

Unit - II

one dimensional Inviscid Incompressible flow

5M [Equation of continuity - Stream tube flow

Definition :- Stream tube flow

The series of stream lines cannot be crossed by the fluid. Then they act as a solid boundary with the fluid ^{flowing within them}. This is called a stream tube.

The cross sectional areas of tube are small in comparison with its length so that the velocity can be assumed constant across any sectional area A.

Note :-

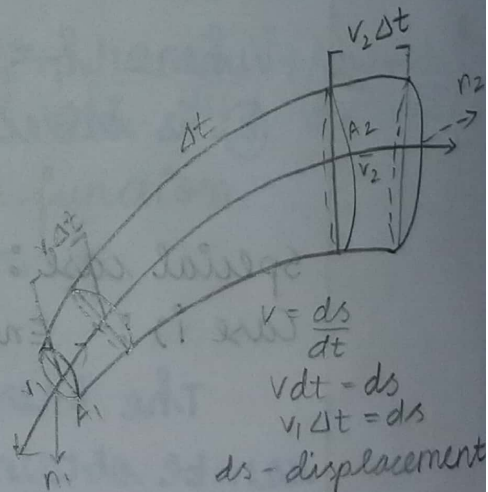
The stream tube maintained is the same shape for steady flow.

For an unsteady flow the stream tube changes its shape. 5M

(5M) Equation of continuity :-

Consider a stream tube with two arbitrary cross sections A_1 and A_2 with a mass of the fluid entering the tube through A_1 and leaving through A_2 . In an increment of time Δt the fluid particles which entered at A_1 have moved an infinitesimal distance ($v_1 \Delta t$) (displacement = velocity $\times$ time) and those that were at A_2 have moved an amount $v_2 \Delta t$

By conservation of mass no fluid can flow across the stream tube, the mass of the fluid entering the stream tube at A_1 at time Δt



must be exactly equal to mass of the fluid leaving at A_2 during the same interval of time Δt . This can be expressed mathematically

$$\rho V_1 A_1 \Delta t = \rho V_2 A_2 \Delta t$$

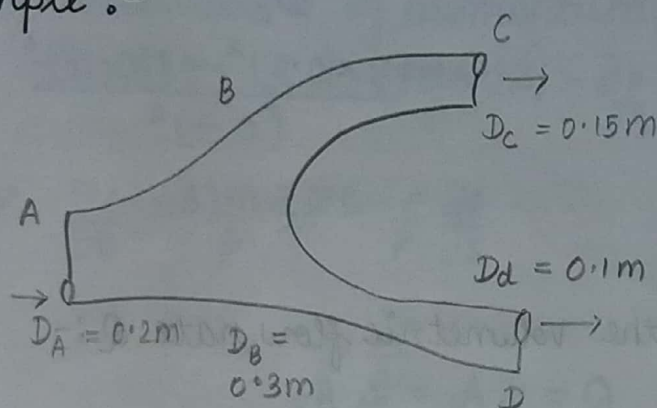
$$V_1 A_1 = V_2 A_2 = VA = Q = (\text{constant}) \quad \text{--- (1)}$$

① is called equation of continuity for one dimensional inviscid incompressible flow.

[The constant Q represents the volume of fluid which passes through each cross section of the stream tube per unit time. It is called the volumetric flow rate.]

Example :-

1.



This figure is shown water flowing through a circular tube (stream tube); It enters at A and leaves at C and D. If the velocity at B is $V_B = 1 \text{ m/s}$ and velocity at C is $V_C = 3 \text{ m/s}$. Calculate the velocities at D and A and the volumetric flow rate. (Friction can be ignored)

Solution :-

From equation of continuity

$$V_1 A_1 = V_2 A_2$$

$$V_A = ? \quad V_D = ? \quad Q = ?$$

$$A_1 = \pi r_1^2 = \pi \left(\frac{D_A}{2} \right)^2$$

$$A_2 = \pi r_2^2 = \pi \left(\frac{D_B}{2} \right)^2$$

$$V_A \pi \left(\frac{D_A}{2}\right)^2 = V_B \pi \left(\frac{D_B}{2}\right)^2$$

$$V_A (D_A)^2 = V_B (D_B)^2$$

$$\Rightarrow V_A = \frac{V_B D_B^2}{D_A^2}$$

$$= \frac{1 \times (0.3)^2}{(0.2)^2}$$

$$V_A = 2.25 \text{ m/s}$$

Similarly $V_B \pi \left(\frac{D_B}{2}\right)^2 = V_C \pi \left(\frac{D_C}{2}\right)^2 + V_D \pi \left(\frac{D_D}{2}\right)^2$

$$V_B D_B^2 = V_C D_C^2 + V_D D_D^2$$

$$V_D = \frac{V_B D_B^2 - V_C D_C^2}{D_D^2}$$

$$= \frac{1 \times (0.3)^2 - 3(0.15)^2}{(0.1)^2}$$

$$V_D = 2.25 \text{ m/s}$$

To find the Volumetric flow rate Q :-

$$Q = V_1 A_1 = V_2 A_2$$

$$\Rightarrow V_A \pi \frac{D_A^2}{4} = V_C \pi \left(\frac{D_C}{2}\right)^2 + V_D \pi \left(\frac{D_D}{2}\right)^2$$

$$V_A \pi \frac{D_A^2}{4} = 2.25 \times \frac{22}{7} \times \frac{(0.2)^2}{4} = 0.0707 \text{ m}^3/\text{s}$$

$$V_C \pi \frac{D_C^2}{4} + V_D \pi \frac{D_D^2}{4} = 3 \times \frac{22}{7} \times \frac{(0.15)^2}{4} + 2.25 \times \frac{22}{7} \times \frac{(0.1)^2}{4}$$

$$= 0.0707 \text{ m}^3/\text{s}$$

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(Tue)

Equation of Motion :- Euler's equation

Let us consider an elementary section of a stream tube. The mass of the stream tube element of length ds is $\rho dA ds$ and

the acceleration of the fluid flowing along ds is $\frac{DV}{Dt}$

The forces acting on the element are of two types, the body force and the force exerted by the hydrostatic pressure on the ends of the tube.

The total force along the path ds is

$$\Sigma F = X_s(\rho dA ds) + p dA - (p + \frac{\partial p}{\partial s} ds) dA$$

$$= X_s(\rho dA ds) - \frac{\partial p}{\partial s} ds dA$$

By conservation of momentum, $F = ma$

$$(\rho dA ds) \frac{DV}{Dt} = X_s(\rho dA ds) - \frac{\partial p}{\partial s} ds dA$$

$$\Rightarrow \frac{DV}{Dt} = \frac{1}{\rho} X_s - \frac{1}{\rho} \frac{\partial p}{\partial s}$$

$$= X_s - \frac{1}{\rho} \frac{\partial p}{\partial s}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{q} \cdot \nabla$ where $\bar{q} = V$ $\nabla = \frac{\partial}{\partial s}$

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial s} \right) V = X_s - \frac{1}{\rho} \frac{\partial p}{\partial s}$$

$$\boxed{\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial s} = X_s - \frac{1}{\rho} \frac{\partial p}{\partial s}} \quad \text{--- (1)}$$

is called Euler's equation of motion.

Application of Euler's equation :-

Bernoulli's equation :-

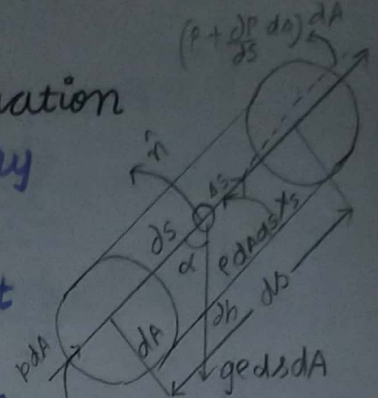
Assuming that the body force per unit mass

X_s is the gravitational force per unit mass

$$\therefore X_s = -g \cos \alpha$$

$$X_s(\rho dA ds) = (-g \cos \alpha) \rho dA ds$$

where $\cos \alpha = \frac{\partial h}{\partial s}$



2M(5)
+
2M(5)
+
5M(5)

$$X = r \cos \theta \quad X_s = -g \cos \alpha$$

2 M (5)

From Euler's equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} = X_s - \frac{1}{\rho} \frac{\partial p}{\partial s}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} = -g \left(\frac{\partial h}{\partial s} \right) - \frac{1}{\rho} \frac{\partial p}{\partial s}$$

For steady flow $\frac{\partial v}{\partial t} = 0$

$$v \frac{\partial v}{\partial s} = -g \frac{\partial h}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s}$$

$$\frac{1}{2} \frac{\partial}{\partial s} (v^2) + g \frac{\partial h}{\partial s} + \frac{1}{\rho} \frac{\partial p}{\partial s} = 0$$

$$\frac{\partial}{\partial s} \left(\frac{v^2}{2} + gh + \frac{p}{\rho} \right) = 0$$

$$\Rightarrow \frac{v^2}{2} + gh + \frac{p}{\rho} = K \text{ (a constant)} \quad \text{--- (2)}$$

This is called Bernoulli's equation for steady incompressible inviscid fluid. 5M/5

Deduction :-

Assume that the gravitational body force is negligible. Then the Bernoulli's equation reduces to

$$\frac{v^2}{2} + \frac{p}{\rho} = \text{constant}$$

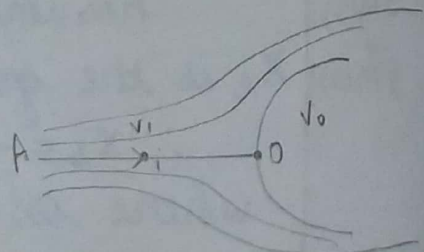
$$\times \rho \Rightarrow \frac{1}{2} \rho v^2 + p = \text{constant} \quad \text{--- (3)}$$

[The term $\frac{1}{2} \rho v^2$ has units same as those of pressure. However this pressure can exist only if the fluid is in motion.]

$\therefore$ We define this term as $\frac{1}{2} \rho v^2 = p_d$ where p_d is called dynamic pressure of the fluid

Stagnation point :-

(3) can be written as Bernoulli's equation with



negligible body force is

$$\frac{1}{2} \rho v^2 + p = \text{constant} \quad \text{--- (1)}$$

Writing (1) for the points 1 and 0

$$\frac{1}{2} \rho v_1^2 + p_1 = \frac{1}{2} \rho v_0^2 + p_0 = \text{constant}$$

On stream line A the flow is brought to rest at point 0. Hence $v_0 = 0$

[This point of zero velocity is called the stagnation point] $2 M(s) + 2 M(s)$

The pressure at point 0 is called the stagnation pressure and is given by

or (total pressure)

$$p_0 = \frac{1}{2} \rho v_1^2 + p_1$$

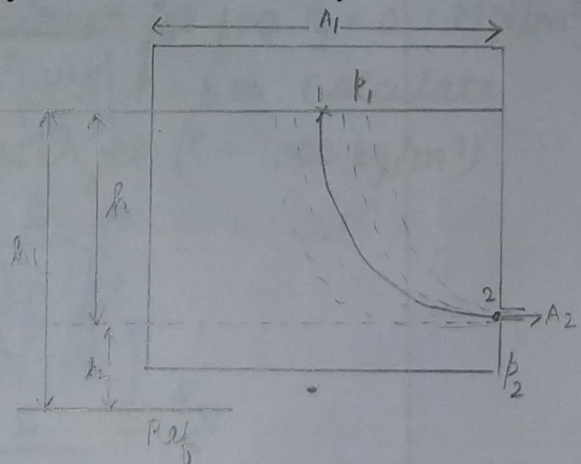
$$= p_{d1} + p_1$$

[$\therefore$ For an inviscid incompressible fluid with negligible body force, the stagnation pressure p_0 is equivalent to the sum of the dynamic pressure and static pressure and is constant throughout the fluid.]

Applications of Bernoulli's equation :-

1. Flow from a tank through a small Orifice

[A liquid filled tank which is sealed except for a small orifice near its base is shown in this figure. It is desired to determine the velocity of a efflux of the liquid from the tank when the orifice is open.]



from the tank when the orifice is open.]

Let the cross sectional area of the tank be A_1 and that of orifice be A_2 . The line connecting point 1 on the liquid with point 2 in the jet representing a stream line of the flow. Writing Bernoulli's

Equation between the point 1 and 2

$$\frac{v_1^2}{2} + gh_1 + \frac{p_1}{\rho} = \frac{v_2^2}{2} + gh_2 + \frac{p_2}{\rho}$$

$$\frac{v_1^2}{2} = \frac{v_2^2}{2} + gh_2 - gh_1 + \frac{p_2}{\rho} - \frac{p_1}{\rho}$$

$$= \frac{v_2^2}{2} + g(h_2 - h_1) + \frac{1}{\rho}(p_2 - p_1)$$

$$\because h_1 - h_2 = h$$

$$\frac{v_1^2}{2} = \frac{v_2^2}{2} - gh + \frac{1}{\rho}(p_2 - p_1) \quad \text{--- (1)}$$

From equation of continuity

$$v_1 A_1 = v_2 A_2$$

$$\Rightarrow v_1 = \frac{v_2 A_2}{A_1} \quad \text{--- (2)}$$

Substituting (2) in (1)

$$\frac{\left(\frac{v_2 A_2}{A_1}\right)^2}{2} = \frac{v_2^2}{2} - gh + \frac{1}{\rho}(p_2 - p_1)$$

$$\frac{1}{2} \left(\left(\frac{v_2 A_2}{A_1} \right)^2 - v_2^2 \right) = -gh + \frac{1}{\rho}(p_2 - p_1)$$

$$\frac{v_2^2}{2} \left(\left(\frac{A_2}{A_1} \right)^2 - 1 \right) = -gh + \frac{1}{\rho}(p_2 - p_1)$$

$$\Rightarrow v_2^2 = \frac{2}{\left[\left(\frac{A_2}{A_1} \right)^2 - 1 \right]} \left[-gh + \frac{1}{\rho}(p_2 - p_1) \right]$$

$$= \frac{2}{\left[1 - \left(\frac{A_2}{A_1} \right)^2 \right]} \left[gh - \frac{1}{\rho}(p_2 - p_1) \right]$$

$$v_2^2 = \frac{2}{\left[1 - \left(\frac{A_2}{A_1} \right)^2 \right]} \left[gh + \frac{1}{\rho}(p_1 - p_2) \right]$$

$$v_2 = \sqrt{\frac{2}{1 - \left(\frac{A_2}{A_1} \right)^2} \left[gh + \frac{1}{\rho}(p_1 - p_2) \right]} \quad \text{--- (3)}$$

v_2 is the relation for the velocity of efflux from the tank through the orifice

Case i) :- When $A_1 \gg A_2$

$$\text{then } \Rightarrow \left(\frac{A_2}{A_1} \right) \ll 1 \quad \text{--- (i)}$$

If the tank is vented to the atmosphere or has an open surface $p_1 = p_2$ --- (ii)

③ becomes

$$v_2 = \sqrt{2gh}$$

Statement :-

Thus [the velocity of efflux from the vented tank is equal to that of a rigid body falling freely from height h . This statement is known as Torricelli's theorem.] Proof :-

Case ii) :-

5M(S) ✗

$$\text{When } A_1 \gg A_2 \Rightarrow \left(\frac{A_2}{A_1} \right) \ll 1 \quad \text{--- (i)}$$

$$\frac{p_1 - p_2}{\rho} \gg gh \quad \text{--- (ii)}$$

$$\text{Then } \textcircled{3} \Rightarrow v_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho}} \quad] \text{ 5M(S)}$$

Example :-

1. Calculate the velocity of water jet for $p_2 = 0.1 \text{ MN/m}^2$, $p_1 = 0.2 \text{ MN/m}^2$, $A_2 = 0.01$ and $h = 5 \text{ m}$. Calculate the velocity of A_1 the water jet ($\rho = 1000 \text{ kg/m}^3$)

Solution :-

The velocity of water jet

$$v_2 = \sqrt{\frac{2}{1 - \left(\frac{A_2}{A_1} \right)^2} \left[gh + \frac{1}{\rho} (p_1 - p_2) \right]}$$

$$p_2 = 0.1 \text{ MN/m}^2 = 0.1 \times 10^6 \text{ N/m}^2$$

$$p_1 = 0.2 \text{ MN/m}^2 = 0.2 \times 10^6 \text{ N/m}^2$$

$$\rho = 1000 \text{ kg/m}^3$$

$$g = 9.81$$

$$v_2 = \sqrt{\frac{2}{1-(0.01)^2} \left[(9.81 \times 5) + \frac{(0.2-0.1) \times 10^3}{1 \times 10^3} \right]}$$

$$= \sqrt{\frac{2}{1-0.0001} (49.05 + 100)}$$

$$= \sqrt{\frac{2(149.05)}{1-0.0001}}$$

$$= \sqrt{\frac{2(149.05)}{0.9999}}$$

$$\begin{array}{r} 0.9999 \\ 1.0000 \\ -0.0001 \\ \hline 0.9999 \end{array}$$

$$v_2 = 17.3 \text{ m/s}$$

If $p_1 = p_2$ then

$$v_2 = \sqrt{2gh}$$

$$= \sqrt{2 \times 9.81 \times 5}$$

$$= 9.9 \text{ m/s}$$

Note :-

In order to take friction into account a correction factor can be introduced as

$$v_2 = c \sqrt{2gh} \quad \text{where } c \text{ is the}$$

the coefficient of discharge can be determined experimentally.

It varies from $0.59 \leq c \leq 1$

For a sharp edged opening c is taken as

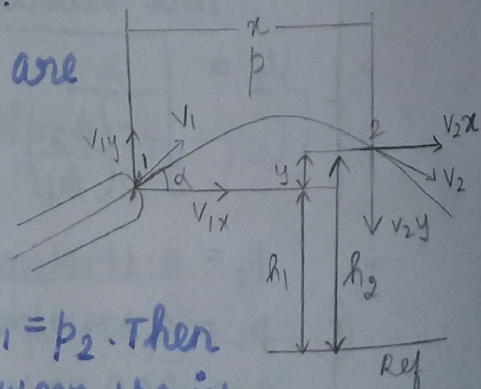
$$c = 0.6$$

2. Trajectory of ~~Flow~~ ^{Free} jet

Consider a liquid jet.

If the atmospheric ^{viscous} effects are neglected the jet can be considered as a stream line. Since the entire

jet is in the atmosphere $p_1 = p_2$. Then the Bernoulli's equation between the jet exit 1 and an arbitrary point 2 on the stream line



$$\frac{v_1^2}{2} + gh_1 = \frac{v_2^2}{2} + gh_2 \quad (\because p_1 = p_2) \quad \text{--- (1)}$$

We have to determine the velocity of the jet at any point on its trajectory as a function of jet flow rate Q , the horizontal distance x to point 2 and the initial inclination angle α . $v_2 = f(x, Q, \alpha)$

Assume the exit area of the nozzle to be A_1 . Then from Equation of continuity

$$Q = v_1 A_1$$

$$\Rightarrow v_1 = \frac{Q}{A_1} \quad \text{--- (2)}$$

$$\text{From (1)} \quad \frac{v_1^2}{2} + g(h_1 - h_2) = \frac{v_2^2}{2} \quad h_2 - h_1 = y$$

$$\frac{v_1^2}{2} + 2g(-y) = \frac{v_2^2}{2} \quad \text{--- (*)}$$

Substituting (2),

$$v_2^2 = \left(\frac{Q}{A_1}\right)^2 - 2gy \quad \text{--- (3)}$$

To evaluate y in terms of x and α by using the laws of kinematics

The equation can be utilized for any motion that can be described as being either a constant velocity motion or a constant acceleration.

The relation between the velocity components at points 1 and 2 are

$$v_{2x} = v_{1x} \quad (\text{or}) \quad v = v_0 + at$$

$$v_{2y} = v_{1y} - gt \quad \text{where } a \text{ is the acceleration due to gravity.} \quad \text{--- (5)}$$

Co-ordinates at the point (2) are

$$(4) \Rightarrow \frac{dx}{dt} = v_{1x}$$

$$x = v_{1x}t \quad \text{--- (6)}$$

$$(5) \Rightarrow \frac{dy}{dt} = v_{1y} - gt$$

$$y = v_{1y}t - \frac{gt^2}{2} \quad \text{--- (7)}$$

$$(x, y) = \left(v_{1x}t, v_{1y}t - \frac{gt^2}{2} \right)$$

$$(6) \Rightarrow t = \frac{x}{v_{1x}}$$

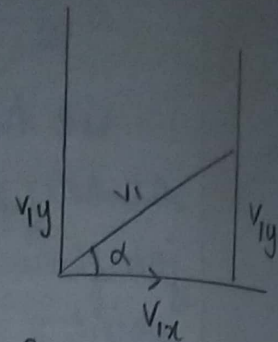
put (6) in (7)

$$y = v_{1y} \left(\frac{x}{v_{1x}} \right) - \frac{g}{2} \left(\frac{x}{v_{1x}} \right)^2$$

$$y = x \left(\frac{v_{1y}}{v_{1x}} \right) - \frac{gx^2}{2} \left(\frac{1}{v_{1x}^2} \right)$$

[using (8)]

$$y = x \tan \alpha - \frac{gx^2}{2} \left(\frac{1}{v_1^2 \cos^2 \alpha} \right)$$



$$(8) \begin{cases} \cos \alpha = \frac{v_{1x}}{v_1} \\ \sin \alpha = \frac{v_{1y}}{v_1} \end{cases}$$

$$\Rightarrow v_{1x} = v_1 \cos \alpha$$

$$\tan \alpha = \frac{v_{1y}}{v_{1x}}$$

$$(3) \Rightarrow v_2^2 = \left(\frac{Q}{A_1} \right)^2 - 2g \left[x \tan \alpha - \frac{gx^2}{2v_1^2 \cos^2 \alpha} \right]$$

$$\Rightarrow v_2 = \left[\left(\frac{Q}{A_1} \right)^2 - 2g \left[x \tan \alpha - \frac{gx^2}{2v_1^2 \cos^2 \alpha} \right] \right]^{1/2}$$

5m/s
+
5m/s

problem :-

- 3) 1. If a water jet is discharged from a nozzle (inclined) at an angle 60° with the horizontal at 5m/s. Calculate the horizontal distance required for the jet striking the ground which is 1m below the horizontal line of the nozzle. What is the velocity of the jet just before reaching the ground?

Solution :-

Given $\alpha = 60^\circ$ $v_1 = 5\text{m/s}$ $y = -1$

The horizontal and vertical components of the jet velocity at the nozzle exit are

$$v_{1x} = v_1 \cos \alpha = 5 \cos 60^\circ = 5 \left(\frac{1}{2} \right) = 2.5\text{m/s}$$

$$v_{1y} = v_1 \sin \alpha = 5 \sin 60^\circ = 5 \left(\frac{\sqrt{3}}{2} \right) = 4.33\text{m/s}$$

To find : the horizontal distance 'x'

$$y = \left(\frac{v_{1y}}{v_{1x}} \right) x - \frac{g}{2} \left(\frac{x}{v_{1x}} \right)^2$$

$$= \left(\frac{4.33}{2.5} \right) x - \frac{9.81}{2} \left(\frac{x}{2.5} \right)^2$$

$$-1 = y = 1.732x - 0.7848x^2$$

$$0.7848x^2 - 1.732x - 1 = 0$$

$$x = \frac{1.732 \pm \sqrt{(1.732)^2 - 4(0.7848)(-1)}}{2(0.7848)}$$

$$= \frac{1.732 \pm \sqrt{6.139024}}{1.5696}$$

$$= \frac{1.732 \pm 2.47770539}{1.5696}$$

(distance cannot be negative
so (-) neglected)

$$= 2.682$$

$$x = 2.682$$

ie) The horizontal distance between the nozzle and the jet striking the ground is

$$x = 2.682 \text{ m}$$

To find :- v_2

$$\text{By } (*) \Rightarrow v_2^2 = v_1^2 - 2gy$$

$$v_2 = \sqrt{v_1^2 - 2gy}$$

$$= \sqrt{25 - 2(9.81)(-1)}$$

$$v_2 = 6.68 \text{ m/s}$$

The Momentum Theorem :- (Application of Euler's equation)

The Application of Euler's equation to a particular problem involves the solution of differential equations satisfying appropriate boundary conditions

In many technically important problems they can be solved when the behaviour of fluid elements at the boundaries of a certain well defined region in the flow fluid is known, instead of motion of individual particle as given by Euler's equation.

This approach to problems is based on the momentum theorem

From the Euler's equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} = -g \frac{\partial h}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s} \quad \text{--- (1)}$$

$$\textcircled{1} \times \rho A \Rightarrow \rho A \frac{\partial v}{\partial t} + \rho A v \frac{\partial v}{\partial s} = -\rho A g \frac{\partial h}{\partial s} - \frac{\rho A}{\rho} \frac{\partial p}{\partial s} \quad \text{--- (2)}$$

From continuity equation of stream tube
mass can neither be created nor destroyed $m = \rho A \Delta t$

$$\frac{D(\rho A \Delta t)}{Dt} = 0 \quad \text{where}$$

$$\frac{D}{Dt} \quad \textcircled{3} \quad \Delta t = A \Delta s \quad \text{and} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial s}$$

③ becomes

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial s} \right) (\rho A \Delta s) = 0$$

$$\frac{\partial}{\partial t} (\rho A \Delta s) + v \frac{\partial}{\partial s} (\rho A \Delta s) = 0 \quad \text{--- (4)} \quad v = \frac{\Delta s}{\Delta t}$$

④ becomes

$$\frac{\partial}{\partial t} (\rho A v \Delta t) + v \frac{\partial}{\partial s} (\rho A v \Delta t) = 0 \quad \Delta s = v \Delta t$$

$$\left[v \frac{\partial}{\partial t} (\rho A) + v \frac{\partial}{\partial s} (\rho A v) \right] \Delta t = 0$$

$$\Rightarrow v \frac{\partial}{\partial t} (\rho A) + v \frac{\partial}{\partial s} (\rho A v) = 0 \quad \text{--- (5)}$$

② + ⑤ $\Rightarrow$

$$\rho A \frac{\partial v}{\partial t} + \rho A v \frac{\partial v}{\partial s} + v \frac{\partial}{\partial t} (\rho A) + v \frac{\partial}{\partial s} (\rho A v) = -\rho A g \frac{\partial h}{\partial s} - A \frac{\partial p}{\partial s}$$

$$\frac{\partial}{\partial t} (\rho A v) + v \frac{\partial}{\partial s} (\rho A v) + \rho A v \frac{\partial v}{\partial s} = -\rho A g \frac{\partial h}{\partial s} - A \frac{\partial p}{\partial s}$$

$$\frac{\partial}{\partial t} (\rho A v) + 2 \rho A v \frac{\partial v}{\partial s} = -\rho A g \frac{\partial h}{\partial s} - A \frac{\partial p}{\partial s} \quad \text{--- (6)}$$

$$\text{but} \quad \frac{\partial}{\partial s} (\rho A v^2) = 2 \rho A v \frac{\partial v}{\partial s}$$

$$\frac{\partial}{\partial s} (A p) = A \frac{\partial p}{\partial s} + p \frac{\partial A}{\partial s}$$

$$\Rightarrow A \frac{\partial p}{\partial s} = \frac{\partial}{\partial s} (Ap) - p \frac{\partial A}{\partial s}$$

$\therefore$ (6) becomes

$$\frac{\partial}{\partial t} (PAV) + \frac{\partial}{\partial s} (PAV^2) = -PAg \frac{\partial h}{\partial s} - \left[\frac{\partial}{\partial s} (Ap) - p \frac{\partial A}{\partial s} \right]$$

$$\rho \frac{\partial}{\partial t} (AV) + \rho \frac{\partial}{\partial s} (AV^2) = p \frac{\partial A}{\partial s} - \rho Ag \frac{\partial h}{\partial s} - \frac{\partial}{\partial s} (Ap) \quad \text{--- (7)}$$

Integrating (7) with respect to s between any two points 1 and 2

$$\begin{aligned} \rho \int_1^2 \frac{\partial}{\partial t} (AV) ds + \rho \int_1^2 \frac{\partial}{\partial s} (AV^2) ds \\ = \int_1^2 p \frac{\partial A}{\partial s} ds - \rho g \int_1^2 A \frac{\partial h}{\partial s} ds - \int_1^2 \frac{\partial}{\partial s} (Ap) ds \end{aligned}$$

$$\rho \int_1^2 \frac{\partial}{\partial t} (AV) ds + \rho (AV^2)_1^2 = (p) \int_1^2 dA - \rho g \int_1^2 A dh - (Ap)_1^2$$

$$\rho \int_1^2 \frac{\partial}{\partial t} (AV) ds + \rho (A_2 v_2^2 - A_1 v_1^2) = \int_1^2 p \cdot dA - \rho g \int_1^2 A dh - (A_2 p_2 - A_1 p_1) \quad \text{--- (8)}$$

The LHS of (8) is rate of change of momentum in space between the sections 1 and 2 and RHS is the total force in the s -direction, due to the pressure on the ends and the walls and of gravity respectively.

For steady flow with gravity neglected,

(8) becomes $\Rightarrow$

$$\rho (A_2 v_2^2 - A_1 v_1^2) = \int_1^2 p \cdot dA + A_1 p_1 - A_2 p_2 \quad \text{--- (9)}$$

From Equation of continuity

$$Q = A_1 v_1 = A_2 v_2$$

$$A_1 = \frac{Q}{v_1} \quad A_2 = \frac{Q}{v_2}$$

Substituting A_1 and A_2 in (9)

$$\rho \left(\frac{Q}{v_2} v_2^2 - \frac{Q}{v_1} v_1^2 \right) = \int_1^2 p dA + \frac{Q}{v_1} p_1 - \frac{Q}{v_2} p_2$$

$$\rho Q (v_2 - v_1) = \int_1^2 p \cdot dA - Q \left[\left(\frac{p_2}{v_2} \right) - \left(\frac{p_1}{v_1} \right) \right]$$

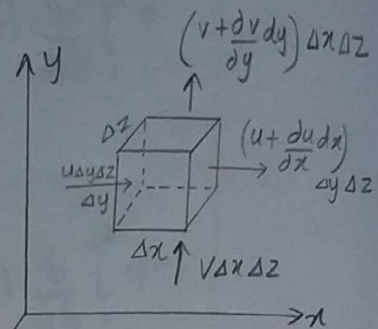
Two and three dimensional Inviscid incompressible flow :-

Basic equations and concepts of flow :-

(2M) [i) Equation of continuity

For an incompressible fluid the equation of continuity is $\nabla \cdot \vec{q} = 0$.

The net amount of fluid flowing through a surface $\Delta y \Delta z$ in a unit time is $\frac{\partial u}{\partial x} \Delta x \Delta y \Delta z$. Similarly the net out flows in the y and z directions are $\frac{\partial v}{\partial y} \Delta x \Delta y \Delta z$ and $\frac{\partial w}{\partial z} \Delta x \Delta y \Delta z$.



$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \Delta x \Delta y \Delta z = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\text{ie) } \nabla \cdot \vec{q} = 0]$$

ii) [Eulerian equation of motion :-

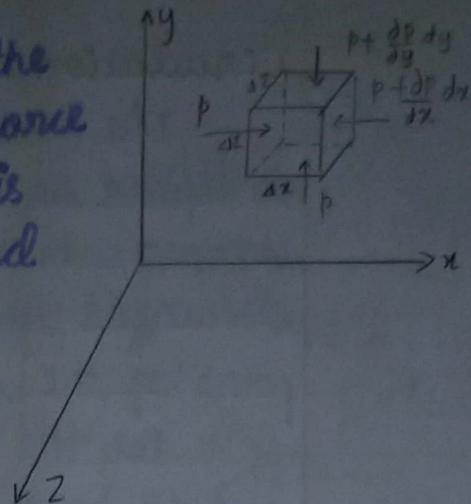
Euler's equation of motion is valid for both compressible and incompressible flows

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = \vec{X} - \frac{1}{\rho} \nabla p$$

In cartesian co-ordinates: consider a fluid element of dimension $\Delta x, \Delta y, \Delta z$. The forces acting on this fluid element consist of the body force $\vec{X}$ and the normal pressure force exerted by the fluid around it. Let X_x be the

5M(15)
+
2M(15)

Body force per unit mass in the x -direction. Then the total force acting on the x -direction is $X_x \rho \Delta x \Delta y \Delta z - \frac{\partial p}{\partial x} \Delta x \Delta y \Delta z$ and



the momentum along x -direction is ΔM_x is $\rho \Delta x \Delta y \Delta z u$.

By Newton's second law the total force acting on the mass of the fluid = rate of change of linear momentum $m a = F$

$$\frac{D}{Dt} (\Delta M_x) = X_x \rho \Delta x \Delta y \Delta z - \frac{\partial p}{\partial x} \Delta x \Delta y \Delta z$$

$$\text{i.e. } \frac{D}{Dt} (\rho \Delta x \Delta y \Delta z u) = \left(X_x \rho - \frac{\partial p}{\partial x} \right) \Delta x \Delta y \Delta z$$

$$\frac{D}{Dt} (\rho u) = X_x \rho - \frac{\partial p}{\partial x}$$

For an incompressible fluid, ρ - constant

$$\rho \frac{Du}{Dt} = X_x \rho - \frac{\partial p}{\partial x}$$

$\div \rho$

$$\frac{Du}{Dt} = X_x - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad \text{--- (1)}$$

Similarly in the y and z directions are

$$\frac{Dv}{Dt} = X_y - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad \text{--- (2)}$$

$$\frac{Dw}{Dt} = X_z - \frac{1}{\rho} \frac{\partial p}{\partial z} \quad \text{--- (3)}$$

In vector form ①, ②, ③ can be written as

$$\frac{D\vec{q}}{Dt} = \vec{x} - \frac{1}{\rho} \nabla p$$

$$\text{where } \frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{q} \cdot \nabla)]$$

2M19

[Circulation :- concept

The circulation around the contour C is defined as the summation of the velocity components tangential to the C times the elemental length of the contour $d\vec{r}$ at every point $\vec{r}$ of the contour.]

Let the velocity field be $\vec{q}$ at any arbitrary point and $d\vec{r}$ be the directed element of C at the same point. Then the circulation is given by

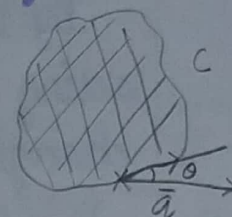
$$\Gamma = \oint_C \vec{q} \cdot d\vec{r} \quad a \cdot b = ab \cos \theta$$

$$= \oint_C q \cos \theta \, dr$$

If $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ then

$$\Gamma = \oint (u dx + v dy + w dz)$$

where the symbol $\oint$ denote the line integral taken around the ^{closed curve in the} anti clockwise direction.



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10M15)

Stoke's Theorem :-

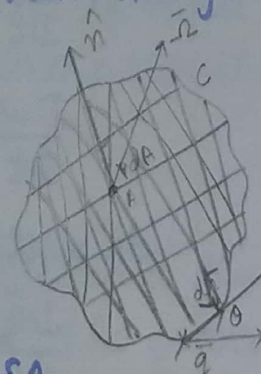
Let A be a surface bounded by the contour C and $\hat{n}$ be the unit vector normal to the surface at point P . Stoke's theorem may be expressed as

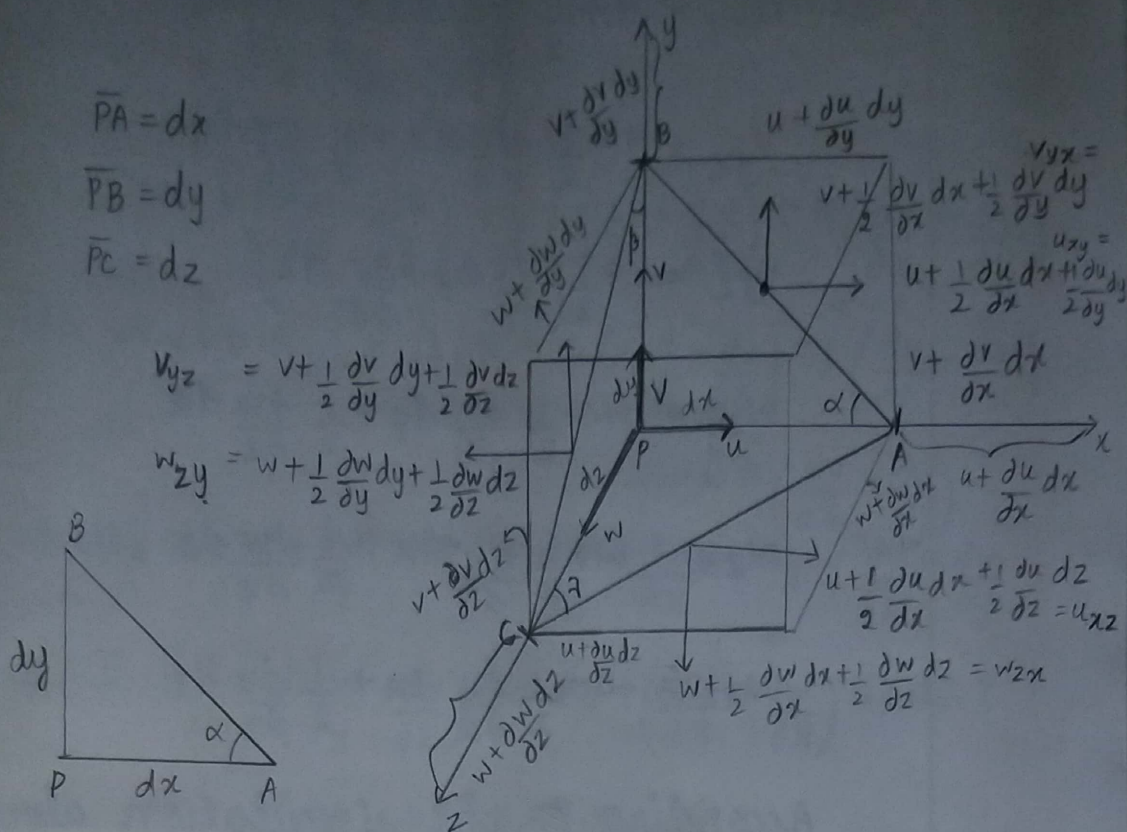
$$\Gamma = \oint_C \vec{q} \cdot d\vec{r} = \iint_A \vec{\omega} \cdot \hat{n} \, dA$$

Proof :-

Let us divide the space A into a very large number of infinitesimally small triangles δA

If lines are drawn from the vertices of these triangles parallel to x, y and z axes, a series of infinitesimally small tetrahedrons is formed





PABC be the one of the elementary tetrahedrons with edges PA, PB and PC equal to $\overline{PA} = dx$ $\overline{PB} = dy$ $\overline{PC} = dz$ respectively. From ΔPAB

$$\cos \alpha = \frac{\overline{PA}}{AB} = \frac{dx}{AB}$$

$$\sin \alpha = \frac{\overline{PB}}{AB} = \frac{dy}{AB}$$

$$dx = AB \cos \alpha$$

$$dy = AB \sin \alpha$$

$$AB = \frac{dx}{\cos \alpha}$$

$$AB = \frac{dy}{\sin \alpha}$$

Similarly from ΔPBC and ΔPAC

$$BC = \frac{dy}{\cos \beta}$$

$$BC = \frac{dz}{\sin \beta}$$

$$AC = \frac{dz}{\cos \gamma}$$

$$AC = \frac{dx}{\sin \gamma}$$

ie) AB, BC and CA equal to

$$\frac{dx}{\cos \alpha} = \frac{dy}{\sin \alpha} \quad \frac{dy}{\cos \beta} = \frac{dz}{\sin \beta} \quad \frac{dz}{\cos \gamma} = \frac{dx}{\sin \gamma}$$

0805-10-38
(contd.)

The velocity components at the mid points of AB, BC and CA are

$$ie) u_{xy} = u + \frac{1}{2} \frac{\partial u}{\partial x} dx + \frac{1}{2} \frac{\partial u}{\partial y} dy$$

$$v_{yx} = v + \frac{1}{2} \frac{\partial v}{\partial x} dx + \frac{1}{2} \frac{\partial v}{\partial y} dy$$

$$v_{yz} = v + \frac{1}{2} \left(\frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz \right)$$

$$w_{zy} = w + \frac{1}{2} \frac{\partial w}{\partial y} dy + \frac{1}{2} \frac{\partial w}{\partial z} dz$$

$$w_{zx} = w + \frac{1}{2} \frac{\partial w}{\partial x} dx + \frac{1}{2} \frac{\partial w}{\partial z} dz$$

$$u_{xz} = u + \frac{1}{2} \frac{\partial u}{\partial x} dx + \frac{1}{2} \frac{\partial u}{\partial z} dz$$

According to the circulation along the sides of the triangle ABCA is given by (circulation is considered positive if it rotates according to Right Handed Screw rule with outward normal)

$$\begin{aligned} d\Gamma &= -u_{xy} \cos \alpha (AB) + v_{yx} \sin \alpha (AB) \\ &\quad - v_{yz} \cos \beta (BC) + w_{zy} \sin \beta (BC) \\ &\quad - w_{zx} \cos \gamma (CA) + u_{xz} \sin \gamma (CA) \\ &= -u dx - \frac{1}{2} \frac{\partial u}{\partial x} dx dx - \frac{1}{2} \frac{\partial u}{\partial y} dy dx \\ &\quad + v dy + \frac{1}{2} \frac{\partial v}{\partial x} dx dy + \frac{1}{2} \frac{\partial v}{\partial y} dy dy - v dz - \frac{1}{2} \frac{\partial v}{\partial y} dy dz - \frac{1}{2} \frac{\partial v}{\partial z} dz dy \\ &\quad + w dz + \frac{1}{2} \frac{\partial w}{\partial y} dy dz + \frac{1}{2} \frac{\partial w}{\partial z} dz dz - w dx - \frac{1}{2} \frac{\partial w}{\partial x} dx dx - \frac{1}{2} \frac{\partial w}{\partial z} dz dx \\ &\quad + u dx + \frac{1}{2} \frac{\partial u}{\partial x} dx dx + \frac{1}{2} \frac{\partial u}{\partial z} dz dx \\ &= -\frac{1}{2} \frac{\partial u}{\partial y} dx dy + \frac{1}{2} \frac{\partial u}{\partial z} dz dx + \frac{1}{2} \frac{\partial v}{\partial x} dx dy - \frac{1}{2} \frac{\partial v}{\partial z} dz dy \\ &\quad - \frac{1}{2} \frac{\partial w}{\partial x} dx dz + \frac{1}{2} \frac{\partial w}{\partial y} dy dz \end{aligned}$$

$$= \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) dz dy + \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) dz dx + \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

23-01-2020
(Thurs)

consider

$$2(\vec{n} \cdot \hat{n}_{dA}) = 2(\vec{n}_x \hat{i} + \vec{n}_y \hat{j} + \vec{n}_z \hat{k}) \left(\hat{i} \frac{dy dz}{2} + \hat{j} \frac{dz dx}{2} + \hat{k} \frac{dx dy}{2} \right)$$

$$2(\vec{n} \cdot \hat{n}_{dA}) = \vec{n}_x dy dz + \vec{n}_y dz dx + \vec{n}_z dx dy$$

where $\vec{n} = \nabla \times \vec{q}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \hat{j} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \hat{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\text{ie) } \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k} = \hat{i} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \hat{j} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \hat{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\Rightarrow \Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \quad \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$2(\vec{\Omega} \cdot \hat{n} dA) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) dy dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) dz dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy \quad \text{--- (2)}$$

Substituting (2) in (1)

$$d\Gamma = \frac{1}{2} \times 2(\vec{\Omega} \cdot \hat{n} dA)$$

Integrating on both sides

$$\Gamma = \oint_A \vec{\Omega} \cdot \hat{n} dA$$

Hence proved Stoke's theorem.

5M(5) Kelvin's theorem :- ^{constancy} consistency of circulation

5M(5) In an ideal homogeneous fluid, the circulation is independent of time along a closed fluid line where the external forces are conservative and are derived from a single valued potential

Proof :-

By definition of circulation

$$\Gamma = \oint \vec{q} \cdot d\vec{n}$$

$$\frac{D}{Dt}(\Gamma) = \oint \frac{D}{Dt}(\vec{q} \cdot d\vec{n}) \quad \text{--- (1)}$$

$$\frac{D}{Dt}(\vec{q} \cdot d\vec{n}) = \vec{q} \cdot \frac{D}{Dt}(d\vec{n}) + d\vec{n} \cdot \frac{D\vec{q}}{Dt} \quad \text{--- (2)}$$

From the ^{Eulerian} equation of motion

$$\frac{D\bar{q}}{Dt} = \bar{X} - \frac{1}{\rho} \nabla p$$

$$\bar{F} = -\nabla \phi$$

Given that $\bar{X}$ is conservative

$$\bar{X} = -\nabla \phi_f$$

$$\therefore \frac{D\bar{q}}{Dt} = -\nabla \phi_f - \frac{1}{\rho} \nabla p$$

Taking dot product with $d\bar{n}$

$$d\bar{n} \cdot \frac{D\bar{q}}{Dt} = -d\bar{n} \cdot \nabla \phi_f - \frac{1}{\rho} (d\bar{n} \cdot \nabla p)$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= -(dx\hat{i} + dy\hat{j} + dz\hat{k}) \cdot \left(\frac{\partial \phi_f}{\partial x} \hat{i} + \frac{\partial \phi_f}{\partial y} \hat{j} + \frac{\partial \phi_f}{\partial z} \hat{k} \right)$$

$$- \frac{1}{\rho} \left[(dx\hat{i} + dy\hat{j} + dz\hat{k}) \cdot \left(\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \right) \right]$$

$$= - \left(\frac{\partial \phi_f}{\partial x} dx + \frac{\partial \phi_f}{\partial y} dy + \frac{\partial \phi_f}{\partial z} dz \right)$$

$$- \frac{1}{\rho} \left(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz \right)$$

$$= -d\phi_f - \frac{1}{\rho} dp \quad \text{--- (3)}$$

Substituting (3) in (2)

$$\frac{D}{Dt} (\bar{q} \cdot d\bar{n}) = \bar{q} \cdot d\bar{q} - d\phi_f - \frac{1}{\rho} dp$$

$$\frac{D}{Dt} = \frac{d}{dt}$$

$$\frac{D}{Dt} (d\bar{n}) = \frac{d}{dt} (d\bar{n})$$

$$= d \left(\frac{d\bar{n}}{dt} \right)$$

$$= \frac{d\bar{q}}{dt}$$

Integrating along a closed curve C on both sides

$$\oint_C \frac{D}{Dt} (\bar{q} \cdot d\bar{n}) = \oint_C \bar{q} \cdot d\bar{q} - \oint_C d\phi_f - \oint_C \frac{1}{\rho} dp$$

$$\frac{D}{Dt} \oint_C \bar{q} \cdot d\bar{n} = \frac{q^2}{2} - \phi_f - \frac{1}{\rho} p$$

$$\frac{D\Gamma}{Dt} = - \left[\phi_f + \frac{p}{\rho} - \frac{q^2}{2} \right] \quad \text{--- (3)}$$

If as assumed ϕ_f, p, q are single valued functions then RHS is zero.

$$\Rightarrow \frac{D\Gamma}{Dt} = 0 \Rightarrow \Gamma \text{ is a constant.}$$

This Γ is independent of time.

(5) Problem :-

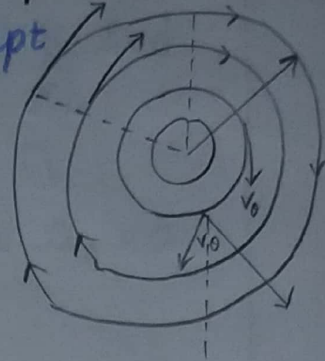
- * 1. Calculate the circulation Γ of a circulatory flow ($v_r = v_z = 0$) with constant vorticity $\Omega_z = \text{constant}$ from both the circulation concept and Stoke's theorem i) By circulation concept

From the vorticity equation

$$\Omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta}$$

given that $v_r = 0$

$$\Omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} = \text{constant (given)}$$



From equation of continuity in cylindrical co-ordinates,

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0$$

given $v_r = v_z = 0$

$$\Rightarrow \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0$$

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

$$\Rightarrow v_\theta = f(r)$$

$$\Gamma = \oint \vec{q} \cdot d\vec{r}$$

$$\vec{q} \cdot d\vec{r} = (v_r, v_\theta, v_z) \cdot (dr, r d\theta, dz)$$

$$= v_\theta r d\theta$$

$$= r f(r) d\theta$$

$$\Gamma = \oint r f(r) d\theta = r f(r) \oint d\theta$$

$$= r f(r) \cdot 2\pi$$

ii) By Stoke's theorem

$$\Gamma = \iint_A \vec{\Omega} \cdot \hat{n} dA$$

$$= \int_0^{2\pi} \int_0^n \Omega_2 n dn d\theta$$

$$= \int_0^{2\pi} \int_0^n \left(\frac{dv_0}{dn} + \frac{v_0}{n} \right) n dn d\theta$$

$$\Gamma = \int_0^{2\pi} \int_0^n \left(n \frac{dv_0}{dn} + v_0 \right) dn d\theta$$

but $\frac{d}{dn} (n v_0) = v_0 + n \frac{dv_0}{dn}$

$$\Gamma = \int_0^{2\pi} \int_0^n \frac{d(n v_0)}{dn} dn d\theta$$

$$= \int_0^{2\pi} [n v_0]_0^n d\theta$$

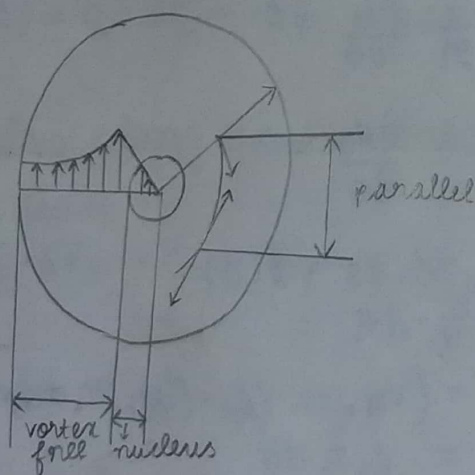
$$= n f(n) \int_0^{2\pi} d\theta$$

$$\Gamma = 2\pi n f(n)$$

$$= 2\pi n v_0$$

The circulatory velocity $v_0 = \frac{\Gamma}{2\pi n}$

24-01-2020 Example :-
(Fni)



The fluid motion in concentric circles is shown in this figure. In this case we consider a flow for which an integral portion of fluid bounded by a circle (the core) is undergoing rigid body motion with constant angular velocity Ω_2 , while a concentric outer portion of the fluid

without rotation i.e. $\Omega_z = 0$ flow
represents a vortex free circulatory free
Calculate the tangential velocity in both regions
From vorticity equation

$$\Omega_z = \frac{dv_\theta}{dr} + \frac{v_\theta}{r} - \frac{1}{r} \frac{dv_r}{d\theta}$$

Since $v_r = 0$

$$\Omega_z = \frac{dv_\theta}{dr} + \frac{v_\theta}{r} \quad \text{--- (1)}$$

$$v_\theta = f(r) \quad \text{--- (2)}$$

i) In the outer region of flow $\Omega_z = 0$

$$\textcircled{1} \Rightarrow \frac{dv_\theta}{dr} + \frac{v_\theta}{r} = 0$$

$$\frac{dv_\theta}{dr} = -\frac{v_\theta}{r}$$

$$\frac{dv_\theta}{v_\theta} = -\frac{dr}{r}$$

Integrating both sides

$$\ln v_\theta = -\ln r + \ln c$$

$$\ln v_\theta + \ln r = \ln c$$

$$\ln (v_\theta r) = \ln c$$

$$v_\theta r = c$$

$$\Rightarrow v_\theta = \frac{c}{r}$$

This represents a vortex free circulatory flow.

ii) In the inner region $\Omega_z = A_1$

$$\textcircled{1} \Rightarrow \frac{dv_\theta}{dr} + \frac{v_\theta}{r} = A_1 \quad \text{where } A_1 \text{ is a constant}$$

$$\frac{d}{dr} (v_\theta r) = \frac{dv_\theta}{dr} \cdot r + v_\theta$$

$$\Rightarrow \frac{1}{r} \frac{d}{dr} (v_\theta r) = \frac{1}{r} \left(\frac{dv_\theta}{dr} r + v_\theta \right) = \frac{dv_\theta}{dr} + \frac{v_\theta}{r}$$

$$\text{ie) } \frac{1}{r} \frac{d}{dr} (v_\theta r) = A_1$$

$$\frac{d}{dr} (v_\theta r) = A_1 r$$

Integrating with respect to r

$$v_\theta r = \frac{A_1 r^2}{2} + C$$

$$\Rightarrow v_\theta = \frac{A_1 r}{2} + C_1 \quad \text{where } C_1 = \frac{C}{r}$$

$$\text{When } r=0 \quad v_\theta = 0$$

$$0 = 0 + C_1$$

$$\Rightarrow C_1 = 0$$

Substituting C_1 in v_θ

$$v_\theta = \frac{A_1 r}{2} = A r \quad \text{where } A = \frac{A_1}{2}$$

$$\text{ie) } v_\theta = A r$$

This represents either a rotating solid ~~body~~ boundary or a fluid rotating like a solid ~~body~~ boundary in the nucleus

This concept is employed in a 2D airfoil theory where in vortex free flow exists outside the boundary of a solid body such as an airfoil

Velocity potential - ^{2m/s} [Irrotational flow :-]

The ^{ve}moment of ideal fluids is in general of two types. Each differing from other mathematically and physically. They are called rotational and irrotational flow.

From the definition of circulation

$$\Gamma = \oint_C \vec{q} \cdot d\vec{r} \quad - (1)$$

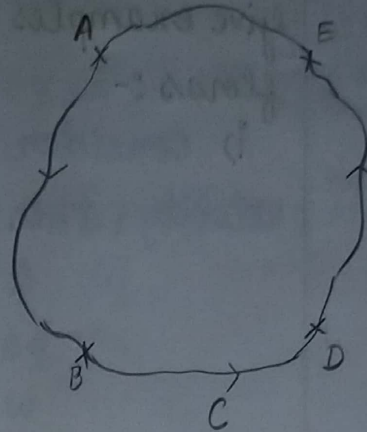
$$\text{By Stoke's theorem } \oint_C \vec{q} \cdot d\vec{r} = \iint_A \vec{\omega} \cdot \hat{n} dA \quad - (2)$$

①, ② $\Rightarrow \Gamma = \oint \bar{q} \cdot d\bar{n} = \iint \bar{\omega} \cdot \hat{n} dA$ where $\bar{\omega} = \nabla \times \bar{q}$
 For irrotational flow $\bar{\omega} = 0$

ie) $\nabla \times \bar{q} = 0$

$\Rightarrow \Gamma = 0$

Let us consider a closed contour C ~~ABDEA~~
 with the circuit ~~ABDEA~~.
 The circuit ~~ABDEA~~ lie in a
 single valued function
simply connected region



$\int \bar{q} \cdot d\bar{n} = 0$

ABDEA

$\int_{ABD} \bar{q} \cdot d\bar{n} + \int_{DEA} \bar{q} \cdot d\bar{n} = 0$

$\Rightarrow \int_{ABD} \bar{q} \cdot d\bar{n} = - \int_{DEA} \bar{q} \cdot d\bar{n}$

$\int_{ABD} \bar{q} \cdot d\bar{n} = \int_{AED} \bar{q} \cdot d\bar{n}$
 $= \int_A^{D(x,y,z)} \bar{q} \cdot d\bar{n}$

For irrotational flow $\bar{q} = \nabla \phi$
 $= \int_A^{D(x,y,z)} \nabla \phi \cdot d\bar{n} = \int_A^D d\phi$

$= [\phi]_A^D$
 $= \phi_D - \phi_A$

where ϕ_D is a scalar quantity of space whose value depends only on the position of D and not on the path from A to D .

$\bar{q} \cdot d\bar{n} = d\phi$

$\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$d\bar{n} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

ie) $u dx + v dy + w dz = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$

$\Rightarrow u = \frac{\partial \phi}{\partial x}$

$v = \frac{\partial \phi}{\partial y}$

$w = \frac{\partial \phi}{\partial z}$

Thus when the motion is irrotational the velocity vector is the gradient of the scalar function ϕ . ϕ is called the velocity potential.

Example:-

(5) Give examples of rotational and irrotational flows:-

i) Consider a parallel flow with uniform velocity. Then we have the velocity components

$$u = Ax$$

$$v = 0$$

$$w = 0$$

$$\begin{array}{l} \xrightarrow{(u)} u = Ax \\ \xrightarrow{\quad} v = 0 \\ \xrightarrow{\quad} w = 0 \end{array}$$

The vorticity about z axis

$$\vec{\Omega} = \nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax & 0 & 0 \end{vmatrix} = 0$$

$$\Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k} = \vec{\Omega}$$

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= 0$$

Hence the flow is irrotational.

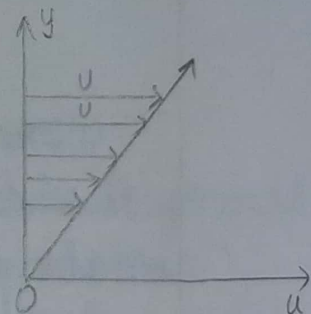
ii) A 2D shear flow is an example for a rotational flow.

In this case the velocity components are

$$u = Ay$$

$$v = 0$$

$$w = 0$$



The vorticity and shearing stress are

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -A$$

$$\tau_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = A$$

$$\vec{\Omega} = \nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 0 \end{vmatrix} = -A \hat{k}$$

Integration of the equation of motion :-
(Bernoulli's equation)

Equation of motion for inviscid fluid is

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla) \bar{q} = \bar{x} - \frac{1}{\rho} \nabla p \quad \text{--- (1)}$$

We know that

$$\nabla(\bar{A} \cdot \bar{B}) = \bar{A} \times (\nabla \times \bar{B}) + \bar{B} \times (\nabla \times \bar{A}) + (\bar{A} \cdot \nabla) \bar{B} + (\bar{B} \cdot \nabla) \bar{A}$$

$$\text{If } \bar{A} = \bar{B} = \bar{q}$$

$$\nabla(\bar{q} \cdot \bar{q}) = \bar{q} \times (\nabla \times \bar{q}) + \bar{q} \times (\nabla \times \bar{q}) + (\bar{q} \cdot \nabla) \bar{q} + (\bar{q} \cdot \nabla) \bar{q}$$

$$\nabla(q^2) = 2 \bar{q} \times (\nabla \times \bar{q}) + 2(\bar{q} \cdot \nabla) \bar{q}$$

$$\nabla \times \bar{q} = \bar{\omega}$$

$$\frac{1}{2} \nabla(q^2) - (\bar{q} \times \bar{\omega}) = (\bar{q} \cdot \nabla) \bar{q}$$

$$\nabla\left(\frac{q^2}{2}\right) - (\bar{q} \times \bar{\omega}) = (\bar{q} \cdot \nabla) \bar{q} \quad \text{--- (2)}$$

$$\bar{x} = -\nabla \phi_f \quad \text{--- (3)}$$

Substituting (2) and (3) in (1)

$$\frac{d\bar{q}}{dt} + \left[\nabla\left(\frac{q^2}{2}\right) - (\bar{q} \times \bar{\omega}) \right] = -\nabla \phi_f - \frac{1}{\rho} \nabla p$$

$$\frac{\partial \bar{q}}{\partial t} - (\bar{q} \times \bar{\omega}) = -\nabla\left(\frac{q^2}{2} + \phi_f + \frac{p}{\rho}\right) \quad \text{--- (4)}$$

Case i) :- Steady motion

For steady flow $\frac{\partial \bar{q}}{\partial t} = 0$

$\therefore$ (4) becomes

$$-(\bar{q} \times \bar{\omega}) = -\nabla\left(\frac{q^2}{2} + \phi_f + \frac{p}{\rho}\right)$$

Taking dot product with $d\bar{n}$

$$(\bar{q} \times \bar{\omega}) \cdot d\bar{n} = \nabla\left(\frac{q^2}{2} + \frac{p}{\rho} + \phi_f\right) \cdot d\bar{n}$$

$$\nabla \phi \cdot d\bar{n} = d\phi$$

$$= d\left(\frac{q^2}{2} + \frac{p}{\rho} + \phi_f\right)$$

Integrate this equation along the stream line
($\bar{q} \cdot d\bar{n} = 0$)

LHS of the above equation becomes

$$(\bar{q} \times \bar{\omega}) \cdot d\bar{n} = \bar{\omega} \cdot (\bar{q} \times d\bar{n})$$

$$= 0$$

$$\therefore \text{ie) } d\left(\frac{q^2}{2} + \frac{p}{\rho} + \phi_f\right) = 0 \quad (\text{irrotational flow})$$

$$\Rightarrow \frac{q^2}{2} + \frac{p}{\rho} + \phi_f = C_s$$

where C_s is the constant along the stream line.

Case ii) :- [Irrotational flow] ^{2M/15}

For an irrotational flow $\vec{\omega} = 0 \Rightarrow \vec{q} = \nabla \phi$

Then (4) becomes

$$\frac{\partial \vec{q}}{\partial t} = \frac{\partial}{\partial t}(\nabla \phi) = -\nabla\left(\frac{q^2}{2} + \phi_f + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \phi_f + \frac{p}{\rho}\right) = 0$$

Integrating,

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \phi_f + \frac{p}{\rho} = F(t) \quad \text{--- (5)}$$

where $F(t)$ is an arbitrary function of time.

In ^{$\partial \phi / \partial t = 0$} steady irrotational flow the equation

(5) reduces to [Bernoulli's equation for]

incompressible fluid ^{2M/15}

$$\text{ie) } \frac{q^2}{2} + \phi_f + \frac{p}{\rho} = C$$

The Momentum Theorem

10M/15

From the law of conservation of momentum $\vec{F} = m \vec{a}$

$$\underbrace{\iiint_V \rho \frac{d\vec{q}}{dt} d\tau}_{\vec{F}} = \vec{F}_S + \vec{F}_B \quad \text{--- (1)}$$

where F_S and F_B are the resultant's of surface and body forces respectively. The term on the LHS represents the rate of change of linear momentum and its integrand can be expressed as follows:

$$\rho \frac{d\bar{q}}{dt} = \rho \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \bar{q}$$

$$= \rho \left(\frac{\partial \bar{q}}{\partial t} + u \frac{\partial \bar{q}}{\partial x} + v \frac{\partial \bar{q}}{\partial y} + w \frac{\partial \bar{q}}{\partial z} \right) \quad \text{--- (2)}$$

consider

$$\frac{\partial}{\partial x} (u\bar{q}) + \frac{\partial}{\partial y} (v\bar{q}) + \frac{\partial}{\partial z} (w\bar{q})$$

$$= \left(u \frac{\partial \bar{q}}{\partial x} + \bar{q} \frac{\partial u}{\partial x} \right) + \left(v \frac{\partial \bar{q}}{\partial y} + \bar{q} \frac{\partial v}{\partial y} \right) + \left(w \frac{\partial \bar{q}}{\partial z} + \bar{q} \frac{\partial w}{\partial z} \right)$$

$$\Rightarrow \frac{\partial}{\partial x} (u\bar{q}) + \frac{\partial}{\partial y} (v\bar{q}) + \frac{\partial}{\partial z} (w\bar{q}) - \bar{q} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

$$= u \frac{\partial \bar{q}}{\partial x} + v \frac{\partial \bar{q}}{\partial y} + w \frac{\partial \bar{q}}{\partial z} \quad \text{--- (3)} \quad \hookrightarrow \nabla \cdot \bar{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Substituting (3) in (2)

$$\rho \frac{d\bar{q}}{dt} = \rho \left[\frac{\partial \bar{q}}{\partial t} + \frac{\partial}{\partial x} (u\bar{q}) + \frac{\partial}{\partial y} (v\bar{q}) + \frac{\partial}{\partial z} (w\bar{q}) - \bar{q} (\nabla \cdot \bar{q}) \right]$$

For an incompressible fluid $\nabla \cdot \bar{q} = 0$

$$\rho \frac{d\bar{q}}{dt} = \rho \left\{ \frac{\partial \bar{q}}{\partial t} + \frac{\partial}{\partial x} [u(\hat{u} + v\hat{j} + w\hat{k})] + \frac{\partial}{\partial y} [v(u\hat{i} + v\hat{j} + w\hat{k})] + \frac{\partial}{\partial z} [w(u\hat{i} + v\hat{j} + w\hat{k})] \right\}$$

$$\rho \frac{d\bar{q}}{dt} = \rho \left\{ \frac{\partial \bar{q}}{\partial t} + \hat{i} \left[\frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial y} (uv) + \frac{\partial}{\partial z} (wu) \right] + \hat{j} \left[\frac{\partial}{\partial x} (uv) + \frac{\partial}{\partial y} (v^2) + \frac{\partial}{\partial z} (vw) \right] + \hat{k} \left[\frac{\partial}{\partial x} (uw) + \frac{\partial}{\partial y} (vw) + \frac{\partial}{\partial z} (w^2) \right] \right\} \quad \text{--- (4)}$$

Consider

$$\hat{i} [\nabla \cdot u\bar{q}] = \hat{i} \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (u\hat{i} + v\hat{j} + w\hat{k})$$

$$= \hat{i} \left[\frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial y} (uv) + \frac{\partial}{\partial z} (uw) \right]$$

$$\text{(4) can be written as}$$

$$\rho \frac{d\bar{q}}{dt} = \rho \left[\frac{\partial \bar{q}}{\partial t} + \hat{i} (\nabla \cdot u\bar{q}) + \hat{j} (\nabla \cdot v\bar{q}) + \hat{k} (\nabla \cdot w\bar{q}) \right]$$

taking triple integral over $d\tau$ on both sides

$$\iiint_{\tau} \rho \frac{d\bar{q}}{dt} d\tau = \iiint_{\tau} \rho \frac{\partial \bar{q}}{\partial t} d\tau + \iiint_{\tau} \rho [\hat{i}(\nabla \cdot u \bar{q}) + \hat{j}(\nabla \cdot v \bar{q}) + \hat{k}(\nabla \cdot w \bar{q})]$$

[By Gauss Divergence Theorem

$$\iint_A \bar{F} \cdot \hat{n} dA = \iiint_V \nabla \cdot \bar{F} dv]$$

According to Gauss Divergence Theorem

$$\begin{aligned} \iiint_{\tau} \rho \frac{d\bar{q}}{dt} d\tau &= \iiint_{\tau} \rho \frac{\partial \bar{q}}{\partial t} d\tau + \iint_A \rho (\hat{i}u\bar{q} + v\bar{q}\hat{j} + \hat{k}w\bar{q}) \cdot \hat{n} dA \\ &= \iiint_{\tau} \rho \frac{d\bar{q}}{dt} d\tau + \iint_A \rho (u\hat{i} + v\hat{j} + w\hat{k}) (\bar{q} \cdot \hat{n}) dA \\ &= \iiint_{\tau} \rho \frac{d\bar{q}}{dt} d\tau + \iint_A \rho \bar{q} (\bar{q} \cdot \hat{n}) dA \end{aligned} \quad (5)$$

where $\hat{n}$ is the unit outward normal

Denoting the direction cosines of the normal to dA by (l, m, n)

$$\begin{aligned} (\bar{q} \cdot \hat{n}) dA &= [(u\hat{i} + v\hat{j} + w\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k})] dA \\ &= (ul + vm + wn) dA \\ &= dQ \end{aligned}$$

where dQ represents volumetric flow rate passing through the elementary surface dA .

Then the momentum equation becomes

$$\iiint_{\tau} \rho \frac{d\bar{q}}{dt} d\tau + \iint \rho \bar{q} dQ = F_s + F_B$$

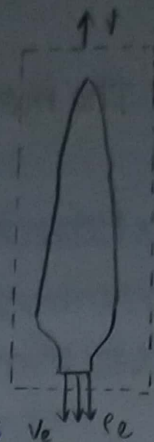
Example

1. Consider a vertical flight of a rocket in a frictionless medium. If the rocket motor is assumed to operate under steady condition so that chamber pressure and the effective exhaust velocity do not change with respect to time. Calculate the velocity of the rocket at any instant time t with reference to a fixed reference.

Solution:-

Let m_0 be the total mass of the rocket system at the beginning of launching.

The mass of the rocket system at any time t after firing $m = m_0 - \alpha t$ where α is the mass of the fuel consumed per unit time.



If A_e, ρ_e, p_e, v_e are chosen to denote the cross sectional area, the mass density of the discharged gas, the discharge pressure and the velocity at the exit, then $\alpha = \rho_e A_e v_e$ — (2)

Since the rocket operates under steady condition the momentum theorem gives

$$m \frac{dv}{dt} = (\rho_e A_e v_e) v_e + p_e A_e - mg \quad (3)$$

$$m \frac{dv}{dt} = \alpha v_e + p_e A_e - mg$$

Substituting (2) in (3)

$$(m_0 - \alpha t) \frac{dv}{dt} = \alpha v_e + p_e A_e - (m_0 - \alpha t)g$$

$$\frac{dv}{dt} = \frac{\alpha v_e + p_e A_e}{m_0 - \alpha t} - g$$

$$dv = \frac{\alpha v_e + p_e A_e}{m_0 - \alpha t} dt - g dt$$

Integrating on both sides between the limits 0 to t

$$v = \left(\frac{\alpha v_e + p_e A_e}{\alpha} \right) \ln \left(\frac{m_0}{m_0 - \alpha t} \right) - gt$$

It represents the velocity of the rocket at any time t after launching.

t	0	t
u	m_0	$m_0 - \alpha t$

$$u = m_0 - \alpha t$$

$$du = -\alpha dt$$

$$dt = -\frac{1}{\alpha} du$$

$$\frac{1}{\alpha} \int \frac{du}{u}$$

$$\frac{1}{\alpha} \ln(u)$$

$$\frac{1}{\alpha} \ln \left(\frac{m_0 - \alpha t}{m_0} \right)$$

(10 M)
(5)

+
(10 M) (5)

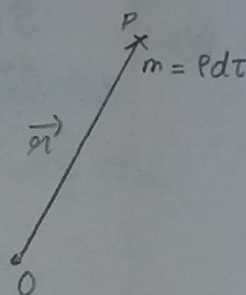
The moment of momentum theorem :-

Newton's law for moment of momentum states that the rate of change of moment of momentum about a point or axis is equal to the moment of the external forces.

Based on this law we can derive the equation of the moment of momentum from the Euler's equation.

Let O be the origin of any reference frame, $\bar{r}$ the position vector of a point P and $\rho d\tau$ the elementary mass at P . The moment of the element of the inertial forces about the origin is $\bar{r} \times \frac{D\bar{q}}{Dt} \rho d\tau$

$$\begin{aligned} &= \left(\bar{r} \times \frac{D\bar{q}}{Dt} + \bar{q} \times \bar{q} \right) \rho d\tau \\ &= \frac{D}{Dt} (\bar{r} \times \bar{q}) \rho d\tau \quad \text{--- (1)} \end{aligned}$$



Hence the rate of change of moment of momentum is

$$\iiint \bar{r} \times \frac{D\bar{q}}{Dt} \rho d\tau = \iiint \frac{D}{Dt} (\bar{r} \times \bar{q}) \rho d\tau \quad \text{--- (2)}$$

The moment of the resultant of the surface and body forces at the origin O as M_s and M_b respectively.

By Newton's law for moment of momentum

$$\iiint \frac{D}{Dt} (\bar{r} \times \bar{q}) \rho d\tau = M_s + M_b \quad \text{--- (3)}$$

$$\frac{D}{Dt} (\bar{r} \times \bar{q}) = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) (\bar{r} \times \bar{q})$$

$$\begin{aligned} &= \frac{\partial}{\partial t} (\bar{r} \times \bar{q}) + u \frac{\partial}{\partial x} (\bar{r} \times \bar{q}) + v \frac{\partial}{\partial y} (\bar{r} \times \bar{q}) \\ &\quad + w \frac{\partial}{\partial z} (\bar{r} \times \bar{q}) \end{aligned}$$

Consider $\frac{\partial}{\partial x} (u \bar{n} \times \bar{q}) = \frac{\partial u}{\partial x} (\bar{n} \times \bar{q}) + u \frac{\partial}{\partial x} (\bar{n} \times \bar{q})$

$$u \frac{\partial}{\partial x} (\bar{n} \times \bar{q}) = \frac{\partial}{\partial x} (u \bar{n} \times \bar{q}) - \frac{\partial u}{\partial x} (\bar{n} \times \bar{q})$$

Similarly $v \frac{\partial}{\partial y} (\bar{n} \times \bar{q}) = \frac{\partial}{\partial y} (v \bar{n} \times \bar{q}) - \frac{\partial v}{\partial y} (\bar{n} \times \bar{q})$

$$w \frac{\partial}{\partial z} (\bar{n} \times \bar{q}) = \frac{\partial}{\partial z} (w \bar{n} \times \bar{q}) - \frac{\partial w}{\partial z} (\bar{n} \times \bar{q})$$

$$\therefore \frac{D}{Dt} (\bar{n} \times \bar{q}) = \frac{\partial}{\partial t} (\bar{n} \times \bar{q}) - (\bar{n} \times \bar{q}) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{\partial}{\partial x} (u \bar{n} \times \bar{q}) + \frac{\partial}{\partial y} (v \bar{n} \times \bar{q}) + \frac{\partial}{\partial z} (w \bar{n} \times \bar{q})$$

For an incompressible fluid $\nabla \cdot \bar{q} = 0$

ie) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Let $\bar{n} \times \bar{q} = \bar{R}$

$$\frac{D\bar{R}}{Dt} = \frac{\partial \bar{R}}{\partial t} + \frac{\partial}{\partial x} (u \bar{R}) + \frac{\partial}{\partial y} (v \bar{R}) + \frac{\partial}{\partial z} (w \bar{R}) \quad \text{--- (4)}$$

where $\bar{R} = R_x \hat{i} + R_y \hat{j} + R_z \hat{k}$

consider

$$\hat{i} [\nabla \cdot (R_x \bar{q})] = \hat{i} \left[\frac{\partial}{\partial x} (R_x u) + \frac{\partial}{\partial y} (R_x v) + \frac{\partial}{\partial z} (R_x w) \right]$$

$$\hat{j} [\nabla \cdot (R_y \bar{q})] = \hat{j} \left[\frac{\partial}{\partial x} (R_y u) + \frac{\partial}{\partial y} (R_y v) + \frac{\partial}{\partial z} (R_y w) \right]$$

$$\hat{k} [\nabla \cdot (R_z \bar{q})] = \hat{k} \left[\frac{\partial}{\partial x} (R_z u) + \frac{\partial}{\partial y} (R_z v) + \frac{\partial}{\partial z} (R_z w) \right]$$

(4) becomes

$$\frac{D\bar{R}}{Dt} = \frac{\partial \bar{R}}{\partial t} + \left[\hat{i} \nabla \cdot (R_x \bar{q}) + \hat{j} \nabla \cdot (R_y \bar{q}) + \hat{k} \nabla \cdot (R_z \bar{q}) \right]$$

30-01-2020
(Thurs)

Multiplying ρ and taking triple integral on both sides

$$\iiint \rho \frac{D}{Dt} (\bar{n} \times \bar{q}) d\tau = \iiint \rho \frac{\partial}{\partial t} (\bar{n} \times \bar{q}) d\tau + \iiint \rho \left[\hat{i} \nabla \cdot (R_x \bar{q}) + \hat{j} \nabla \cdot (R_y \bar{q}) + \hat{k} \nabla \cdot (R_z \bar{q}) \right] d\tau$$

Applying Gauss Divergence Theorem

$$\iiint_V \rho \frac{D}{Dt} (\bar{r} \times \bar{q}) d\tau = \iiint_V \rho \frac{\partial}{\partial t} (\bar{r} \times \bar{q}) d\tau + \iint_A \rho (\hat{i} R_x + \hat{j} R_y + \hat{k} R_z) (\bar{q} \cdot \hat{n}) dA$$

$$= \iiint_V \rho \frac{\partial}{\partial t} (\bar{r} \times \bar{q}) d\tau + \iint_A \rho (\bar{r} \times \bar{q}) d\bar{q}$$

Then the equation of moment of momentum becomes

$$\iiint_V \rho \frac{\partial}{\partial t} (\bar{r} \times \bar{q}) d\tau + \iint_A \rho (\bar{r} \times \bar{q}) d\bar{q} = M_S + M_B$$

This is the Moment of momentum equation.

Example :-

1. A lawn sprinkler which is held in a stationary position while water is discharging at the exit pipe area A is shown in the following figure

Assuming the discharge velocity is constant and R is much larger than the pipe diameter. Calculate the torque of the rotor
Solution :-

The velocity vector relative to the non-rotating frame is

$$\bar{q} = \hat{i}_z v \cos \theta$$

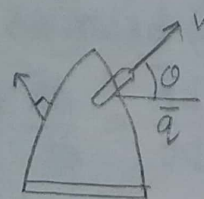
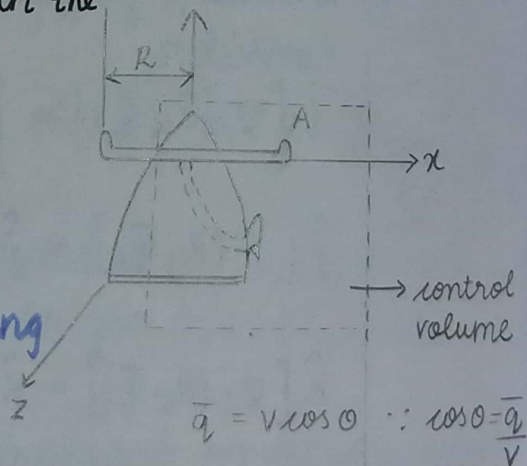
The position vector

$$\bar{r} = \hat{i}_x R$$

$$\bar{r} \times \bar{q} = \hat{i}_x R \times \hat{i}_z v \cos \theta$$

$$= \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ R & 0 & 0 \\ 0 & 0 & v \cos \theta \end{vmatrix} = -\hat{i}_y R v \cos \theta$$

$\therefore \bar{r} \times \bar{q}$ is constant and $M_B = 0$. Then the



1005-10-08
(contd.)

equation of moment of momentum becomes

$$\iint \rho (\vec{n} \times \vec{q}) dQ = M_s$$

$$\text{Torque } T = \rho (\vec{n} \times \vec{q}) \iint dQ$$

$$= \rho (\vec{n} \times \vec{q}) Q \quad \text{--- (1)}$$

$$\because \text{Area of circle} = 2\pi r$$

$$\text{Volumetric flow rate } Q = 2VA$$

$$\text{(1) becomes } T = \rho [-\hat{i}_y R V \cos \theta] 2VA$$

Then the torque T of the rotor is

$$T = -2\rho V^2 A R \cos \theta$$