

(2M)
2M(S)
+
2M(S)

[Laplace equation in cartesian co-ordinates :-]

If we substitute the gradient of velocity potential for the velocity vector $[\bar{q} = \nabla\phi]$ into the equation of continuity, the following second order PDE is obtained $\nabla^2\phi = 0$

$$\text{i.e.) } \nabla \cdot \bar{q} = 0$$

$$\nabla \cdot (\nabla\phi) = 0$$

$$\nabla^2\phi = 0$$

In cartesian co-ordinates

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0$$

This equation is known as Laplace equation and its solution ϕ is called harmonic function.

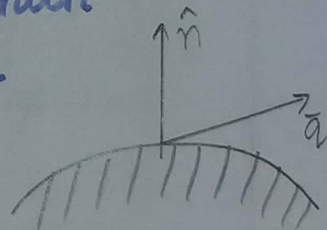
5M [Boundary conditions:-]

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i) [When the fluid is in contact with the solid surface at rest] or another unmixed fluid the boundary conditions which satisfies the Laplace equation

are

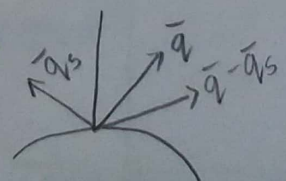
$$\hat{n} \cdot \bar{q} = \frac{\partial \phi}{\partial n} = 0$$



ii) The pressure shall be continuous at the common boundary of fluids] 2M

Condition i) is a kinematical relation which means that the fluid and the surface with which contact is taking place must have the same normal velocity.

If $\bar{q}$ and $\bar{q}_s$ are the velocities of the fluid and moving solid



$$\hat{n} \cdot \bar{q} = \hat{n} \cdot \bar{q}_s$$

$$\hat{n} \cdot (\bar{q} - \bar{q}_s) = 0$$

Hence the velocity of the fluid is tangential to the surface for an inviscid fluid.

Condition ii) If p_1, p_2 are the pressures in the two fluids

$$\hat{n} p_1 dA = \hat{n} p_2 dA$$

$$(\text{or}) p_1 = p_2 \quad 5M$$

Problem :-

1. Show that the velocity potential

$$\phi = \frac{a}{2} (x^2 + y^2 - 2z^2)$$

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5M(S)
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satisfies the Laplace equation and represents the flow against a fixed plane wall

Solution :-

$$\frac{\partial^2 \phi}{\partial x^2} = a \quad \frac{\partial^2 \phi}{\partial y^2} = a \quad \frac{\partial^2 \phi}{\partial z^2} = -2a$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = a + a - 2a = 0$$

Hence the Laplace equation is satisfied.

The stream lines in the yz -plane is given by the differential equation

$$\frac{dy}{v} = \frac{dz}{w} \quad (\text{by definition of stream line})$$

$$\bar{q} = \nabla \phi \quad u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}$$

$$v = ay \quad w = -2az$$

$$\Rightarrow \frac{dy}{ay} = \frac{dz}{-2az}$$

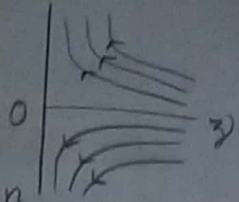
$$-2 \frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides

$$-2 \ln y = \ln z + \ln c$$

$$\ln y^2 + \ln z = \ln c$$

$$y^2 z = c$$



This equation is called a cubic hyperbola. The result of this solution is the flow against a fixed plane wall is shown in this figure.

2M(5) [Laplace equation in cylindrical co-ordinates :-
 $\bar{q} = \nabla \phi$

$$\text{ie) } (v_r, v_\theta, v_z) = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right)$$

$$\Rightarrow v_r = \frac{\partial \phi}{\partial r} \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad v_z = \frac{\partial \phi}{\partial z}$$

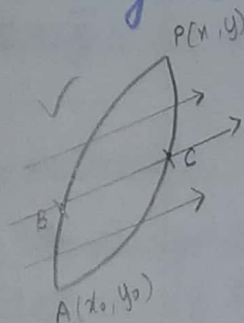
$$\text{Laplace equation is } \nabla^2 \phi = \nabla \cdot \bar{q} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad] \quad 2M(5)$$

2M(5) + 2M(5) Stream function in 2D motion :-

Let $A(x_0, y_0)$ be a fixed point in xy -plane which has unit thickness in the z -direction and $p(x, y)$ be an arbitrary point in the same plane. Two arbitrary paths ABP and ACP are drawn between points A and P.

By the definition of continuity there is no flow created or destroyed within the region bounded by these curves.

Hence the rate of fluid flow entering the region across the curve ABP is equal to the flow rate across the curve ACP.



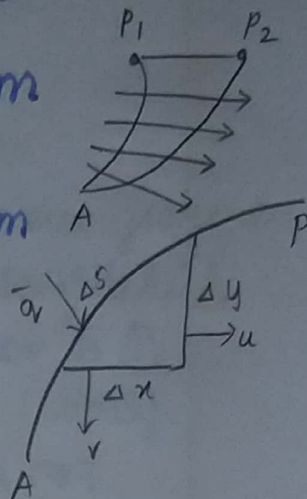
[The rate of flow is called flux].
 2M [The flux across the curve ABP is equal to the flux across any curve joining A to P. Since the point A is fixed the flux is a function of

position of P and time t . If the flux is denoted by ψ . ψ is a function of position of P and time t

$$\psi = \psi(x, y, t)$$

ψ is called the stream function and has ^{2M} units of volume for unit length ~~per~~ unit time]

Consider two points P_1 and P_2 and two curves drawn from them to a fixed point A . Let ψ_1 and ψ_2 represents the values of the stream function at points P_1 and P_2 .



Then the flux across AP_2 is equal to the flux across AP_1 plus that across the curve P_1P_2 . Hence the flux across the curve P_1P_2 is $\psi_2 - \psi_1$. When the points P_1 and P_2 are the points of the same stream line then the $P_1P_2 = \psi_2 - \psi_1 = \Delta\psi$

$$\Delta\psi = 0$$

Since by the definition of stream line there is no flow across that line.

In cartesian co-ordinates the velocity components in the x and y direction are found from the following consideration.

Let Δs be the infinitesimal length of the curve AP whose components in the x and y directions are Δx and Δy respectively.

The flux across the curve AP is

$$\psi = \int_{AP} \vec{q} \cdot \hat{n} ds$$

$$\psi = \int_{AP} (u dy - v dx)$$

or the flux across the arc Δs is

$$d\psi = u dy - v dx \quad \left[\text{total differential } d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \right]$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = u dy - v dx$$

$$\Rightarrow \left[u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \right] \text{---(1)}$$

This is the relation between stream function and velocity component.

$$\begin{aligned} \nabla \cdot \bar{q} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \\ &= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \\ &= 0 \end{aligned}$$

Hence the equation of continuity is satisfied.
① is valid for both rotational and irrotational flow.

For irrotational flow, the Laplace equation for ψ is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\begin{aligned} \bar{q} &= \nabla \cdot \phi \\ u\hat{i} + v\hat{j} &= \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} \end{aligned}$$

$$\Rightarrow u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \text{---(2)}$$

From ① and ②

$$\left[\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \right]$$

This is the relation between the velocity potential and stream function. In cartesian co-ordinates

$$\frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$\frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

(In polar co-ordinates

$$\text{Hence } v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = -\frac{\partial \psi}{\partial r}$$

2M(5)

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

v_r, v_θ are the velocity components in polar co-ordinates.) 2M(5) ↓

∴ Stream function ψ satisfies Laplace equation in polar co-ordinates - 2M(4)

Example :-

1. show that the velocity vector $\vec{q}$ is everywhere tangent to lines in the xy plane along which $\psi(x, y) = \text{constant}$

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Solution :-

Since $\psi(x, y) = \text{constant}$

$$d\psi = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \quad \text{--- (1)}$$

⇒ Relation between the velocity components u and v and stream function ψ

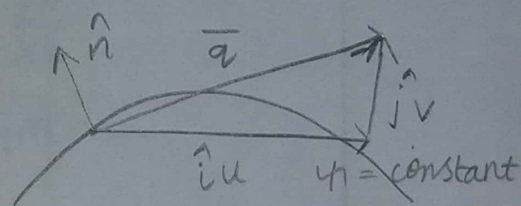
$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \quad \text{--- (2)}$$

Substituting (2) in (1)

$$-v dx + u dy = 0$$

$$-v dx = -u dy$$

$$\frac{dx}{u} = \frac{dy}{v}$$



This is of the form of equation of stream line. By the definition of stream line, the velocity vector $\vec{q}$ is tangent to the line $\psi(x, y) = \text{constant}$.

Note :-

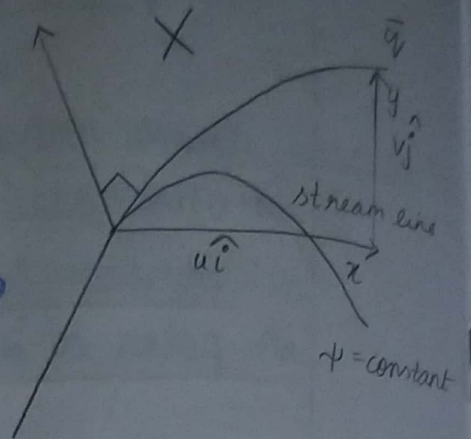
$\hat{n} \cdot \vec{q} = 0$ where $\hat{n}$ is normal to the direction of the stream line

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The Flow Net :-

Stream lines have been defined as lines along which ψ is constant.

Similarly ^{lines} along which ϕ is constant are called potential lines.



We have to show that stream lines and potential lines are orthogonal.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Stream lines $\psi(x, y) = \text{constant}$

$$d\psi = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$$

$$-v dx + u dy = 0$$

$$-v dx = -u dy$$

$$\frac{dy}{dx} = \frac{v}{u}$$

$$m_1 = \frac{v}{u}$$

Potential lines $\phi(x, y) = \text{constant}$

$$d\phi = 0$$

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

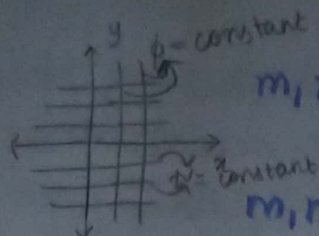
$$u dx + v dy = 0$$

$$u dx = -v dy$$

$$\frac{dy}{dx} = -\frac{u}{v}$$

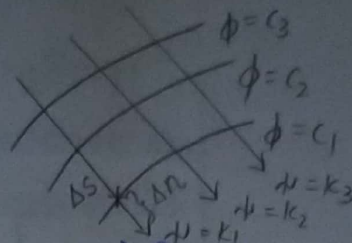
$$m_2 = -\frac{u}{v}$$

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$



$$m_1 m_2 = \frac{v}{u} \times -\frac{u}{v}$$

$$m_1 m_2 = -1$$



Hence stream lines and potential lines are orthogonal.

Since these slopes are negative reciprocals the lines are perpendicular to one another

Thus the potential lines and stream lines form a orthogonal set of lines, which completely describes the flow in a 2D incompressible irrotational flow field.

Such a net is called flow net

10M(15) [Stream function in 3D motion :-] 10M(5)

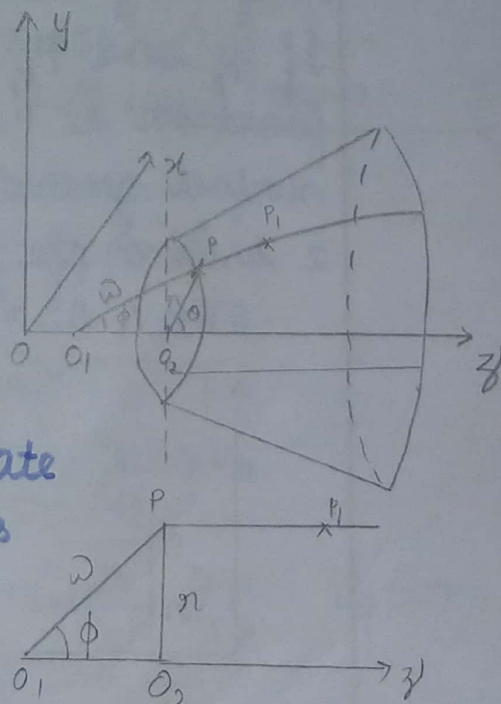
Axially symmetric flow:

The ^{fluid} motion is same in every plane containing a certain axis. ^{This type of flow} is called axially symmetric flow.

The stream function defined only for 3D flow with axial symmetry is called Stoke's stream function

Let z be the axis of symmetry and O_1 be the fixed point on the axis of symmetry. Consider an arbitrary point P lying on the z - r -plane and connected to O_1 by an arbitrary curve. If we rotate the curve $O_1 P$ about the axis of symmetry a surface of revolution is generated.

The flow through the surface of revolution is a function of position of P .



If the flux across the surface of revolution is denoted by $2\pi\psi$, the function ψ at a given time depends on position of P.

$$\psi = \psi(z, r)$$

When the path from O_1 to P is changed the value of ψ is unaltered. (because depends on position of P)

$\therefore$ The flow across O_1P is same as that across O_2P because there is no flow across the axis of symmetry O_1O_2 . Hence when the point P lies on the axis the stream function is zero.

If there is no flux across the curve joining P and P_1 , the value of the stream function at P is equal to that at the point P_1 .

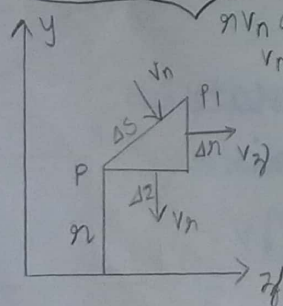
ie) PP_1 is a stream line.

(The surface generated by rotating the stream line PP_1 about the axis of symmetry is called stream tube (or) stream surface.)

5M [To find the relation between ψ and velocity components :-

Let us consider the flow across an infinitesimal surface of revolution in cylindrical co-ordinates. If ψ_p and ψ_{p_1} denote the values of the stream function at P and P_1 , the flux across the surface generated by the revolution about the z-axis of the line PP_1 is

$$2\pi r v_n \Delta S = 2\pi \Delta \psi = 2\pi(\psi_{p_1} - \psi_p)$$



$$r v_n ds = d\psi$$

$$v_n = \frac{1}{r} \frac{\partial \psi}{\partial s}$$

$$= 2\pi(v_z \Delta r - v_r \Delta z)r$$

$$\Rightarrow \Delta \psi = (v_z \Delta r - v_r \Delta z)r$$

Taking the limit $\Delta \psi \rightarrow 0$
 $\Delta r \rightarrow 0$ $\Delta z \rightarrow 0$

$$\lim_{\Delta \psi \rightarrow 0} \Delta \psi = d\psi = r(v_z dr - v_r dz)$$

$$d\psi = r v_z dr - r v_r dz$$

By the definition of total derivative

L ①

$$d\psi = \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial z} dz \quad \text{--- (2)}$$

From (1) and (2)

$$\frac{\partial \psi}{\partial r} = r v_\theta$$

$$\frac{\partial \psi}{\partial z} = -r v_r$$

$$\Rightarrow v_\theta = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

$$\Rightarrow v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}$$

$$v_\phi = 0$$

$$v_r = \frac{1}{r} \frac{\partial \psi}{\partial z}$$

Relation between stream function and velocity potential components in cylindrical co-ordinates :-

$$\therefore v_r = \frac{\partial \phi}{\partial r}$$

$$v_\theta = \frac{\partial \phi}{\partial z}$$

$$v_\phi = 0$$

$$\boxed{\frac{\partial \phi}{\partial r} = -\frac{1}{r} \frac{\partial \psi}{\partial z}}$$

$$\boxed{\frac{\partial \phi}{\partial z} = \frac{1}{r} \frac{\partial \psi}{\partial r}}$$

In spherical co-ordinates

The flux across the surface of revolution along

$$PP_1 \quad 2\pi r \Delta\psi = 2\pi r \sin\theta$$

$$[-v_\theta \Delta r - v_r (-r \Delta\theta)]$$

$$d\psi = -r \sin\theta [v_\theta dr - v_r r d\theta]$$

By the definition of total differential

$$d\psi = \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial \phi} d\phi \quad \text{--- (4)}$$

From (3) and (4)

$$\frac{\partial \psi}{\partial r} = -r \sin\theta v_\theta$$

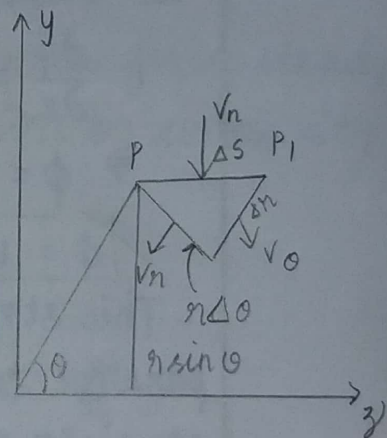
$$\frac{\partial \psi}{\partial \theta} = r^2 \sin\theta v_r$$

$$\Rightarrow v_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}$$

$$v_r = \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial \theta}$$

$$v_\phi = 0$$

Relation between velocity potential and stream function :-



$$v_r = \frac{\partial \phi}{\partial r}$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

$$\frac{\partial \phi}{\partial r} = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}$$

$$\frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{-1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

$$\frac{\partial \phi}{\partial \theta} = \frac{-1}{\sin \theta} \frac{\partial \psi}{\partial r}$$

10 M/s

10 M/s

i)

2D flow :- Examples :-

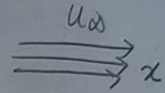
A rectilinear flow :-

5 M/s

5 M/s

If a uniform flow is given parallel to the positive direction of the x-axis, the expressions for velocity components, velocity potential and stream function

$$u = u_\infty = \text{constant} \quad \text{(uniform)}$$



Relation between velocity components, velocity potential, stream function

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

ie) Substituting ① in ②

$$\frac{\partial \phi}{\partial x} = u_\infty$$

$$\frac{\partial \psi}{\partial y} = u_\infty$$

$$\Rightarrow \phi = u_\infty x$$

$$\psi = u_\infty y$$

u_∞ - constant

$$\phi = u_\infty r \cos \theta$$

$$\psi = u_\infty r \sin \theta$$

This streamlines are all straight lines parallel to x axis and potential lines are straight line parallel to y axis.

Similarly the expressions for velocity components, velocity potential and stream function can be obtained for a uniform flow parallel to the positive direction of y axis as follows :

$$u = 0 \quad v = v_\infty = \text{constant} \quad \text{--- ①}$$

Relation between velocity component, velocity potential and stream function :-

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad \text{--- ②}$$

Substituting ① in ②

$$\frac{\partial \phi}{\partial y} = V_{\phi}$$

$$\phi = V_{\phi} y$$

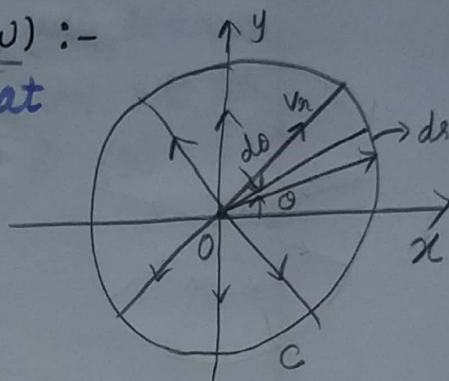
$$-\frac{\partial \psi}{\partial x} = V_{\phi}$$

$$\psi = -V_{\phi} x$$

The stream lines in this case are straight lines parallel to y axis and potential lines are straight lines parallel to x axis.] 5 M(S)

10 M(S) 2 M(S) Source and sink (Radial flow) :-

In a 2D flow a source at a point O is conceived as a point from which fluid emanates in equal amounts along radial paths.] 2 M



Hence the lines of flow will be straight radial lines. $V_{\theta} = 0$, $V_r = V_r(r)$.

From the conservation of mass for a steady incompressible flow, the equation of continuity

$$\nabla \cdot \vec{q} = 0$$

$$\frac{\partial V_r}{\partial r} + \frac{1}{r} \frac{\partial V_{\theta}}{\partial \theta} + \frac{\partial V_z}{\partial z} + \frac{V_r}{r} = 0$$

Since the flow is in 2D $V_z = 0$ $V_{\theta} = 0$

$$\frac{dV_r}{dr} + \frac{V_r}{r} = 0$$

$$\frac{dV_r}{dr} = -\frac{V_r}{r}$$

$$\frac{dV_r}{V_r} = -\frac{dr}{r}$$

Integrating on both sides

$$\ln V_r = -\ln r + \ln B$$

$$\ln V_r + \ln r = \ln B$$

$$\ln (V_r r) = \ln B$$

Taking exponential on both sides

$$v_n r = B$$

$$v_n = \frac{B}{r}$$

where B is a constant to be determined

To find the constant 'B' :-

Let Q be the quantity of fluid per second flowing out through curve 'C' surrounding the source. Then Q must be equal to the total flow across any circle of radius r with its centre at the source

$$\begin{aligned} \text{ie) } Q &= \int_C \vec{q} \cdot \hat{n} \, ds = \int_0^{2\pi} v_n r \, d\theta \\ &= \int_0^{2\pi} \frac{B}{r} r \, d\theta \\ &= B(2\pi) \end{aligned}$$

$$B = \frac{Q}{2\pi}$$

Substituting B in v_n

$$v_n = \frac{Q}{2\pi r}, \quad v_\theta = 0 \quad \left] \begin{array}{l} \text{source} \\ 5M \end{array} \right.$$

Sink :-

$2M$ [A sink is simply the reverse of the source. It is a point into which fluid flows radially in equal amounts from all directions in the field] $2M$

Algebraically a sink is represented as a negative of the source.)

The stream function is found by the direct integration of the velocity components v_n and v_θ

v_θ

$$v_n = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad v_n = \frac{Q}{2\pi r}$$

$$\Rightarrow \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r}$$

$$\partial \psi = \frac{Q}{(2\pi)} \partial \theta$$

$$\psi = \left(\frac{Q}{2\pi}\right)\theta + f(r) \quad \text{--- (1)}$$

$$v_\theta = 0 \quad v_r = -\frac{\partial \psi}{\partial r}$$

$$\frac{\partial \psi}{\partial r} = 0$$

$$\psi = f(\theta) \quad \text{--- (2)}$$

From (1) and (2)

$$\left(\frac{Q}{2\pi}\right)\theta + f(r) = f(\theta)$$

$$\Rightarrow f(\theta) = \left(\frac{Q}{2\pi}\right)\theta \quad f(r) = 0$$

Substituting $f(r) = 0$ in (1)

$$\psi = \left(\frac{Q}{2\pi}\right)\theta$$

$$v_r = \frac{Q}{2\pi r} \quad v_\theta = 0 \quad \text{--- (*)}$$

We have to find the velocity potential ϕ .
From the irrotational vorticity equation $\bar{q} = \nabla \phi$

$$-r_z = \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r} \quad \text{--- (3)}$$

Substituting v_r, v_θ in (3)

$$-r_z = \frac{\partial}{\partial r} (0) - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{Q}{2\pi r}\right) + 0$$

$$-r_z = 0$$

$\therefore$ The velocity potential exist in the case of source and sink flows.

Relation between velocity components and velocity potential :-

$$v_r = \frac{\partial \phi}{\partial r} \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad \text{--- (4)}$$

$\therefore$ By (*)

$$(v_r, v_\theta) = \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}\right) \phi$$

Substituting v_r and v_θ in (4)

$$\frac{\partial \phi}{\partial r} = \frac{Q}{2\pi r} \quad \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$$

Integrating partially
with respect to 'r'

$$\phi = \left(\frac{Q}{2\pi}\right) \ln r + g(\theta) \quad (5)$$

Integrating partially
with respect to θ

$$\phi = g(r) \quad (6)$$

From (5) and (6)

$$\left(\frac{Q}{2\pi}\right) \ln r + g(\theta) = g(r)$$

$$g(r) = \left(\frac{Q}{2\pi}\right) \ln r$$

$$g(\theta) = 0$$

Substituting $g(\theta) = 0$ in (5)

$$\phi = \left(\frac{Q}{2\pi}\right) \ln r + \text{constant}$$

where $Q > 0$ for a source and $Q < 0$ for a sink.

10M(15)

(5M)

Vortex flow :- 2M(5)

5M(5)
+
5M(5)
Laplace
equation
satisfy?
φ and ψ
+
5M(5)
5M(5)

Consider the fluid motion in concentric circles and the flow is irrotational. A circulatory flow free from rotation surrounds a core which acts like a rotating solid body.

Since the flow is irrotational ^{with} circular stream lines $v_r = 0$ $v_\theta = v_\theta(r)$

It must satisfy vorticity equation.

$$\Omega_z = \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}$$

$$\Omega_z = 0$$

$$\therefore \Omega_z = \frac{dv_\theta}{dr} + \frac{v_\theta}{r} = 0$$

$$\Rightarrow \frac{dv_\theta}{dr} = -\frac{v_\theta}{r}$$

$$\frac{dv_\theta}{v_\theta} = -\frac{dr}{r}$$

Integrating on both sides

$$\ln(v_0) = -\ln(r) + \ln(c)$$

$$(v_0 r) = c$$

$$\Rightarrow v_0 = \frac{c}{r} \text{ --- (1)}$$

where c is a constant to be determined

To find c :-

$$\Gamma = \oint_C \vec{q} \cdot \hat{n} ds$$

$$= \int_0^{2\pi} v_0 \cdot r d\theta$$

$$= r v_0 (2\pi)$$

Substituting v_0 $\Gamma = \frac{c}{r} (r) (2\pi)$

$$\Gamma = 2\pi c$$

$$c = \frac{\Gamma}{2\pi}$$

Substituting c in (1)

$$v_0 = \frac{\Gamma}{2\pi r} \quad v_r = 0$$

(2M) To find :- velocity potential ϕ
Relation between velocity components and stream function :-

The velocity potential ϕ is

$$\phi = \int v_\theta r d\theta + \text{constant}$$

$$\left(\because v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad \frac{\partial \phi}{\partial \theta} = r v_\theta \right)$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{1}{r} r v_\theta$$

$$\Rightarrow \phi = \int r v_\theta d\theta + \text{constant}$$

$$= \int \frac{\Gamma}{2\pi r} (r) d\theta + \text{constant}$$

$$\checkmark \phi = \left(\frac{\Gamma}{2\pi} \right) \theta + \text{constant.}$$

(2M/5)

To find :- ψ (stream function)
(2m) Relation between ψ and

$$v_0 = -\frac{\partial \psi}{\partial r}$$

$$v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_r = 0 \quad \frac{\partial \psi}{\partial \theta} = 0$$

$$\Rightarrow \psi = f(r)$$

$$\frac{\partial \psi}{\partial r} = -v_0$$

$$\partial \psi = -v_0 \partial r$$

$$\Rightarrow \psi = -\int v_0 dr$$

$$= -\frac{v_0}{2\pi} \int \frac{dr}{r}$$

$$\checkmark \psi = -\frac{v_0}{2\pi} (\ln r) + \text{constant.}$$

2m(s)

5m(s)

$$\left[\phi = \left(\frac{\Gamma}{2\pi} \right) \theta + \text{constant} \right]$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$+ \frac{1}{r} \frac{\partial \phi}{\partial r}$$

$$= 0$$

$$\psi = -\frac{\Gamma}{2\pi} \ln(r) + \text{constant}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$$

$$= \frac{\Gamma}{2\pi} \frac{1}{r^2} - \frac{\Gamma}{2\pi} \frac{1}{r^2}$$

$$= 0 \quad \psi \text{ and } \phi \text{ are}$$

harmonic
They satisfy
Laplace
eqn.

5m(s)

07-02-2020 Doublet :-

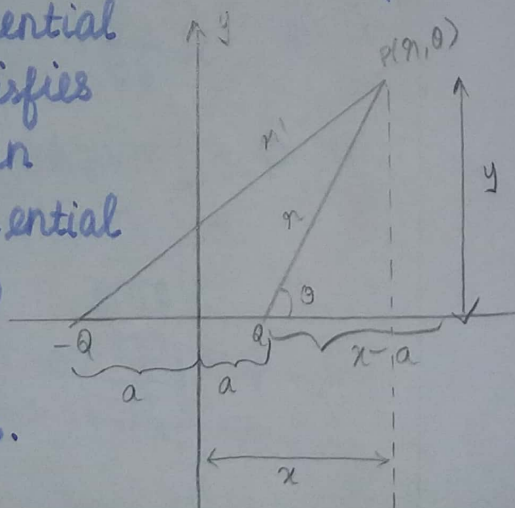
(10m)

+ 10m(s)

(2D)

Let a source and sink of equal strength Q be placed at a distance $2a$ on a path along the x -axis. From any point $P(r, \theta)$ lines r and r' are drawn to the source and sink respectively.

The angles θ and θ' are measured from the positive x axis to lines r and r' respectively. Since the velocity potential for source or sink satisfies Laplace equation, we can obtain the velocity potential for the combined flow by superposition of source and sink flows.



The velocity potential for the source and sink are

$$\phi = \frac{Q}{2\pi} \ln(r) + \text{constant} \quad \text{--- (1)}$$

$$\phi = -\frac{Q}{2\pi} \ln(r') + \text{constant} \quad \text{--- (2)}$$

$\therefore$ Velocity potential for doublet is (1) + (2)

$$\phi = \frac{Q}{2\pi} [\ln(r) - \ln(r')] + \text{constant}$$

$$= -\frac{Q}{2\pi} [\ln(r') - \ln(r)] + \text{constant}$$

$$= -\frac{Q}{2\pi} \ln\left(\frac{r'}{r}\right) + \text{constant}$$

By stream function $\psi = \left(\frac{Q}{2\pi}\right) \theta$ and $\psi = -\frac{Q}{2\pi} \theta'$
 $\therefore$ Stream function for doublet is (3) + (4)

$$\psi = \frac{Q}{2\pi} (\theta - \theta')$$

$$\Rightarrow \theta - \theta' = \frac{2\pi\psi}{Q} \quad \text{--- (5)}$$

$$\tan(\theta - \theta') = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta \tan\theta'}$$

We have to find streamline counter equation :-

$$\text{We know that } \tan\theta = \frac{y}{x-a}$$

$$\tan\theta' = \frac{y}{x+a}$$

$$\begin{aligned} \tan(\theta - \theta') &= \frac{\left(\frac{y}{x-a}\right) - \left(\frac{y}{x+a}\right)}{1 + \left(\frac{y}{x-a}\right)\left(\frac{y}{x+a}\right)} \\ &= \frac{y \left[\frac{x+a - x-a}{(x+a)(x-a)} \right]}{\frac{(x-a)(x+a) + y^2}{(x+a)(x-a)}} \end{aligned}$$

$$= \frac{2ay}{x^2 + y^2 - a^2}$$

Using (5)

$$\tan\left(\frac{2\pi\psi}{Q}\right) = \frac{2ay}{x^2 + y^2 - a^2}$$

$$\text{let } k = \cot\left(\frac{2\pi\psi}{Q}\right)$$

$$\frac{1}{k} = \frac{2ay}{x^2 + y^2 - a^2}$$

$$x^2 + y^2 - a^2 = 2kay$$

$$x^2 + y^2 - 2akay - a^2 = 0$$

Add and subtract $k^2 a^2$

$$x^2 + (y^2 - 2akay + k^2 a^2) - k^2 a^2 - a^2 = 0$$

$$x^2 + (y - ka)^2 = a^2(1 + k^2) \quad \text{--- (6)}$$

$$(x - a)^2 + (y - b)^2 = r^2$$

$$\text{centre } (0, ka) \quad r = a\sqrt{1+k^2}$$

(6) is the equation of a circle centre at $(0, ka)$ and radius $= a\sqrt{1+k^2}$

Hence the stream lines are co-axial circle passing through Q and $-Q$

2nd (Consider a source and sink of equal strength are allowed to approach each other while the product of their strength and distance apart is held constant.

This combination of source and sink of infinite strength at an infinitesimal distance apart is called doublet of strength μ .) 2nd

To find the equations describing doublet :-

$$\text{Let } x' = x + \Delta x \quad \text{and } Q' = Q - \Delta Q \quad \text{and } 2a = \Delta s$$

Substituting these expressions into velocity potential ϕ of doublet is

$$\phi = -\frac{Q}{2\pi} \ln\left(\frac{r'}{r}\right) + \text{constant}$$

$$= -\frac{Q}{2\pi} \ln\left(\frac{r+\Delta r}{r}\right) + \text{constant}$$

$$= -\frac{Q}{2\pi} \ln\left(1+\frac{\Delta r}{r}\right) + \text{constant}$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$= -\frac{Q}{2\pi} \left[\frac{\Delta r}{r} - \frac{1}{2} \left(\frac{\Delta r}{r} \right)^2 + \dots \right] + \text{constant}$$

$$= -\frac{Q}{2\pi} (\Delta r) + \text{constant} \quad \text{--- (7)}$$

$\therefore \Delta r \rightarrow 0$ neglecting higher order terms

$\therefore \Delta r, \Delta \theta, \Delta S$ are infinitesimal quantity ~~is~~ Substituting

$$r^2 = (\Delta S)^2 + (r+\Delta r)^2 - 2\Delta S(r+\Delta r) \cos(\theta - \Delta \theta)$$

By cosine formula

$$\begin{aligned} r^2 &= \Delta S^2 + (r^2 + \Delta r^2 + 2r\Delta r - (2\Delta S r - 2\Delta^2 S r) \cos(\theta - \Delta \theta)) \\ &= r^2 + \Delta r^2 + 2r\Delta r - 2r\Delta S \cos \theta \end{aligned}$$

$$2r\Delta r = 2r\Delta S \cos \theta \Rightarrow \Delta r = \Delta S \cos \theta$$

Substituting Δr in (7)

$$\phi = -\frac{Q}{2\pi r} (\Delta S \cos \theta) + \text{constant}$$

$$\text{Let } \mu = Q\Delta S \text{ and } \phi = \frac{-\mu \cos \theta}{2\pi r} + \text{constant}$$

It is the velocity potential for the doublet of strength μ .

To find the velocity components :-

$$v_r = \frac{\partial \phi}{\partial r} = \frac{\mu \cos \theta}{2\pi r^2}$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{\mu \sin \theta}{2\pi r^2}$$

Stream function ψ for a doublet :-

$$\begin{aligned} \psi &= -\int v_\theta r dr + \text{constant} \\ &= -\frac{\mu \sin \theta}{2\pi} \int \frac{1}{r^2} dr + \text{constant} \end{aligned}$$

$$\psi = \frac{\mu \sin \theta}{2\pi r} + \text{constant}$$

3D axially Symmetric flow :-

Example :-

a) [uniform flow :-

In a uniform flow u_∞ is given parallel to the positive direction of the z -axis. The expressions for the velocity components are

$$\left. \begin{aligned} v_z &= u_\infty = \text{constant} \\ v_r &= 0 \\ v_\theta &= 0 \end{aligned} \right\} \text{--- (1)}$$

In cylindrical co-ordinates, the relation between velocity components and velocity potential are

$$v_r = \frac{\partial \phi}{\partial r} = 0 \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0 \quad v_z = \frac{\partial \phi}{\partial z} = u_\infty \text{ --- (2)}$$

and the relation between velocity components and the stream function

$$v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \quad v_\theta = 0 \quad v_z = \frac{1}{r} \frac{\partial \psi}{\partial r} = u_\infty \text{ --- (3)}$$

Substituting (3) in (1) and (2)

$$(2) \Rightarrow \frac{\partial \phi}{\partial z} = u_\infty$$

Integrating on both sides,

$$\Rightarrow \phi = u_\infty z + \text{constant}$$

$$z = r \cos \theta$$

$$\phi = u_\infty (r \cos \theta) + \text{constant}$$

$$(3) \Rightarrow \frac{1}{r} \frac{\partial \psi}{\partial r} = u_\infty$$

$$\frac{\partial \psi}{\partial r} = r u_\infty$$

Integrating on both sides

$$\psi = \frac{u_\infty r^2}{2} + \text{constant}$$

$$r = r \sin \theta$$

$$\left. \begin{aligned} r, \theta, z \\ r, \theta, \phi \\ r, \phi, \theta \end{aligned} \right\} \text{ spherical}$$

$$\psi = \frac{u_\infty r^2 \sin^2 \theta}{2} + \text{constant}$$

The velocity components in spherical co-ordinate are

$$v_r = \frac{\partial \psi}{\partial r} = \frac{\partial}{\partial r} (u_\infty r \cos \theta + \text{constant})$$

$$v_r = u_\infty \cos \theta$$

$$v_\theta = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} (u_\infty r \cos \theta + \text{constant})$$

$$= \frac{1}{r} r u_\infty (-\sin \theta)$$

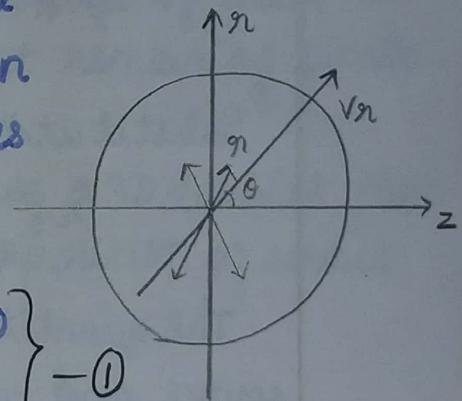
$$= -u_\infty \sin \theta$$

$$v_\phi = 0 \quad] \quad 5M(5)$$

b) Radial flow :-

Source or Sink :-

A simple source in 3D flow is a point from which fluid emanates in equal amounts in all directions. Hence the lines of flow will be straight radial lines.



$$\left. \begin{aligned} v_r &= v_r(r) & v_\theta &= 0 & v_\phi &= 0 \\ & & \frac{\partial}{\partial \theta} &= 0 & & \end{aligned} \right\} \text{--- (1)}$$

From equation of continuity

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{\sin \theta} \frac{\partial v_\phi}{\partial \phi} = 0 \quad \text{--- (2)}$$

Substituting (1) in (2)

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) = 0 \quad \therefore \frac{1}{r^2} \neq 0 \quad \frac{\partial}{\partial r} (r^2 v_r) = 0$$

$$r^2 v_r = K (\text{constant})$$

$$\Rightarrow v_r = \frac{K}{r^2} \quad \text{where } K \text{ is a constant.}$$

Let Q be the quantity of fluid flowing out of the source. By the condition of continuity Q would be equal to quantity of fluid flowing through a sphere of radius R .

To find K :-

$$Q = \iint_A \vec{q} \cdot \hat{n} dA$$

$$= \iint_{\text{sphere}} v_R dA$$

$$= \iint_{\text{sphere}} \frac{K}{R^2} dA$$

$$= \frac{K}{R^2} \iint_{\text{sphere}} dA$$

$$= \frac{K}{R^2} (4\pi R^2)$$

$$Q = 4\pi K$$

$$\Rightarrow K = \frac{Q}{4\pi}$$

Substituting 'K' in v_r

$$v_r = \frac{Q}{4\pi r^2} \text{ is called the radial velocity.}$$

The quantity $+Q$ is called the strength of the source and $-Q$ is the strength of the sink.

To find stream function ψ :- of the source in 3D

$$v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}$$

$$v_r r^2 \sin \theta = \frac{\partial \psi}{\partial \theta}$$

Integrating on both sides with respect to θ

$$\psi = \int v_r r^2 \sin \theta d\theta + \text{constant}$$

$$= \frac{Q}{4\pi r^2} (r^2) \int \sin \theta d\theta + \text{constant}$$

$$\psi = -\frac{Q}{4\pi} \cos \theta + \text{constant}$$

To find the velocity potential ϕ :-

$$v_n = \frac{\partial \phi}{\partial n}$$

$$\frac{Q}{4\pi n^2} = \frac{\partial \phi}{\partial n}$$

$$\partial \phi = \frac{Q}{4\pi n^2} \partial n$$

$$\phi = \frac{Q}{4\pi} \left(\frac{-1}{n} \right) + \text{constant}$$

$$\phi = \frac{-Q}{4\pi n} + \text{constant}$$

where $Q > 0$ for source and $Q < 0$ for sink.

Doublet :- 10 m/s

^{2M} (A doublet is a combination of source and sink of infinite strength at an infinitesimal distance apart with the product of their strength and distance apart remaining finite and constant.) ^{2M}

The velocity potential of a 3D doublet flow can be obtained by the superposition of source and sink flow

$$\phi = \frac{-Q}{4\pi n} + \text{constant}$$

$$\phi = \frac{-Q}{4\pi(n-\Delta n)} - \left[\frac{-Q}{4\pi(n+\Delta n)} \right] + \text{constant}$$

$$= \frac{-Q}{4\pi n} \left[\frac{1}{\left(1 - \frac{\Delta n}{n}\right)} - \frac{1}{\left(1 + \frac{\Delta n}{n}\right)} \right] + \text{constant}$$

$$= \frac{-Q}{4\pi n} \left[\frac{1 + \frac{\Delta n}{n} - 1 + \frac{\Delta n}{n}}{\left(1 - \frac{\Delta n}{n}\right)\left(1 + \frac{\Delta n}{n}\right)} \right] + \text{constant}$$

$$\phi = \frac{-Q}{2\pi n} \left[\frac{2\Delta n}{n} \right] + \text{constant}$$

$$= \frac{-Q}{2\pi n^2} \Delta n + \text{constant}$$

By the Trigonometric relation

$$\Delta n = \frac{\Delta S \cos \theta}{2}$$

$$\phi = \frac{-Q}{2\pi n^2} \left(\frac{\Delta S \cos \theta}{2} \right) + \text{constant}$$

$Q \rightarrow \infty$
 $\Delta S \rightarrow 0$

Taking the limit

$$\phi = -\lim_{\substack{Q \rightarrow \infty \\ \Delta S \rightarrow 0}} \left[\frac{Q}{4\pi n^2} \Delta S \cos \theta \right] + \text{constant}$$

$$\phi = -\frac{\mu}{4\pi n^2} \cos \theta + \text{constant} \quad \checkmark$$

where $\mu = \lim_{\substack{Q \rightarrow \infty \\ \Delta S \rightarrow 0}} Q \Delta S$ is finite.

The velocity components of the doublet are v_n and v_θ

$$v_n = \frac{\partial \phi}{\partial n} = \frac{\mu}{2\pi n^3} \cos \theta$$

$$v_\theta = \frac{1}{n} \frac{\partial \phi}{\partial \theta} = \frac{1}{n} \left[\frac{-\mu}{4\pi n^2} (-\sin \theta) \right] = \frac{1}{n} \frac{\mu \sin \theta}{4\pi n^2}$$

$$= \frac{\mu \sin \theta}{4\pi n^3}$$

The stream function

$$\psi = \int v_n n^2 \sin \theta d\theta + \text{constant}$$

$$= \frac{\mu}{2\pi n^3} (n^2) \int \cos \theta \sin \theta d\theta + \text{constant}$$

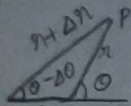
$$= \frac{\mu}{2\pi n} \int \sin \theta d(\sin \theta) + \text{constant}$$

$$= \frac{\mu}{2\pi n} \left(\frac{\sin^2 \theta}{2} \right) + \text{constant}$$

cosine rule



$$b^2 + c^2 - 2bc \cos A = a^2$$



$$n^2 = \left(\frac{\Delta S}{2} \right)^2 +$$

$$(n + \Delta n)^2 - 2 \left(\frac{\Delta S}{2} \right) (n + \Delta n) \cos(\theta - \Delta \theta)$$

$$\cos(\theta - \Delta \theta)$$

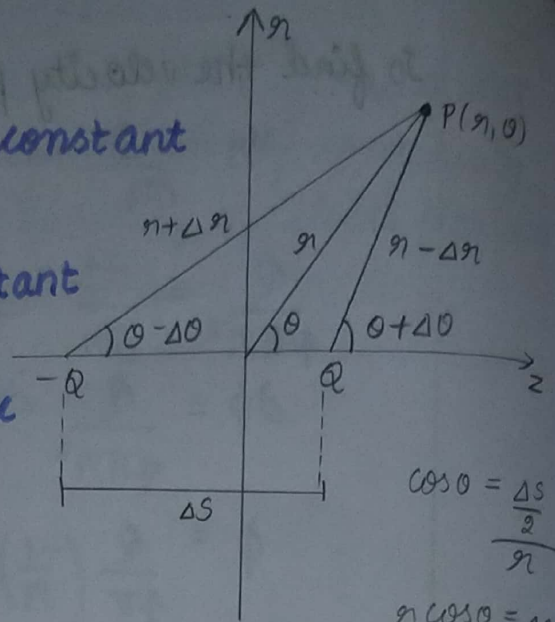
$$n^2 = n^2 + (\Delta n)^2$$

$$+ 2n\Delta n -$$

$$\Delta S (n + \Delta n) \cos(\theta - \Delta \theta)$$

$$2n\Delta n = \Delta S n \cos \theta$$

$$\Delta n = \frac{\Delta S \cos \theta}{2}$$



$$\cos \theta = \frac{\Delta S}{2n}$$

$$n \cos \theta = \frac{\Delta S}{2}$$

$$\psi = \frac{\mu}{4\pi n} \sin^2\theta + \text{constant} \quad \checkmark$$

(10 M)

1.
10 M (S)
+
10 M (S)

Example :-

Verify that the stream function ψ and the velocity potential ϕ of a 3D axially symmetric doublet flow satisfy the Laplace equation. ???

Solution :-

The stream function for a 3D doublet is

$$\psi = \frac{\mu}{4\pi n} \sin^2\theta + \text{constant}$$

To prove :- $\nabla^2\psi = 0$

$$\nabla^2\psi = \frac{1}{\sin\theta} \frac{\partial^2\psi}{\partial n^2} + \frac{1}{n^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial\psi}{\partial\theta} \right) \quad \text{--- ①}$$

$$\begin{aligned} \frac{\partial\psi}{\partial n} &= \frac{\mu}{4\pi} \sin^2\theta \frac{\partial(n^{-1})}{\partial n} \\ &= -\frac{\mu}{4\pi} \sin^2\theta n^{-2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2\psi}{\partial n^2} &= -\frac{\mu}{4\pi} \sin^2\theta (-2n^{-3}) \\ &= \frac{\mu}{4\pi} \sin^2\theta (2n^{-3}) \\ &= \frac{\mu}{2\pi n^3} \sin^2\theta \end{aligned}$$

$$\begin{aligned} \frac{\partial\psi}{\partial\theta} &= \frac{\mu}{4\pi n} 2\sin\theta \cos\theta \\ &= \frac{\mu}{2\pi n} \sin\theta \cos\theta \end{aligned}$$

$$\begin{aligned} \text{①} \Rightarrow \nabla^2\psi &= \frac{1}{\sin\theta} \left(\frac{\mu}{2\pi n^3} \sin^2\theta \right) + \frac{1}{n^2} \frac{\partial}{\partial\theta} \left[\frac{1}{\sin\theta} \left(\frac{\mu}{2\pi n} \sin\theta \cos\theta \right) \right] \\ &= \frac{\mu \sin\theta}{2\pi n^3} + \frac{1}{n^2} \left[\frac{\mu}{2\pi n} \right] (-\sin\theta) \end{aligned}$$

$$\nabla^2\psi = 0$$

Hence stream function ψ satisfies Laplace equation

To prove :- $\nabla^2 \phi = 0$

$$\nabla^2 \phi = \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right)$$

$$\phi = -\frac{\mu}{4\pi r^2} \cos \theta + \text{constant}$$

$$\frac{\partial \phi}{\partial r} = -\frac{\mu}{4\pi} (-2r^{-3}) \cos \theta$$

$$= \frac{\mu}{2\pi r^3} \cos \theta$$

$$\frac{\partial \phi}{\partial \theta} = \frac{\mu}{4\pi r^2} \sin \theta$$

$$\begin{aligned} \nabla^2 \phi &= \frac{\partial}{\partial r} \left(r^2 \frac{\mu}{2\pi r^3} \cos \theta \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\mu}{4\pi r^2} \sin \theta \right) \\ &= -\frac{\mu}{2\pi r^2} \cos \theta + \frac{1}{\sin \theta} \left(\frac{\mu}{4\pi r^2} \right) (2 \sin \theta \cos \theta) \end{aligned}$$

$$\nabla^2 \phi = 0$$

Hence the velocity potential ϕ satisfies Laplace equation.