)

2M(S) + 2M(S)

Stream lines :-

If for a fixed instant of time, a space curve is drawn so that it is tangent everywhere to a velocity vector then this curve is called a stream line.



By definition of stream line $\bar{q}$ is parallel to $d\bar{r}$. ie) $\bar{q} \times d\bar{r} = \bar{0}$

$$\bar{q} \times d\bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ dx & dy & dz \end{vmatrix}$$

$$= \hat{i}(vdz - wdy) + \hat{j}(wdx - udz) + \hat{k}(udy - vdx)$$

$$= \bar{0}$$

$\Rightarrow vdz = wdy \qquad wdx - udz = 0 \qquad udy = vdx$

$\frac{dz}{w} = \frac{dy}{v} \qquad \frac{dx}{u} = \frac{dz}{w} \qquad \frac{dy}{v} = \frac{dx}{u}$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{---} (1)$$

This is known as equation of stream line.

Example :-

1. 5M(S) + 5M(S) + 5M(S) The velocity vector $\bar{q}$ is given by $\bar{q} = \hat{i}x - \hat{j}y$. Determine the equation of stream line

Solution :-

$\bar{q} = u\hat{i} + v\hat{j}$

comparing with $\bar{q} = \hat{i}x - \hat{j}y$

$u = x \quad v = -y$

The Equating of stream line $\frac{dx}{u} = \frac{dy}{v}$

$\frac{dx}{x} = \frac{dy}{-y}$

$\ln(x) = -\ln(y) + \log c$

$\ln(x) + \ln(y) = \ln(c)$

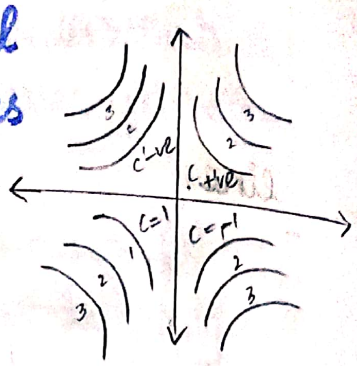
$\ln(xy) = \ln(c)$

Taking exponential on both sides

$$xy = c$$

where c is an arbitrary constant.

It represents the graphical representation of stream lines for a steady 2D flow at 90° corner



rectangular hyperbola

The velocity vector $\bar{q}$ is everywhere tangent to the stream line $xy = c$.

2M(S) + 2M(S)

Path lines :-

A path line is the trajectory of fluid particle of fixed identity which is determined from

$$\frac{dx}{dt} = u(x, y, z, t)$$

$$\frac{dy}{dt} = v(x, y, z, t)$$

$$\frac{dz}{dt} = w(x, y, z, t)$$

with ~~The~~ initial conditions $x(t_0) = x_0$ $y(t_0) = y_0$ $z(t_0) = z_0$ ~~are determined from these equations.~~

For the steady flow the velocity vector $\bar{q}$ is independent of time

Then the path lines are identical to that of stream line

$$\frac{dx}{u} = dt$$

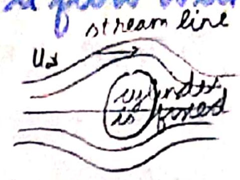
$$\frac{dy}{v} = dt$$

$$\frac{dz}{w} = dt$$

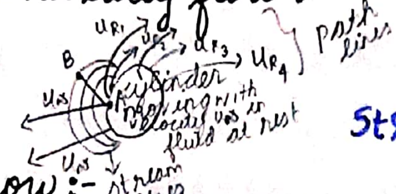
$$\Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

[Case i) Steady flow:-

Example :- Cylinder at rest in a flow with velocity u_∞]



Case ii) Unsteady flow :-



Stream line unsteady flow path lines

Steady flow :-

A solid circular cylinder is placed in a fixed position in a moving fluid. It is an example for steady state flow pattern.

[Unsteady flow :-

The unsteady motion created by a cylinder moving with a uniform velocity u_∞ in a fluid at rest]

Let us consider a particle A on the surface of the cylinder which is initially $t = t_0$ located at x_0. The cylinder is assumed to be moving with a uniform velocity u_∞ in the x-direction. Initially the particle is on stream line 1 and it will move along the stream line 1 with resultant velocity u_∞ until it reaches the point B after the time Δt has elapsed. So during this time the cylinder together with the stream line pattern has also travelled a distance $u_\infty \Delta t$.

Similarly the stream lines 1, 2, 3 all are separated by an amount $u_\infty \Delta t$ in the direction of motion. If the limit Δt to infinitesimal quantity the path of the particle will be the path line.

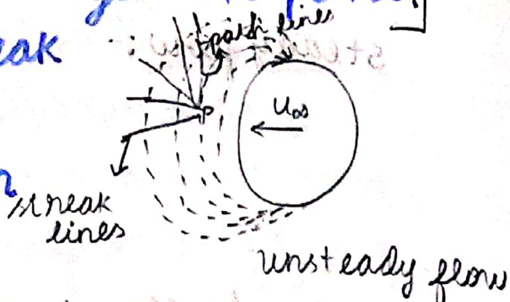
04-12-19 (Wed) 2M(S) + 2M(S)

Streak lines :-

A streak line is the locus of points at a given time t_1 that connects the temporary location of all the particles that have passed through a fixed point in the flow field]

[For example, if dye is injected into a liquid at a fixed point in the flow field then at a later time the dye will indicate the end points of the path lines of particles which have passed through the injection point.]

The equation of streak lines at time t can be derived by the Lagrangian method.



If a fluid particle (x_0, y_0, z_0) passes a fixed point (x_1, y_1, z_1) in the course of time, then the Lagrangian method gives

$$\left.\begin{aligned} f_1(x_0, y_0, z_0, t) &= x_1 \\ f_2(x_0, y_0, z_0, t) &= y_1 \\ f_3(x_0, y_0, z_0, t) &= z_1 \end{aligned}\right\} \text{---} (1)$$

Solving for x_0, y_0, z_0

$$\left.\begin{aligned} [x_0 &= g_1(x_1, y_1, z_1, t) \\ y_0 &= g_2(x_1, y_1, z_1, t) \\ z_0 &= g_3(x_1, y_1, z_1, t)] \end{aligned}\right\} \text{---} (2)$$

If the positions (x, y, z) of the particles which have passed the fixed point (x_1, y_1, z_1) can be determined then a streak line can be drawn through these points.

[The equation of streak lines at time t is given by

$$\left.\begin{aligned} x &= F_1(x_0, y_0, z_0, t) \\ y &= F_2(x_0, y_0, z_0, t) \\ z &= F_3(x_0, y_0, z_0, t) \end{aligned}\right\} \text{---} (3)$$

Using (2) in (3)

$$\left.\begin{aligned} x &= F_1(g_1, g_2, g_3, t) \\ y &= F_2(g_1, g_2, g_3, t) \\ z &= F_3(g_1, g_2, g_3, t) \end{aligned}\right\} \text{---} (4)$$

(4) is called the equations of streak lines.]

Example

1. Let $u = \frac{x}{t}$ $v = y$ $w = 0$ Equation of streak line given the velocity u, v, w

Solution :-

$$\frac{dx}{dt} = \frac{x}{t} \quad \frac{dy}{dt} = y \quad \frac{dz}{dt} = 0$$

$$\frac{dx}{x} = \frac{dt}{t} \quad \frac{dy}{y} = dt \quad dz = 0$$

Integrating

$$\ln x = \ln t + \ln c_1 \quad \ln y = t + c_2 \quad z = c_3$$

$$\frac{x}{t} = c_1 \qquad y = e^{t+c_2}$$

$$\left.\begin{aligned} x &= c_1 t \\ y &= e^{t+c_2} \\ z &= c_3 \end{aligned}\right\} \text{—} (1)$$

Using the initial condition we have to find c_1, c_2, c_3 ie) when $t = t_0$, $x = x_0$, $y = y_0$, $z = z_0$

$$x_0 = c_1 t_0 \qquad y_0 = e^{t_0} . e^{c_2} \qquad z_0 = c_3$$

$$\Rightarrow c_1 = \frac{x_0}{t_0} \qquad \frac{y_0}{e^{t_0}} = e^{c_2}$$

Substituting c_1, c_2, c_3 in (1)

$$x = \left(\frac{x_0}{t_0}\right) t \qquad y = e^{t} \frac{y_0}{e^{t_0}} \qquad z = z_0$$

$$x = x_0 \left(\frac{t}{t_0}\right) \qquad y = y_0 e^{t-t_0} \qquad z = z_0 \text{ — } (2)$$

put $x = x_1$ $y = y_1$ $z = z_1$

The particle at (x_1, y_1) at all times can be obtained from the inverse relation of (2)

$$x_0 = x_1\left(\frac{t_0}{t}\right) \qquad y_0 = \frac{y_1}{e^{t-t_0}} \qquad z_0 = z_1$$

Replacing $x_0 = x_1$ $y = y_1$ $z = z_1$ $t_0 = t_0$ $t = s$

$$x_0 = x_1\left(\frac{t_0}{s}\right) \qquad y_0 = \frac{y_1}{e^{s-t_0}} \qquad z_0 = z_1$$

$$y_0 = y_1 e^{-s+t_0} \qquad \text{— } (3)$$

$t_0 \le s \le t$

where s lies between t_0 and t

Substituting ③ in ② the co-ordinates of streak lines are

$$x = \left[x_1\left(\frac{t_0}{s}\right)\right]\left(\frac{t}{t_0}\right) \qquad y = y_1(e^{-s+t_0})(e^{t-t_0}) \qquad z = z_1$$

$$y = y_1 e^{t-s}$$

$$x = x_1\left(\frac{t}{s}\right)$$

These are the co-ordinates of streak lines

The material derivatives and acceleration

d/dt $\quad d^3/dt^3$

The velocity of particles at any point in the space can be written in Eulerian form

$$\bar{q} = \bar{f}(\bar{r}, t) \quad \text{—} ①$$

The velocity in the x direction at time t can be written in the form

$$u = f_1(x, y, z, t) \quad \text{—} ②$$

At an infinitesimal time interval Δt later this particle will move to the position $(x+\Delta x, y+\Delta y, z+\Delta z)$. The increments $\Delta x, \Delta y, \Delta z$ are

$$\left.\begin{aligned} \Delta x &= u\Delta t \\ \Delta y &= v\Delta t \\ \Delta z &= w\Delta t \end{aligned}\right\}$$

(Rate of change of displacement is velocity) $u = \frac{\Delta x}{\Delta t}$ $\quad v = \frac{\Delta y}{\Delta t}$ $\quad w = \frac{\Delta z}{\Delta t}$

Hence

$$u + \Delta u = f_1(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t)$$
$$= f_1(x+u\Delta t, y+v\Delta t, z+w\Delta t, t+\Delta t)$$

We know that

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \dots$$

$$u + \Delta u = f_1(x, y, z, t) + \left(u\frac{\partial f_1}{\partial x}\Delta t + v\Delta t\frac{\partial f_1}{\partial y} + w\Delta t\frac{\partial f_1}{\partial z} + \Delta t\frac{\partial f_1}{\partial t}\right) + O(\Delta t)$$

$$\cancel{f_1(x, y, z, t)} + \Delta u = \cancel{f_1(x, y, z, t)} + \left(u\frac{\partial f_1}{\partial x} + v\frac{\partial f_1}{\partial y} + w\frac{\partial f_1}{\partial z} + \frac{\partial f_1}{\partial t}\right)\Delta t + O(t)$$

$$\frac{\Delta u}{\Delta t} = \frac{\partial f_1}{\partial t} + u\frac{\partial f_1}{\partial x} + v\frac{\partial f_1}{\partial y} + w\frac{\partial f_1}{\partial z} + O(\Delta t)$$

Taking the limit $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t} = \frac{Du}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u$$

[$\Rightarrow \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

is known as material derivative with respect to time

The first term $\frac{\partial}{\partial t}$ is called the local derivative

and the last three terms $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ is called convective derivative.]

[The symbol $\frac{d}{dt}$ is also used for material derivative.

$$\frac{D}{Dt} = \frac{d}{dt}]$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\bar{q}\cdot\nabla)$$

$$\because \bar{q}\cdot\nabla = (u\hat{i} + v\hat{j} + w\hat{k})\cdot(\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z)$$

$$= u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z$$

The acceleration of a fluid particle of fixed identity can be expressed by the material derivative of the velocity vector $\bar{q}$.

$$\bar{a} = \frac{d\bar{q}}{dt} = (\partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z)\,\bar{q}$$

$$a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = (\partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z)(u\hat{i} + v\hat{j} + w\hat{k})$$

Equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we obtain

$$\left.\begin{aligned} a_x &= (\partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z)\,u \\ a_y &= (\partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z)\,v \\ a_z &= (\partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z)\,w \end{aligned}\right\}$$

These are the acceleration components in cartesian co-ordinates.]

The cylindrical co-ordinates (r, θ, z) with velocity components (v_r, v_θ, v_z). The components of acceleration are

$$a_r = \frac{dv_r}{dt} - \frac{v_\theta^2}{r}$$

$$a_\theta = \frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r}$$

$$a_z = \frac{dv_z}{dz}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + v_r \partial/\partial r + \frac{v_\theta}{r} \partial/\partial\theta + v_z \partial/\partial z$

In spherical co-ordinates (r, θ, ϕ) with velocity components (v_r, v_θ, v_ϕ). The components of acceleration are

$$a_r = \frac{dv_r}{dt} - \left[\frac{v_\theta^2 + v_\phi^2}{r}\right]$$

$$a_\theta = \frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}$$

$$a_\phi = \frac{dv_\phi}{dt} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot\theta}{r}$$

where $\partial/\partial t + u_r \partial/\partial r + \frac{v_\theta}{r} \partial/\partial\theta + \frac{v_\phi}{r\sin\theta} \partial/\partial\phi = \frac{d}{dt}$

Example :-

1. Given the velocity field $\bar{q} = \hat{i}(Ax^2y) + \hat{j}(By^2zt) + \hat{k}(Czt^2)$. Determine the acceleration of a fluid particle of fixed identity.

5M(S) + 10M(S)

Solution :-

$$u\hat{i} + v\hat{j} + w\hat{k} = \hat{i}(Ax^2y) + \hat{j}(By^2zt) + \hat{k}(Czt^2)$$

$$\left.\begin{array}{llllll} u = Ax^2y & & V = By^2zt & & W = Czt^2 & \\ \frac{\partial u}{\partial t} = 0 & \frac{\partial u}{\partial x} = 2Axy & \frac{\partial V}{\partial t} = By^2z & \frac{\partial V}{\partial x} = 0 & \frac{\partial W}{\partial t} = 2Czt & \frac{\partial W}{\partial x} = 0 \\ \frac{\partial u}{\partial y} = Ax^2 & \frac{\partial u}{\partial z} = 0 & \frac{\partial V}{\partial y} = 2Byzt & \frac{\partial V}{\partial z} = By^2t & \frac{\partial W}{\partial y} = 0 & \frac{\partial W}{\partial z} = Ct^2 \end{array}\right\} \quad (1)$$

Acceleration components are

$$\left.\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \text{— (2)}$$

Substituting (1) in (2)

$$a_x = (Ax^2y)(2Axy) + (By^2zt)(Ax^2)$$

$$a_y = By^2z + (Ax^2y)(0) + (By^2zt)(2Byzt) + (czt^2)(By^2t)$$

$$a_z = 2czt + (czt^2)(ct^2)$$

$$\therefore \quad a_x = 2A^2x^3y^2 + ABx^2y^2zt$$

$$a_y = By^2z + 2B^2y^3z^2t^2 + BCy^2zt^3$$

$$a_z = 2czt + c^2zt^4$$

2. Determine the acceleration of a fluid particle from flow field $\bar{q} = \hat{i}(Axy^2t) + \hat{j}(Bx^2yt) + \hat{k}(cxyz)$

5M(S)

Solution :-

$$u\hat{i} + v\hat{j} + w\hat{k} = \hat{i}(Axy^2t) + \hat{j}(Bx^2yt) + \hat{k}(cxyz)$$

$$\text{(3)} — \left\{\begin{array}{llllll} u = Axy^2t & & v = Bx^2yt & & w = cxyz & \\ \frac{\partial u}{\partial t} = Axy^2 & \frac{\partial u}{\partial x} = Ay^2t & \frac{\partial v}{\partial t} = Bx^2y & \frac{\partial v}{\partial x} = 2Bxyt & \frac{\partial w}{\partial t} = 0 & \frac{\partial w}{\partial x} = cyz \\ \frac{\partial u}{\partial y} = 2Axyt & \frac{\partial u}{\partial z} = 0 & \frac{\partial v}{\partial y} = Bx^2t & \frac{\partial v}{\partial z} = 0 & \frac{\partial w}{\partial y} = cxz & \frac{\partial w}{\partial z} = cxy \end{array}\right.$$

Substituting (3) in (2),

$$a_x = Axy^2 + (Axy^2t)(Ay^2t) + (Bx^2yt)(2Axyt)$$

$$= Axy^2 + A^2xy^4t^2 + 2ABx^3y^2t^2$$

$$a_y = Bx^2y + (Axy^2t)(2Bxyt) + (Bx^2yt)(Bx^2t)$$

$$= Bx^2y + 2ABx^2y^3t^2 + B^2x^4yt^2$$

$$a_z = (Axy^2t)(cxy) + (Bx^2yt)(cxz) + (cxyz)(cxy)$$

$$= ACxy^3t + BCx^3yzt + c^2x^2y^2z$$

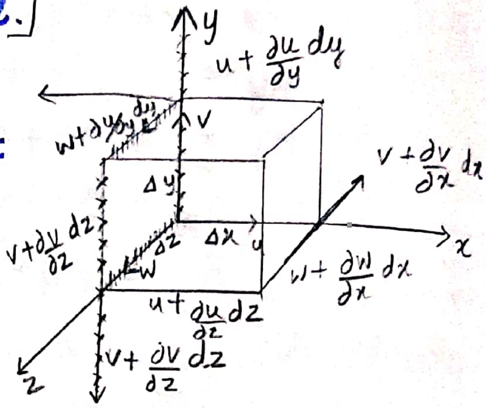
9-12-19
(Mon)

Vorticity:-

Vorticity is a antisymmetric tensor and its three distinct elements transform the components of a vector in cartesian co-ordinates.

[Definition

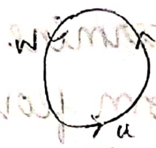
Vorticity is defined as the curl of velocity vector $\bar{q}$ ($\nabla \times \bar{q}$) and hence it is the measure of the local rotation of the fluid.]



The angular velocities of the segment Δy and Δz about the x axis as follows:

Angular velocity of the segment Δy

$$\Delta y = \frac{\cancel{w} + \frac{\partial w}{\partial y} \cancel{\Delta y} - \cancel{w}}{\cancel{\Delta y}} = \frac{\partial w}{\partial y}$$



Similarly the angular velocity of the segment Δz

$$\Delta z = \frac{-\left(v + \frac{\partial v}{\partial z}\Delta z\right) + v}{\Delta z} = \frac{-\partial v}{\partial z}$$

$\therefore$ The average angular velocity component in the x-direction is

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad \text{— (1)}$$

Angular velocity of the segment Δz and Δx about y axis as follows:

Angular velocity of the segment of the segment Δz

$$\Delta z = \frac{u + \frac{\partial u}{\partial z}\Delta z - u}{\Delta z} = \frac{\partial u}{\partial z}$$

Angular velocity of the segment Δx

$$\Delta x = \frac{-\left(w + \frac{\partial w}{\partial x}\Delta x\right) + w}{\Delta x} = -\frac{\partial w}{\partial x}$$

$\therefore$ The average angular velocity component in the y-direction is

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Similarly the average angular velosity component in the z-direction is

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Hence the angular velocity $\bar{\omega}$

$$\bar{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\bar{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right] \quad \text{—①}$$

$\therefore$ The angular velocity $\bar{\omega}$ of a fluid element in terms of velocity field

We know that $\bar{\Omega} = \nabla \times \bar{q}$

$$\bar{\Omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\bar{\omega} = \frac{1}{2}\bar{\Omega}$$

ie) $\bar{\Omega} = 2\bar{\omega}$

This is the relation between vorticity vector and angular velocity vector.

Vortex line :-

If a line is drawn in the fluid so that the tangent to it at each point is in the direction of vorticity vector $\bar{\Omega}$ at that point, the line is called the vortex line]. If dI is in the direction of the vorticity vector then

$\bar{\Omega} \times d\bar{I} = \bar{0}$

where $\bar{\Omega} = \Omega_x\hat{i} + \Omega_y\hat{j} + \Omega_z\hat{k}$

$dI = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\bar{\Omega} \times d\bar{I} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ dx & dy & dz \end{vmatrix}$$

$$= \hat{i}(\Omega_y dz - \Omega_z dy) - \hat{j}(\Omega_x dz - \Omega_z dx) + \hat{k}(\Omega_x dy - \Omega_y dx)$$

$$= \bar{0}$$

$\Omega_y dz = \Omega_z dy \qquad \Omega_x dz = \Omega_z dx \qquad \Omega_x dy = \Omega_y dx$

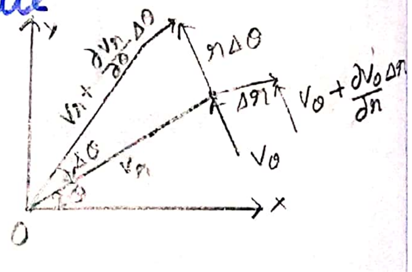
$$\frac{dz}{\Omega_z} = \frac{dy}{\Omega_y} \qquad \frac{dx}{\Omega_x} = \frac{dz}{\Omega_z} \qquad \frac{dx}{\Omega_x} = \frac{dy}{\Omega_y}$$

[$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_y}$$

This equation is called the equation of vortex line.]

Vorticity in polar co-ordinates :-

In the 2D polar co-ordinate system the angular velocity of the segment Δr



$$\Delta r = \frac{\cancel{V_\theta} + \frac{\partial V_\theta}{\partial r}\cancel{\Delta r} - \cancel{V_\theta}}{\cancel{\Delta r}}$$

$$\Delta r = \frac{\partial V_\theta}{\partial r}$$

The angular velocity of the segment $r\Delta\theta$

$$= \frac{-\left(V_r + \frac{\partial V_r}{\partial \theta}\Delta\theta\right) + V_r}{r\Delta\theta} = \frac{-1}{r}\frac{\partial V_r}{\partial \theta}$$

The additional angular velocity about the origin is $\frac{V_\theta}{r}$

[$\because$ Vorticity in polar co-ordinates is

$$\Omega_z = \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}]$$ 2M (S)

10M (S) [Vorticity in orthogonal curvilinear co-ordinate:

The general form of vorticity components can be written in terms of the orthogonal curvilinear co-ordinates are,

$$\left.\begin{aligned} \hat{i}_x \Omega_x &= \frac{\hat{i}_x h_1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 q_3) - \frac{\partial}{\partial x_3}(h_2 q_2)\right] \\ \hat{i}_y \Omega_y &= \frac{\hat{i}_y h_2}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_3}(h_1 q_1) - \frac{\partial}{\partial x_1}(h_3 q_3)\right] \\ \hat{i}_z \Omega_z &= \frac{\hat{i}_z h_3}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 q_2) - \frac{\partial}{\partial x_2}(h_1 q_1)\right] \end{aligned}\right\} \text{— (1)}$$

Case i) :- Cylindrical co-ordinate

Let $(x_1, x_2, x_3) = (r, \theta, z)$ and $h_1 = 1$ $h_2 = r$, $h_3 = 1$ $(q_1, q_2, q_3) = (v_r, v_\theta, v_z)$ — (2)

Substituting (2) in (1) we get

$$\Omega_r = \frac{1}{r}\left[\frac{\partial}{\partial \theta}(1 \cdot v_z) - \frac{\partial}{\partial z}(r v_\theta)\right]$$

$$\Omega_\theta = \frac{r}{r}\left[\frac{\partial}{\partial z}(v_r) - \frac{\partial}{\partial r}(v_z)\right]$$

$$\Omega_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial}{\partial \theta}(v_r)\right] = \frac{1}{r}\left[r\frac{\partial v_\theta}{\partial r} + v_\theta(1) - \frac{\partial v_r}{\partial \theta}\right]$$

which is

$$\Omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}$$

$$\Omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$

$$\Omega_z = \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}$$

Case ii) :- Spherical co-ordinates :-

$$\left.\begin{aligned} &\text{Let } (x_1, x_2, x_3) = (r, \theta, \phi) \quad h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta \\ &(q_1, q_2, q_3) = (v_r, v_\theta, v_\phi) \end{aligned}\right\} \text{(3)}$$

Substituting ③ in ①

$$\Omega_r = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta\, v_\phi) - \frac{\partial}{\partial\phi}(r v_\theta)\right]$$

$$= \frac{r}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{v_\phi}\right]$$

$$= \frac{1}{r\sin\theta}\left[\cos\theta\, v_\phi + \sin\theta\frac{\partial v_\phi}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right]$$

$$\Omega_r = \frac{v_\phi}{r}\cot\theta + \frac{1}{r}\frac{\partial v_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}$$

$$\Omega_\theta = \frac{r}{r^2\sin\theta}\left[\frac{\partial}{\partial\phi}(v_r) - \frac{\partial}{\partial r}(r\sin\theta\cdot v_\phi)\right]$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial v_r}{\partial\phi} - (1)\sin\theta\, v_\phi - r\sin\theta\frac{\partial v_\phi}{\partial r}\right]$$

$$\Omega_\theta = \frac{1}{r\sin^2\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\phi}{r} - \frac{\partial v_\phi}{\partial r}$$

$$\Omega_\phi = \frac{\cancel{r\sin\theta}}{r^{\cancel{2}}\cancel{\sin\theta}}\left[\frac{\partial}{\partial r} r v_\theta - \frac{\partial}{\partial\theta}(v_r)\right]$$

$$= \frac{1}{r}\left[v_\theta + r\frac{\partial v_\theta}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]$$

$$\Omega_\phi = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}$$ 10 M (S)

Example :-

1. 5M(S) The velocity vector in the flow field

$\bar{q} = \hat{i}(Az - By) + \hat{j}(Bx - Cz) + \hat{k}(Cy - Ax)$

where A, B, C are non-zero constant. Determine the equation of vortex line.

Solution :-

The equation of vortex lines are

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \quad \text{where } \bar{\Omega} = \nabla\times\bar{q}$$

$$\Omega_x\hat{i} + \Omega_y\hat{j} + \Omega_z\hat{k} = \nabla\times\bar{q}$$

$$\nabla \times \bar{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

$$\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

given,

$$u\hat{i} + v\hat{j} + w\hat{k} = \hat{i}(Az - By) + \hat{j}(Bx - Cz) + \hat{k}(Cy - Ax)$$

$u = Az - By$ $\quad$ $v = Bx - Cz$ $\quad$ $w = Cy - Ax$

$\frac{\partial u}{\partial y} = -B$ $\quad$ $\frac{\partial v}{\partial x} = B$ $\quad$ $\frac{\partial w}{\partial x} = -A$

$\frac{\partial u}{\partial z} = A$ $\quad$ $\frac{\partial v}{\partial z} = -C$ $\quad$ $\frac{\partial w}{\partial y} = C$

$$\Omega_x = C - (-C) = 2C$$

$$\Omega_y = A - (-A) = 2A$$

$$\Omega_z = B - (-B) = 2B$$

$2B\,dx = 2C\,dz$
$Bx - Cz = C$
$x - \frac{C}{B}z = C_2$
$\llcorner$ 2

$$\frac{dx}{2C} = \frac{dy}{2A} = \frac{dz}{2B}$$

$2A\,dz - 2B\,dy = 0$
$A\,dz = B\,dy$
$Az - By = C$
$z - \frac{B}{A}y = C_1$
$\llcorner$ 1

$$\frac{dx}{2C} = \frac{dy}{2A} \Rightarrow A\,dx = C\,dy$$

$2C\,dy - 2A\,dx = 0$
$Cy - Ax = C$
$y - \frac{A}{C}x = C_3$
$\llcorner$ 3

$$Ax - Cy = C_1$$

$$x - \left(\frac{C}{A}\right)y = C_1$$

1, 2, 3 $\rightarrow$ equation of vortex line

$$\frac{dy}{2A} = \frac{dz}{2B} \Rightarrow B\,dy = A\,dz$$

$$By - Az = C_2$$

$$y - \left(\frac{A}{B}\right)z = C_2$$

$$\frac{dx}{2C} = \frac{dz}{2B} \Rightarrow B\,dx = C\,dz$$

$$Bx - Cz = C_3$$

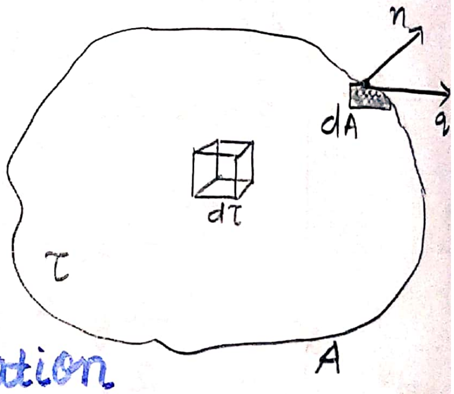
$$\left(\frac{B}{C}\right)x - z = C_3$$

$$z - \left(\frac{B}{C}\right)x = C_3$$

Fundamental equation of the flow of viscous compressible fluid :-

5M(S) + 5M(S) [Equation of continuity :- Conservation of mass

The law of conservation of mass states that mass can neither be created nor destroyed. Consider a moving surface A enclosing a mass of fluid within volume τ lying entirely in fluid.



If $\rho\Delta\tau$ is an element of mass, then by the law of conservation of mass

$$\frac{D}{Dt}(\rho\Delta\tau) = 0 \quad \text{--- (1)}$$

This is one form of equation of continuity

(1) can be written as

$$\frac{D\rho}{Dt}(\Delta\tau) + \rho\frac{D}{Dt}(\Delta\tau) = 0$$

$\div \rho$

$$\frac{1}{\rho}\frac{D\rho}{Dt}(\Delta\tau) + \frac{D}{Dt}(\Delta\tau) = 0 \quad \text{--- (2)}$$

If we consider the total mass of the fluid is enclosed in the ~~on~~ surface A then (1) becomes,

$$\lim_{\Delta\tau \to 0} \Sigma \left[\frac{1}{\rho}\frac{D\rho}{Dt}(\Delta\tau) + \frac{D}{Dt}(\Delta\tau)\right] = 0 \quad \text{--- (3)}$$

$$\lim_{\Delta\tau \to 0} \Sigma \frac{1}{\rho}\frac{D\rho}{Dt}\Delta\tau = \iiint \frac{1}{\rho}\frac{D\rho}{Dt}\,d\tau \quad \text{--- (4)}$$

$$\lim_{\Delta \tau \to 0} \sum \frac{D}{Dt}(\Delta \tau) = \iiint \bar{\nabla} \cdot \bar{q}\, d\tau \quad \text{— (5)}$$

combining (4) and (5)

$$\iiint \frac{1}{\rho}\frac{D\rho}{Dt}\, d\tau + \iiint \bar{\nabla}\cdot\bar{q}\, d\tau = 0$$

$$\Rightarrow \iiint \frac{1}{\rho}\left[\frac{D\rho}{Dt} + \rho(\bar{\nabla}\cdot\bar{q})\right] d\tau = 0 \quad \text{— } (*)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\bar{q}\cdot\bar{\nabla})$

$$\iiint \frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + (\bar{q}\cdot\bar{\nabla}\rho) + \rho(\bar{\nabla}\cdot\bar{q})\right] d\tau = 0$$

$$\iiint \frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + \bar{\nabla}(\rho\cdot\bar{q})\right] d\tau = 0 \quad \text{— (6)}$$

from which the equation of continuity is obtained

$$\Rightarrow \frac{\partial\rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{q}) = 0 \quad \text{— (7)}$$

(7) is called equation of continuity for compressible fluid.

Case i) :-

For incompressible fluid $\frac{\partial\rho}{\partial t} = 0$

then $(*)$ becomes

$$\iiint \frac{1}{\cancel{\rho}}\cancel{\rho}(\bar{\nabla}\cdot\bar{q})\, d\tau = 0$$

$$\bar{\nabla}\cdot\bar{q} = 0$$

ie) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ — (8) is known as equation of continuity. 5M(S)

Example :-

1. 5M(S) + 5M(S) + 5M(S) Consider a 2D incompressible steady flow field with velocity components in rectangle co-ordinate is given by $u(x,y) = \frac{k(x^2-y^2)}{(x^2+y^2)^2}$ $v(x,y) = \frac{2kxy}{(x^2+y^2)^2}$ where k is an

arbitrary non-zero constant. Is the equation of continuity satisfied?

Solution :-

Equation of continuity for an incompressible 2D flow field is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Differentiate u and v with respect to x and y

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)^2(2xk) - k(x^2-y^2)2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{2xk[(x^2+y^2)^2 - (x^2-y^2)2(x^2+y^2)]}{(x^2+y^2)^4}$$

$$= \frac{2xk.(x^2+y^2)}{(x^2+y^2)^4}[x^2+y^2-2(x^2-y^2)]$$

$$= \frac{2xk}{(x^2+y^2)^3}(-x^2+3y^2) \quad \text{—} ①$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)^2(2kx) - (2kxy)2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{2kx(x^2+y^2)}{(x^2+y^2)^4}[(x^2+y^2)-4y^2]$$

$$= \frac{2kx}{(x^2+y^2)^3}(x^2-3y^2) \quad \text{—} ②$$

① + ② $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{2xk}{(x^2+y^2)^3}(-\cancel{x^2}+\cancel{3y^2}+\cancel{x^2}-\cancel{3y^2})$

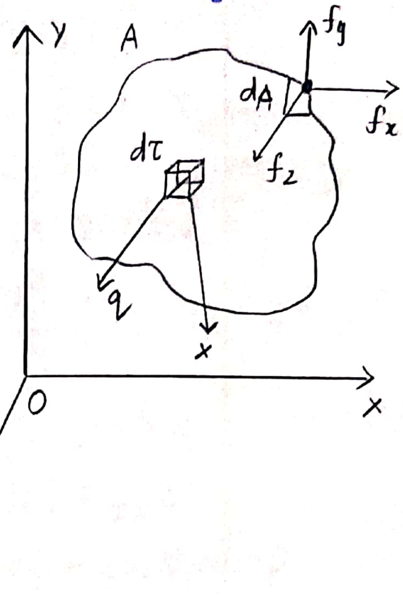
$= 0$

* (10M) (5) + 10M(5)

Equation of motion - Conservation of Momentum [Navier-Stokes Equations]

By Newton second law, "the total force acting on a fluid mass enclosed in an arbitrary volume fixed in space is equal to the time rate of change of linear momentum".

Consider a fluid mass in motion, which at time t occupies a volume τ bounded by a surface A



Let $d\tau$ be an element of volume and $\rho d\tau$ the elementary mass moving with velocity $\bar{q}$

$F = ma$

Then $\underbrace{-\rho d\tau}_{m}\underbrace{\left(\frac{D\bar{q}}{Dt}\right)}_{a}$ is the force of of inertia ($\because$ from D'Alembert's principle) acting upon the elementary mass. The resultant of inertial forces or the time rate of change of linear momentum is

$$\bar{F}_I = -\iiint \rho\left(\frac{D\bar{q}}{Dt}\right) d\tau \quad \text{—— (1)}$$

If the body force ~~for~~ per unit mass acting upon the elementary volume is denoted by $\bar{X}$ then $\bar{X}\, d\tau\rho$ will be the elementary body force.

The resultant of the body force is

$$\bar{F}_B = \iiint \bar{X} \rho\, d\tau \quad \text{—— (2)}$$

The elementary surface force acting on an element of surface, dA is expressed by the ~~force~~ vector

$$\bar{F} = \hat{i}_x f_x + \hat{i}_y f_y + \hat{i}_z f_z$$
$$= \hat{i}_x(\bar{P}_x \cdot d\bar{A}) + \hat{i}_y(\bar{P}_y \cdot d\bar{A}) + \hat{i}_z(\bar{P}_z \cdot d\bar{A})$$

where $\bar{P}_x$, $\bar{P}_y$, $\bar{P}_z$ are the pressure acting in the direction of x, y and z respectively. Then the resultant of the surface force is

$$\bar{F}_S = \iint_S \hat{i}_x(\bar{P}_x \cdot d\bar{A}) + \hat{i}_y \iint_S(\bar{P}_y \cdot d\bar{A}) + \hat{i}_z \iint_S \bar{P}_z \cdot d\bar{A}$$

By Gauss divergence theorem

$$\iint \bar{F} \cdot d\bar{S} = \iiint \operatorname{div} F\, dv = \iiint \nabla \cdot \bar{F}\, dv$$

Using Gauss divergence theorem

$$\bar{F}_s = \hat{i}_x \iiint \nabla\cdot\bar{P}_x\, d\tau + \hat{i}_y \iiint \nabla\cdot\bar{P}_y\, d\tau + \hat{i}_z \iiint_\tau \nabla\cdot\bar{F}_z\, d\tau \quad \text{③}$$

using conservation of mass,

① + ② + ③ = 0

$$\iiint\left[-\rho\frac{D\bar{q}}{Dt} + \rho\bar{X} + \hat{i}_x(\nabla\cdot\bar{P}_x) + \hat{i}_y(\nabla\cdot\bar{P}_y) + \hat{i}_z(\nabla\cdot\bar{P}_z)\right]d\tau = 0$$

$$\Rightarrow \rho\frac{D\bar{q}}{Dt} = \rho\bar{X} + \hat{i}_x(\nabla\cdot\bar{P}_x) + \hat{i}_y(\nabla\cdot\bar{P}_y) + \hat{i}_z(\nabla\cdot\bar{P}_z) \quad \text{④}$$

where $\bar{P}_x = \hat{i}_x\sigma_{xx} + \hat{i}_y\sigma_{xy} + \hat{i}_z\sigma_{xz}$

$\bar{P}_y = \hat{i}_x\sigma_{yx} + \hat{i}_y\sigma_{yy} + \hat{i}_z\sigma_{yz}$

$\bar{P}_z = \hat{i}_x\sigma_{zx} + \hat{i}_y\sigma_{zy} + \hat{i}_z\sigma_{zz}$

where

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p$$

where $\bar{q} = (u, v, w)$

$\bar{X} = (X_x, X_y, X_z)$

$$\sigma_{xy} = \mu\gamma_{\underset{uv}{xy}} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{xz} = \mu\gamma_{\underset{uw}{xz}} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{yx} = \mu\gamma_{\underset{vu}{yx}} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p$$

$$\sigma_{yz} = \mu\gamma_{\underset{vw}{yz}} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\sigma_{zx} = \mu\gamma_{\underset{wu}{zx}} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\sigma_{zy} = \mu\gamma_{\underset{wv}{zy}} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\sigma_{zz} = 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p$$

$$\nabla\bar{P}_x = \left(\hat{i}_x\,\partial/\partial x + \hat{i}_y\,\partial/\partial y + \hat{i}_z\,\partial/\partial z\right)\cdot\left(\hat{i}_x\sigma_{xx} + \hat{i}_y\sigma_{xy} + \hat{i}_z\sigma_{xz}\right)$$

$$= \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{xy}) + \frac{\partial}{\partial z}(\sigma_{xz})$$

The components of ④ is

$$\rho\frac{Du}{Dt} = X_x\rho + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p\right] + \frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mu\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{Du}{Dt} = X_x\rho - \frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x} - \frac{1}{3}\nabla\cdot\bar{q}\right)\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \quad (5)$$

$$\nabla\cdot\bar{P}_y = \left(\hat{i}_x\frac{\partial}{\partial x} + \hat{i}_y\frac{\partial}{\partial y} + \hat{i}_z\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}_x\sigma_{yx} + \hat{i}_y\sigma_{yy} + \hat{i}_z\sigma_{yz}\right)$$
$$= \frac{\partial}{\partial x}(\sigma_{yx}) + \frac{\partial}{\partial y}(\sigma_{yy}) + \frac{\partial}{\partial z}(\sigma_{yz})$$
$$= \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$\rho\frac{Dv}{Dt} = X_y\rho + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

ie) $$\rho\frac{Dv}{Dt} = X_y\rho - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$
$$+ 2\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial y} - \frac{1}{3}\nabla\cdot\bar{q}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \quad (6)$$

Similarly,

$$\nabla\cdot\bar{P}_z = \left(\hat{i}_x\frac{\partial}{\partial x} + \hat{i}_y\frac{\partial}{\partial y} + \hat{i}_z\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}_x\sigma_{zx} + \hat{i}_y\sigma_{zy} + \hat{i}_z\sigma_{zz}\right)$$
$$= \frac{\partial}{\partial x}(\sigma_{zx}) + \frac{\partial}{\partial y}(\sigma_{zy}) + \frac{\partial}{\partial z}(\sigma_{zz})$$
$$= \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p\right]$$

$$\rho \frac{Dw}{Dt} = X_z \rho + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\,\nabla\cdot\bar{q} - p\right]$$

ie) $$\rho \frac{Dw}{Dt} = X_z \rho - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right] + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z} - \frac{1}{3}\nabla\cdot\bar{q}\right)] \quad - (7)$$

(5), (6), (7) are known as Navier Stroke's equation of motion of a viscous compressible fluid.

Reduced form of Navier Stroke's Equations :-

Case i) viscous compressible fluid with constant viscosity

If the coefficient of viscosity μ is constant then (5), (6), (7) are reduced to

$$\rho\frac{Du}{Dt} = X_x \rho - \frac{\partial p}{\partial x} + 2\mu\left[\frac{\partial^2 u}{\partial x^2} - \frac{1}{3}\frac{\partial}{\partial x}(\nabla\cdot\bar{q})\right] + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial x}\right) \quad - (8)$$

$$\rho\frac{Dv}{Dt} = X_y \rho - \frac{\partial p}{\partial y} + 2\mu\left[\frac{\partial^2 v}{\partial y^2} - \frac{1}{3}\frac{\partial}{\partial y}(\nabla\cdot\bar{q})\right] + \mu\left(\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial x^2}\right) + \mu\left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial y}\right) \quad - (9)$$

$$\rho\frac{Dw}{Dt} = X_z \rho - \frac{\partial p}{\partial z} + 2\mu\left[\frac{\partial^2 w}{\partial z^2} - \frac{1}{3}\frac{\partial}{\partial z}(\nabla\cdot\bar{q})\right] + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z}\right) + \mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z}\right) \quad - (10)$$

The vector form of (8), (9) and (10) are

$$\rho\frac{Dq}{Dt} = \rho\bar{X} - \nabla p + \mu\nabla^2 q + \frac{1}{3}\mu\nabla(\nabla\cdot\bar{q}) \quad - (11)$$

Case ii) :- Viscous incompressible fluid

For incompressible fluid where both the density ρ and viscosity μ are constants. Then (11) reduces to $\nabla \cdot \bar{q} = 0$ — (12)

$$\frac{(11)}{\rho} \Rightarrow \frac{D\bar{q}}{Dt} = \bar{X} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \bar{q} \quad \text{— (13)}$$

[$\because$ For an incompressible fluid $\nabla \cdot \bar{q} = 0$]

※ [$\nu = \frac{\mu}{\rho}$ is known as kinematic viscosity]

From (8)

$$\mu\left[\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial z \partial x}\right]$$

$$= \mu\left[\frac{\partial^2 u}{\partial x^2}\left(1 - \frac{2}{3}\right) + \frac{\partial^2 v}{\partial x \partial y}\left(1 - \frac{2}{3}\right) + \frac{\partial^2 w}{\partial z \partial z}\left(1 - \frac{2}{3}\right)\right]$$

$$= \frac{1}{3}\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial z \partial x}\right]$$

$$= \frac{1}{3}\mu \nabla \cdot \bar{q}$$

$$(9) \Rightarrow \frac{1}{3}\mu \frac{\partial}{\partial y}\nabla \cdot \bar{q}$$

$$(10) \Rightarrow \frac{1}{3}\mu \frac{\partial}{\partial z}\nabla \cdot \bar{q}$$

Part-A

Case iii) :- Inviscid fluid and Incompressible fluid

For inviscid fluid the external shearing stress does not exist.

ie) $\mu = 0$

(13) reduces to $\frac{D\bar{q}}{Dt} = \bar{X} - \frac{1}{\rho}\nabla p$

This Equation is known as Euler's equation.

Example :-

1. 5M(S) Consider an incompressible steady flow with constant viscosity. The velocity component are given by

$$\left.\begin{aligned} u(y) &= y\frac{U}{h} + \frac{h^2}{2\mu}\left(\frac{-dp}{dx}\right)\frac{y}{h}\left(1 - \frac{y}{h}\right) \\ v &= w = 0 \end{aligned}\right\} \quad \text{— (1)}$$

If the body force is neglected, does $u(y)$ satisfy the equation of motion? $h, U, \frac{dp}{dx}$ are constants.

Solution :-

$$(13) \Rightarrow \left(\frac{\partial}{\partial t} + \bar{q}\cdot\nabla\right)\bar{q} = X - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \bar{q}$$

$\bar{X} = 0$ and for steady flow $\frac{\partial \bar{q}}{\partial t} = 0$

$$(\bar{q}\cdot\nabla)\bar{q} = \frac{-1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 q$$

$$= \frac{-1}{\rho}\nabla p + \nu\nabla^2\bar{q} \quad \text{—— ②}$$

In cartesian co-ordinates $\bar{q} = (u, v, w)$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

② can be written as

$$\left[(u,v,w)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right](u,v,w) = \frac{-1}{\rho}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)p$$

$$+ \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(u,v,w)$$

$$\left.\begin{aligned}
\Rightarrow \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u &= \frac{-1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v &= \frac{-1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\\
\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w &= \frac{-1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}\right\} \text{③}$$

Substituting ① in ③

$$0 = \frac{-1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{1}{\cancel{\rho}}\frac{dp}{dx} = \frac{\mu}{\cancel{\rho}}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{dp}{dx} = \mu\frac{\partial^2 u}{\partial y^2}$$

$$u(y) = y\frac{U}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\frac{y}{h} - \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\frac{y^2}{h^2}$$

$$\frac{\partial u}{\partial y} = \frac{U}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\frac{1}{h} - \frac{\cancel{h^2}}{\cancel{2}\mu}\left(-\frac{dp}{dx}\right)\left(\frac{\cancel{2}y}{\cancel{h^2}}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Hence satisfies the equation of motion.

10M(S) + 10M(S)

Energy Equation :- Conservation of Energy

By the first law of thermodynamics the energy added to a closed system increases the internal energy per unit mass of the fluid. The energy is added both by heat or by work done on the fluid.

Let Q be the heat added per unit mass of gas and E be the internal energy per unit mass of gas. The rate of work done by the stresses on a unit volume of gas is $dE = \delta Q + \delta W$

$$\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx}$$

The Mathematical formulation of the first law of thermodynamics is

$$\rho\frac{DQ}{Dt} + (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx}) = \rho\frac{DE}{Dt} \quad \text{—①}$$

where

$$\left.\begin{aligned}
\sigma_{xx} &= 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p \\
\sigma_{yy} &= 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\bar{q}\,\mu - p \\
\sigma_{zz} &= 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\nabla\cdot\bar{q} - p \\
\varepsilon_{xx} &= \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \\
\sigma_{xy}\gamma_{xy} &= (\mu\gamma_{xy})(\gamma_{xy}) = \mu\gamma_{xy}^2 = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\
\sigma_{yz}\gamma_{yz} &= (\mu\gamma_{yz})(\gamma_{yz}) = \mu\gamma_{yz}^2 = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \\
\sigma_{zx}\gamma_{zx} &= (\mu\gamma_{zx})(\gamma_{zx}) = \mu\gamma_{zx}^2 = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2
\end{aligned}\right\} \text{—②}$$

Substituting ② in ①

$$\rho\frac{DQ}{Dt}+\frac{\partial u}{\partial x}\left[2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\nabla\cdot\bar{q}-p\right]$$

$$+\frac{\partial v}{\partial y}\left[2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\nabla\cdot\bar{q}-p\right]$$

$$+\frac{\partial w}{\partial z}\left[2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\nabla\cdot\bar{q}-p\right]+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

$$+\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2=\rho\frac{DE}{Dt}$$

$$\rho\frac{DQ}{Dt}-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$-\frac{2}{3}\mu\nabla\cdot\bar{q}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\mu\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right.$$

$$\left.+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]=\rho\frac{DE}{Dt}$$

$$\rho\frac{DQ}{Dt}+\Phi=\rho\frac{DE}{Dt}+p\nabla\bar{q} \quad \text{--- (3)}$$

where $\Phi=2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]$

$$-\frac{2}{3}\mu(\nabla\cdot\bar{q})^2+\mu\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]$$

Φ is called dissipation function.

From equation of continuity

$$\frac{d\rho}{dt}+\rho(\nabla\bar{q})=0$$

$$\rho(\nabla\cdot\bar{q})=-\frac{d\rho}{dt}$$

$$\nabla\cdot\bar{q}=\frac{-1}{\rho}\frac{d\rho}{dt} \quad \text{--- (4)}$$

$$(4)\times\frac{p}{\rho}\Rightarrow\frac{p}{\rho}(\nabla\cdot\bar{q})=\frac{-p}{\rho^2}\frac{d\rho}{dt} \quad \text{--- (5)}$$

consider $\frac{d}{dt}\left(\frac{p}{\rho}\right)=\frac{\rho\frac{dp}{dt}-p\frac{d\rho}{dt}}{\rho^2}=\frac{\cancel{\rho}}{\rho^{\cancel{2}}}\frac{dp}{dt}-\frac{p}{\rho^2}\frac{d\rho}{dt}$

$$= \frac{1}{\rho}\frac{dp}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}$$

$$\frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\frac{dp}{dt} = -\frac{p}{\rho^2}\frac{d\rho}{dt} \quad \text{—⑥}$$

Substituting ⑥ in ⑤

$$\frac{p}{\rho}(\nabla\cdot\bar{q}) = \frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\left(\frac{dp}{dt}\right)$$

$$\Rightarrow p(\nabla\cdot\bar{q}) = \rho\frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{\rho}{\rho}\frac{dp}{dt}$$

$$p(\nabla\cdot\bar{q}) = \rho\frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{dp}{dt} \quad \text{—⑦}$$

Substituting ⑦ in ③

$$\rho\frac{dQ}{dt} + \Phi = \rho\frac{dE}{dt} + \rho\frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{dp}{dt}$$

$$\rho\frac{dQ}{dt} + \Phi = \rho\frac{d}{dt}\left(E + \frac{p}{\rho}\right) - \frac{dp}{dt}$$

$$\rho\frac{dQ}{dt} + \Phi = \rho\frac{dh}{dt} - \frac{dp}{dt} \quad \text{—⑧}$$

where $h = E + \frac{p}{\rho}$ is called enthalpy

⑧ is called Energy equation.

Special case :-

Case i) :- Energy equation for Non viscous fluid:

The Energy equation for a non viscous fluid can be obtained by putting $\Phi = 0$ in ⑧

$$\rho\frac{dQ}{dt} + \Phi = \rho\frac{dh}{dt} - \frac{dp}{dt}$$

Case ii) :-

For an adiabatic flow (temperature is constant)

ie) Q is a constant $\Rightarrow \frac{dQ}{dt} = 0$

⑨ reduces to $\rho\frac{dh}{dt} = \frac{dp}{dt}$