$$\boxed{\nabla^2 \phi = 0}$$

UNIT - IV

Laminar Flow of viscous Incompressible Fluids:

Similiarity of Flows - The Reynolds Number

For viscous Incompressible Flow (Navier's stoke equation), The Equation of motion is given as follows:

$$\rho \frac{\partial \bar{q}}{\partial t} + \rho (\bar{q} \cdot \nabla) \bar{q} = \rho \bar{X} - \nabla p + \mu \nabla^2 \bar{q} \rightarrow ①$$

Let q_∞, p_∞, l and $t_0 = \frac{l}{q_\infty}$ denote the characteristic reference magnitudes of velocity and pressure at infinity, a typical linear dimension of the body and time scale equal to l/q_∞ respectively.

Introducing the Following Non-dimensional Quantities:

$t^* = \frac{t}{t_0}$, $\underset{\text{dou}}{\Delta^*} = l\nabla$, $q^* = \frac{\bar{q}}{q_\infty}$, $p^* = \frac{p}{p_\infty}$

$x^* = \frac{\bar{x}}{x_0}$

$\Rightarrow t = t_0 t^*$; $\nabla = \frac{\overset{\text{dou}}{\Delta}}{l}$; $\bar{q} = q^* q_\infty$; $t = t_0 t^*$

$p = p^* p_\infty$; $\bar{x} = x^* x_0$, $\Rightarrow t = \frac{l}{q_\infty} t^*$

sub ② in ①, we get $\hookrightarrow$ ②

$$① \Rightarrow \rho \left[\frac{\partial (q^* q_\infty)}{\partial \left(\frac{l}{q_\infty} t^* \right)} + \left(q^* q_\infty \cdot \frac{\nabla^*}{l} \right) (q^* q_\infty) \right]$$

$$= \rho x^* x_0 - \frac{\nabla^*}{l} p^* p_\infty + \mu \nabla^{*2} l^2 (q^* q_\infty)$$

$$\Rightarrow \rho \left[\frac{q_\infty}{(l/q_\infty)} \frac{\partial q^*}{\partial t^*} + \frac{q_\infty q_\infty}{l} (q^* \cdot \nabla^*) q^* \right]$$

$$= x^* (\rho x_0) - \frac{p_\infty}{l} (\nabla^* p^*) + \frac{\mu q_\infty}{l^2} \nabla^{*2} q^*$$

$\div$ by $\frac{\rho q_\infty^2}{l}$ (or) multiply by $l/\rho q_\infty^2$

$$\frac{\rho q_\infty^2}{l} \times \frac{l}{\rho q_\infty^2} \cdot \frac{\partial q^*}{\partial t^*} + \frac{\rho q_\infty^2}{l} \times \frac{l}{\rho q_\infty^2} (q^* \cdot \nabla^*) q^*$$

$$= \frac{x^* l}{\rho q_\infty^2} (\rho x_0) - \frac{p_\infty}{l} \times \frac{l}{\rho q_\infty^2} (\nabla^* p^*) + \frac{\mu q_\infty}{l^2} \times \frac{l}{\rho q_\infty^2} \nabla^{*2} q^*$$

$$\Rightarrow \frac{\partial q^*}{\partial t^*} + (q^* \cdot \nabla^*) q^* = \frac{x^* l x_0}{q_\infty^2} - \frac{p_\infty}{\rho q_\infty^2} (\nabla^* p^*) + \frac{\mu}{\rho l q_\infty} \nabla^{*2} q^*$$

$$= \frac{x^* x_0}{(q_\infty^2)} - \frac{p_\infty}{\rho q_\infty^2} \nabla^* p^* + \frac{\mu (q_\infty / l)}{\rho q_\infty^2} \ldots$$

is of the order unity. If X_0 represents the Gravitational Force per unit mass, then the co-efficient of the 1st term of the RHS of (3) may be interpreted as the ratio of the gravitational Force to the dynamic Force which is related to the Inverse of the "Froude Number"

(i.e) $\boxed{F = \frac{q_\infty^2}{gl}} = \frac{(mq_\infty^2)/l}{mg}$

$$\boxed{F = \frac{mq_\infty^2/l}{X_0}}$$

The Froude Number is a very useful parameter in open channel Flow, when the flow is mainly due to gravity.

The Froude Number is very large, if the gravitational Force is negligibly small.

The 2nd term in RHS $\frac{p_\infty}{\rho q_\infty^2}$ represents the ratio of the pressure to twice the dynamic pressure, it is unimportant for the similiarity of flows because constant density in the incompressible Flow.

The co-efficient of Last term of RHS is very important and its inverse value is known as the "Reynolds Number" "Re" (i.e) $\left(Re = \frac{\rho q_\infty^2}{\mu q_\infty/l} = \frac{\rho q_\infty l}{\mu} \quad \left(\because \frac{\mu}{\rho} = \nu\right)\right.$

$$\left.\boxed{Re = \frac{q_\infty l}{\nu}}\right)$$

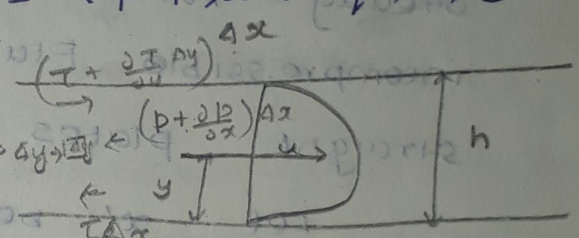
Hence (Reynolds Number may be interpre-ted as the ratio of the dynamic pressure to the shearing stress and is a parameter of viscousity).

8.3 Flow between Parallel Flat plates

For a viscous incompressiable Fluid For a steady ($\partial u/\partial t$) Flow, the Navier's Stoke equation with negligible body Force,

$$\rho(\bar{q}\cdot\nabla)\bar{q} = -\nabla p + \mu\nabla^2\bar{q} \rightarrow (1)$$

In cartesian Form,



(1) can be written as

WKT $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

2 plates at rest

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

|||ly in y-Direction,

$$\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

|||'ly in z-Direction

$$\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$\partial^2 w/\partial z^2$

(2). For incompressible Fluid, the continuity equation, $\nabla\cdot\bar{q} = 0$.

In cartesian Form, $\rightarrow$ (2)a

Nonlinear second order Differential Equation and there is No known general Method For solving them.

Hence we reduce equation ② into linear Differential Equation.

Hence some exact solutions of the Navier's stoke equations in special cases are obtained.

Then consider the 2 - Dimensional steady Laminar Flow of a viscous incompressible Fluid between 2 parallel straight plates

Let 'x' be the Direction of Flow 'y', the Direction Normal to the Flow and the width of the plates Parallel to the z - Direction be large compared with the Distance 'h' between the plates.

Then $u = u(y)$, $v = 0$, $w = 0$.

and $\partial/\partial z = 0 \longrightarrow$ ③

Sub ③ in ②, we get

$$\left.\begin{aligned} &\Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right) = 0 \\ &\Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}(0) = 0 \\ &\quad \hookrightarrow -\frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \end{aligned}\right\} \longrightarrow ④$$

④ $\Rightarrow \dfrac{\partial p}{\partial y} = 0$

$\Rightarrow \boxed{p = f(x)}$

④ $\Rightarrow -\frac{1}{\rho}\frac{dp}{}$

$$\Rightarrow \frac{dp}{dx} = \mu \frac{d^2u}{dy^2}$$

$$\Rightarrow \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Integrate with respect to 'y'

$$\frac{du}{dy} = \left(\frac{1}{\mu}\frac{dp}{dx}\right) y + A$$

once again Integrate with repect to 'y'

$$\boxed{u(y) = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} + Ay + B} \rightarrow (5)$$

where A & B are arbitrary constants to be determine by the boundary conditions.

a) Couette Flow

Flow between 2 parallel flat plates, one of which is at rest and the other moving with velocity U parallel to the Fixed plate.

The 2 arbitrary constants A and B in equation (5) can be determined from the boundary conditions, y = 0 ; u = 0 } (moving)

uniform velocity U y = h

and y = h ; u = U } → (6)

$$\therefore u(y) = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} + Ay + B$$

y = 0 (rest)

using (6) in (5),

$$\therefore u(y) = 0 = \frac{1}{\mu}\left(\frac{dp}{dx}\right) 0 + A(0) + B$$

$$\Rightarrow B = 0$$

sub in (5), we get

$$u(y) = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} + Ay \rightarrow (7)$$

using (6) in (7),

$$U = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{2} + Ah$$

$$\Rightarrow \quad U - \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{h^2}{2} = Ah$$

$$\Rightarrow \quad A = \frac{U}{h} - \frac{1}{\mu h}\frac{dp}{dx}\frac{h^2}{2}$$

$$A = \frac{U}{h} - \frac{1}{\mu}\frac{dp}{dx}\frac{h}{2}$$

Sub A in (7) we get,

$$u(y) = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} + \left(\frac{U}{h} - \frac{1}{\mu}\frac{dp}{dx}\frac{h}{2}\right)y$$

$$= \left(\frac{Uy}{h} + \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} - \frac{1}{2\mu}\left(\frac{dp}{dx}\right)hy\right.$$

$$= \frac{Uy}{h} + \frac{1}{\mu}\frac{dp}{dx}y\left(\frac{y}{2} - \frac{h}{2}\right)$$

$$= \frac{Uy}{h} \overset{(+)}{+} \frac{1}{2\mu}\frac{dp}{dx}y(y-h)$$

$$= \frac{Uy}{h} + \frac{1}{2\mu}\frac{dp}{dx}y(-h)\left(1-\frac{y}{h}\right)$$

$$= \frac{Uy}{h} - \frac{1}{2\mu}\frac{dp}{dx}yh\left(1-\frac{y}{h}\right)$$

Multiply & Divided by h^2.

$$u(y) = \frac{Uy}{h} - \frac{1}{2\mu}\frac{h^2}{h^2}\frac{dp}{dx}yh\left(1-\frac{y}{h}\right)$$

$$u = \frac{Uy}{h} - \frac{1}{2\mu}h^2\frac{dp}{dx}\frac{y}{h}\left(1-\frac{y}{h}\right)$$

$\div U$,

$$\Rightarrow \quad \frac{u}{U} = \frac{y}{h} - \frac{h^2}{2\mu U}\frac{dp}{dx}\frac{y}{h}\left(1-\frac{y}{h}\right)$$

$$\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}\left(1-\frac{y}{h}\right)$$

where $\alpha = \left(-\frac{h^2}{2\mu U}\right)\frac{dp}{dx} \quad \longrightarrow (8)$

$\therefore$ 'α' is the Non-Dimensional Pressure gradient.

For $\alpha > 0$, the pressure is decreasing in the Direction of Flow.

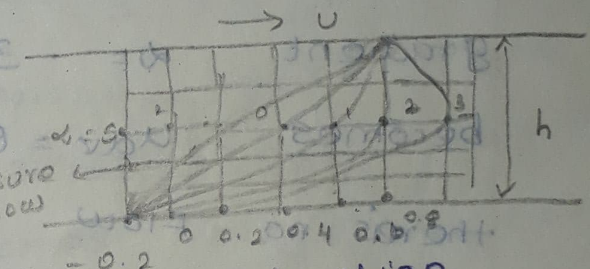
Hence the velocity is (+)ve over the whole width between the plates.

The pressure is ~~de~~ increasing in the Direction of Flow, when $\alpha < 0$, and the reverse Flow begins to occur near the stationary wall, as the value of 'α' becomes less than -1. $\alpha < -1$.

when $\alpha = 0$, equation (8) reduces to

$$\frac{u}{U} = \frac{y}{h}$$

and it is known as <u>simple couitte Flow</u>.



(i) The average and Maximum velocity

The average velocity Distribution of the couitte Flow between 2 $||^{el}$ straight walls can be calculated by

$$u_{av} = \frac{1}{h}\int_0^h u\,dy \longrightarrow (9) \qquad \int_0^h u\,dy$$

WKT $\frac{u}{(U)} = \frac{y}{h} + \alpha\frac{y}{h}\left(1 - \frac{y}{h}\right)$

$$\Rightarrow u = \frac{Uy}{h} + \frac{\alpha U y}{h}\left(1 - \frac{y}{h}\right) \rightarrow (10)$$

sub ~~(8)~~ (10) in (9)

$$u_{av} = \int_0^h U\left(\frac{y}{h}\right) + \alpha U\left(\frac{y}{h}\right)\left(1 - \frac{y}{h}\right) d\left(\frac{y}{h}\right)$$

$$= U\int_0^h \left(\frac{y}{h}\right) d\left(\frac{y}{h}\right) + \alpha U \int_0^h \left(\frac{y}{h} - \frac{y^2}{h^2}\right) d\left(\frac{y}{h}\right)$$

$$= U\left[\left(\frac{y}{h}\right)^2 \ldots\right]_0^h + \alpha U\left[\left(\frac{y}{h}\right)^2 - \left(\frac{y}{h}\right)^3\right]^h$$

$$= U\frac{1}{2} + \alpha U\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= U\frac{1}{2} + \alpha U\left(\frac{3-2}{6}\right)$$

$$= U\frac{1}{2} + \alpha U\frac{1}{6}$$

$$\boxed{u_{av} = U\left(\frac{1}{2} + \frac{\alpha}{6}\right)}$$ average velocity

In case of simple Couette Flow ($\alpha = 0$)

then $\boxed{u_{av} = \frac{U}{2}}$

When the non-Dimensional pressure gradient $\alpha = -3$ then average velocity becomes $\boxed{u_{av} = 0}$. This means that (i.e) there's no Flow between plates.

The Volumetric Flow $Q = h\,u_{av}$

$$\boxed{Q = hU\left(\frac{1}{2} + \frac{\alpha}{6}\right)}$$

<u>Maximum velocity</u>:

The maximum or minimum velocity occurs at the position of y where the slope of velocity is equal to zero co

(i.e) $\frac{du}{dy} = 0$

(10) $\Rightarrow$ $u = \frac{Uy}{h} + \frac{\alpha U y}{h}\left(1 - \frac{y}{h}\right)$

$$= \frac{Uy}{h} + \frac{\alpha U y}{h} - \frac{\alpha U y^2}{h^2}$$

du/dy

$$\frac{du}{dy} = 0 \Rightarrow \left(\frac{U}{h} + \frac{\alpha U}{h} - \frac{\alpha U}{h^2}2y\right) = 0$$

$$\frac{U}{h} = -\frac{\alpha U}{h}\left(1 - \frac{2y}{h}\right)$$

$$\frac{y}{h} = \frac{\cancel{U}(1+\alpha)}{2\alpha \cancel{U}} = \frac{1}{2\alpha} + \frac{\alpha}{2\alpha}$$

non-Dimensional

$$= \frac{1}{2\alpha} + \frac{1}{2}$$

$$\therefore \boxed{\frac{y}{h} = \frac{1}{2} + \frac{1}{2\alpha}} \rightarrow (11)$$

The maximum velocity for $\alpha = 1$, occurs at $\frac{y}{h} = \frac{1}{2} + \frac{1}{2} = \boxed{1 = \frac{y}{h}}$

And the Minimum velocity for $\alpha = -1$ occurs at $\frac{y}{h} = \frac{1}{2} - \frac{1}{2} = \boxed{0 = \frac{y}{h}}$

This implies that, $\alpha = -1$, the velocity gradiant at the stationary wall is zero (0), & it becomes Negative For any value of $\alpha < -1$.

Equation (11) breaks down when α lies between -1 and 1 ($-1 < \alpha < 1$) because the maximum and minimum value of $\frac{y}{h}$ have been reached at $\alpha = 1$ and $\alpha = -1$ respectively.

The values of Maximum and minimum velocity is can be obtained From by substitute the value of 'y' equation (11) into equation (10).

$$u = U\left(\frac{1}{2} + \frac{1}{2\alpha}\right) + \alpha U\left(\frac{1}{2} + \frac{1}{2\alpha}\right) - \alpha U\left(\frac{1}{2} + \frac{1}{2\alpha}\right)^2$$

$$= U\frac{1}{2} + \frac{U}{2\alpha} + \frac{\alpha U}{2} + \frac{\alpha U}{2\cancel{\alpha}} - \left[\frac{\alpha U}{4} + \frac{\alpha U}{4\alpha^2} + \frac{\cancel{2}\alpha U}{\cancel{2}\cdot 2\cancel{\alpha}}\right]$$

$$u = \frac{U}{2} + \frac{U}{2\alpha} + \frac{\alpha U}{2} + \frac{U}{2} - \frac{\alpha U}{4} - \frac{U}{4\alpha} - \frac{U}{2}$$

$$= \frac{U}{2} + \frac{U}{2\alpha}\left(1 - \frac{1}{2}\right) + \frac{\alpha U}{2}\left(1 - \frac{1}{2}\right)$$

$$= \frac{U}{2} + \frac{U}{4\alpha} + \frac{\alpha U}{4} \qquad u = \frac{U}{4}\frac{(1+\alpha)^2}{\alpha}$$

$$= \frac{U}{4}\left(\frac{1+\alpha^2+2\alpha}{\alpha}\right)$$

$$= \frac{U}{4\alpha} + \frac{U\alpha^2}{4\alpha} + \frac{U}{2}$$

$$= \frac{U}{4\alpha} + \frac{U\alpha}{4} + \frac{U}{2}$$

$$\boxed{u = \frac{U}{4}\frac{(1+\alpha)^2}{\alpha}}$$

$$u_{max} = \frac{U(1+\alpha)^2}{4\alpha} \quad \text{for} \quad \alpha \geq 1$$

$$u_{min} = \frac{U(1+\alpha)^2}{4\alpha} \quad \text{for} \quad \alpha \leq -1$$

ii) Shearing stress : $\tau = \mu \frac{du}{dy}$

The shearing stress distribution in a couttie Flow can be obtained by,

$$\tau = \mu \frac{du}{dy}$$

$$\tau = \mu \left(\frac{U}{h} + \frac{\alpha U}{h} - \frac{\alpha U}{h^2} 2y\right) \rightarrow (12)$$

For a simple couttie Flow when $\alpha = 0$ then equation (12) becomes

$$\tau = \frac{\mu U}{h} = \text{constant}$$

It is a constant across the passage between plates

At the position $\frac{y}{h} = \frac{1}{2}$, the equation (12) reduces to,

$$\tau = \mu \quad \frac{U}{} + \alpha U$$

$$\tau = \frac{\mu U}{h}$$

(i.e) Hence the shearing stress is independent of 'α', at the center of the passage.

The shearing stress at the stationary wall ($y = 0$) is (+)ve, (i.e) $\frac{y}{h} = 0$ then equation (12) reduces to,

$$\tau = \mu\left(\frac{U}{h} + \frac{\alpha U}{h}\right)$$

$\therefore$ The shearing stress at the stationary wall is (+)ve for $\alpha > -1$ and (-)ve for $\alpha < -1$

Since the velocity gradient at the stationary wall is zero for $\alpha = -1$, the value of the shearing stress is zero (0) for this condition.

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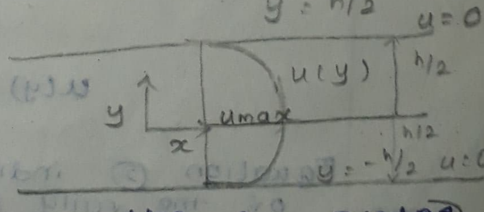
b) plane poiseuille Flow

If the 2 ||el plates are both stationary (at rest) the fully Developed flow between the plates requires the boundary condition

$y = \pm \frac{h}{2}$, $u = 0$, $\rightarrow$ (1)



where the x - axis is along the center between the 2 plates.

From (5), $u(y) = U\left(\frac{y}{h}\right)\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} + Ay + B$

use the Boundary condition (1) in (5),

$$0 = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8} + A\left(\pm\frac{h}{2}\right) + B$$

$$\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8} + \frac{Ah}{2} + B$$

$$A\frac{h}{2} + B = -\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8}$$

$$-A\frac{h}{2} + B = -\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8}$$

$$2B = -\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{2h^2}{8}$$

$$B = -\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{2h^2}{8\times 2}$$

$$\boxed{B = -\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8}}$$

$$A\frac{h}{2} - \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8} = -\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8}$$

$$A\frac{h}{2} = 0$$

$$\boxed{A = 0}$$

Sub A and B in (5),

$$\therefore u(y) = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} + -\left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{h^2}{8}$$

$$= \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} - \frac{h^2}{8\mu}\frac{dp}{dx}$$

$$= -\frac{h^2}{8\mu}\frac{dp}{dx}\left(1 - \frac{4y^2}{h^2}\right)$$

$$\boxed{u(y) = -\frac{h^2}{8\mu}\frac{dp}{dx}\left[1 - 4\left(\frac{y}{h}\right)^2\right]}$$

Equation (2) indicates that the velocity profile of the fully developed laminar flow between (2) 2 ||el plates is parabolic

(i) Maximum and average velocities

$$\frac{du}{dy} = 0 \Rightarrow -\frac{h^2}{8\mu}\frac{dp}{dx}\left(-\frac{4}{h^2}2y\right) = 0$$

$$= +\frac{1}{\mu}\frac{dp}{dx}y = 0$$

Sub $y = 0$ in u (eq 2)

$$\boxed{u_{max} = -\frac{h^2}{8\mu}\frac{dp}{dx}} \longrightarrow (3) \quad \frac{dp}{dx} = -u_{max}\frac{8\mu}{h^2}$$

The average velocity :

To Find u_{av}

WKT $u_{av} = \frac{1}{h}\int_{-h/2}^{h/2} u(y)\, dy$

where ;

$\therefore u(y) = u_{max}\left[1 - 4\left(\frac{y}{h}\right)^2\right]$

$$\therefore u_{av} = \frac{1}{h}\int_{-h/2}^{h/2} u_{max}\left[1 - 4\left(\frac{y}{h}\right)^2\right] dy$$

$$= \frac{1}{h} u_{max}\left[\int_{-h/2}^{h/2} dy - \frac{4}{h^2}\int_{-h/2}^{h/2} y^2 dy\right]$$

$$= \frac{u_{max}}{h}\left\{[y]_{-h/2}^{h/2} - \frac{4}{h^2}\left.\frac{y^3}{3}\right|_{-h/2}^{h/2}\right\}$$

$$= \frac{u_{max}}{h}\left\{\left(\frac{h}{2} + \frac{h}{2}\right) - \frac{4}{3h^2}\left[\left(\frac{h}{2}\right)^3 + \left(\frac{h}{2}\right)^3\right]\right\}$$

$$= \frac{u_{max}}{h}\left(h - \frac{\cancel{4}}{3h^{\cancel{2}}}\frac{\cancel{2}h^{\cancel{3}}}{\cancel{8}}\right)$$

$$= \frac{u_{max}}{h}\left(h - \frac{h}{3}\right)$$

$$= \frac{u_{max}}{\cancel{h}}\left(\frac{2\cancel{h}}{3}\right)$$

$$\boxed{u_{av} = \frac{2}{3}u_{max}} \longrightarrow (4)$$

To Find Pressure Drop From (3) & (4),

$$u_{av} = \frac{\cancel{2}}{3}\cdot\frac{-h^2}{\cancel{8}_4\mu}\frac{dp}{dx}$$

$$\therefore \boxed{\frac{dp}{dx} = -\frac{12\mu}{h^2}u_{av}}$$

→ pressure drop

$\frac{dp}{dx}$ is known as pressure drop.

(ii) Shearing Stress

The shearing stress at the wall for the plane poiseuille' flow can be determined from the velocity gradient,

$$(\tau_{yx})_{\frac{h}{2}} = -\left(\mu \frac{du}{dy}\right)_{\frac{h}{2}}$$

$$= -\left[\mu\left(\frac{1}{\mu}\frac{dp}{dx}\, y\right)\right]_{\frac{h}{2}}$$

$$= -\frac{h}{2}\frac{dp}{dx}$$

From (3), $\frac{dp}{dx} = -u_{max}\frac{8\mu}{h^2}$

$$(\tau_{yx})_{\frac{h}{2}} = -\frac{h}{2}\left(-u_{max}\frac{8\mu}{h^2}\right)$$

$$\boxed{(\tau_{yx})_{\frac{h}{2}} = \frac{4\mu}{h}u_{max}}$$

The Local Frictional coefficient

$$C_f = \frac{(\tau_{yx})_{\frac{h}{2}}}{\rho\left(\frac{u_{av}^2}{2}\right)}$$

$$= \frac{\frac{4\mu}{h}u_{max}}{\rho\left(\frac{u_{av}^2}{2}\right)} = \frac{\frac{4\mu}{h}\frac{3}{2}u_{av}}{\rho\left(\frac{u_{av}^2}{2}\right)} \quad [\because (4)]$$

$$= \frac{2\mu\,(3)(2)}{h\rho\, u_{av}}$$

$$= \frac{12\mu}{h\rho\, u_{av}}$$

$$C_f = \frac{12\nu}{\quad}$$

where $Re = \frac{\nu}{h u_{av}}$ (i.e) $Re = \frac{h u_{av}}{\nu}$ is the Reynold's number of the flow based on the average velocity and channel height

Problem:

1. Water at 20°C flows between 2 large ||el plates at a distance of 1.5 mm (h) apart. If the average velocity is (u_{av}) 0.15 m/s. Find (i) the maximum velocity (ii) the pressure drop (iii) the wall shearing stress (iv) The Frictional coefficient. (μ at 20°C $= 1.01$ gm/s)

(10m)

Sol:

(i) The maximum velocity, $u_{max} = \frac{3}{2} u_{av}$

$$= \frac{3}{2}(0.15)$$

$$u_{max} = 0.225 \text{ m/s}$$

(ii) The pressure drop $\frac{dp}{dx} = -\frac{12\mu}{h^2} u_{av}$

since $\mu = 1.01$ gm/s $= 1.01 \times 10^{-3}$

$$\therefore \frac{dp}{dx} = \frac{-12 \times 10^{-3} \times 1.01 \times 0.15}{(1.5 \times 10^{-3})^2}$$

$$\boxed{\frac{dp}{dx} = -0.808 \text{ N/m}^3}$$

(iii) The wall shearing stress,

$$(\tau_{yx})_{\frac{h}{2}} = \frac{4\mu}{h} u_{max}$$

$$= \frac{4 \times 1.01 \times 10^{-3} \times 0.225}{ \times 10^{-3}}$$

$$\boxed{(\tau_{yx})_{\frac{b}{2}} = 0.606 \ N/m^2}$$

iv) Frictional coefficient,

$$C_f = \frac{(\tau_{yx})_{b/2}}{\rho\left(\frac{u_{av}^2}{2}\right)}$$

Since $\rho = 1 \times 10^3$

$$\therefore C_f = \frac{0.606}{1\times10^3\left(\frac{0.15^2}{2}\right)}$$

$$= \frac{0.606 \times 2}{.0225 \times 10^3}$$

$$= \frac{1.212}{0.0225 \times 10^3}$$

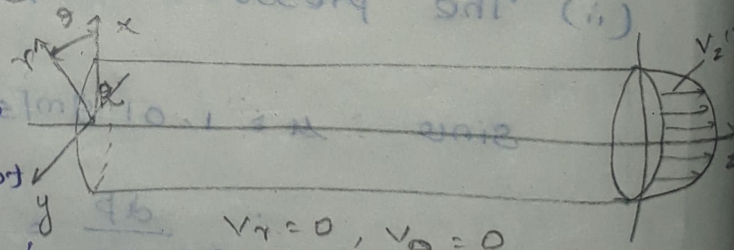
$$\boxed{C_f = 0.0538}$$

Steady Flow in pipes:

(a) Flow through a pipe:

The Hagon poiseuille Flow

Let z be the Direction of Flow along the axis of the pipe and 'r' denote the radial Direction measured outwards from the z-axis.



The velocity components $V_r = V_\theta = 0$ then the equation of continuity for cylinarical coordinates,

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{\rho v_r}{r} = 0$$

This equation reduces to,

$$\frac{\partial v_z}{\partial z} = 0 \rightarrow (1) [\text{since, } v_r = v_\theta = 0]$$

$$\Rightarrow v_z = v_z(r) \longrightarrow (2)$$

Navier stokes equation in cylindrical coordinates are,

$$\rho\left[\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r}\right] = \rho x_r - \frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) \rightarrow (3)$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) = \rho x_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) \rightarrow (4)$$

$$\rho\frac{Dv_z}{Dt} = \rho x_z - \frac{\partial p}{\partial z} + \mu\nabla^2 v_z \rightarrow (5)$$

Equation (3), (4), (5) Neglecting the body force,

Equation (3) reduces to, $[\because v_r = v_\theta = 0]$

$$\frac{\partial p}{\partial r} = 0 \rightarrow (a)$$

(4) reduces to,

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$$

$$\frac{1}{r} \neq 0 \quad \therefore \frac{\partial p}{\partial \theta} = 0 \rightarrow (b)$$

From (a) & (b), $\Rightarrow p = f(z)$

(i.e) p is a function of 'z'

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

in cylindrical coordinates, & $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{q}\cdot\nabla$

and $\nabla = (\partial/\partial r, \frac{1}{r}\partial/\partial\theta, \partial/\partial z)$

$\bar{q}\cdot\nabla = u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z$

Then equation (5) reduces to,

$$\rho\left(\frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\partial/\partial\theta + v_z\,\partial/\partial z\right)v_z = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)$$

Steady
can't change

$$0 = -\frac{dp}{dz} + \mu\left[\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}\right]$$

[$\because$ v is a function of 'r' alone & the flow is steady $\frac{\partial v_z}{\partial t} = 0$]

$$\mu\left(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}\right) = \frac{dp}{dz} \quad \rightarrow (6)$$

Function of 'r' — Function of 'z'

This equation can't be valid unless the pressure is a linear function of 'z'.

is equal RHS: LHS then under integration both are constant

<u>To Find the velocity of the flow v_z.</u>

Equation (6) reduces to,

$$r\frac{d^2 v_z}{dr^2} + \frac{dv_z}{dr} = \frac{1}{\mu}\frac{dp}{dz} r$$

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz} r$$

Integrating both sides with respect to 'r'

$$r\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2} + A \rightarrow (7)$$

$$\Rightarrow \frac{dv_z}{dr} = \frac{1}{\mu r}\frac{dp}{dz}\frac{r^2}{2} + \frac{A}{r}$$

$$\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz} r + \frac{A}{r}$$

Integrate again

$$\Rightarrow v_z = \frac{1}{2\mu}\frac{dp}{dz}\frac{r^2}{2} + A\ln r + B$$

$$\therefore \boxed{v_z(r) = \frac{1}{4\mu}\frac{dp}{dz} r^2 + A\ln r + B}$$

where A and B are constants to be determine by using the Boundary conditions.

The First Boundary condition is found from the symmetry of the flow which requires $r = 0$, $\frac{dv_z}{dr} = 0$

The Second Boundary condition is no slip condition at the wall

(7) $\Rightarrow 0 = \frac{1}{2\mu}(0) + A$

$$\boxed{A = 0}$$

$\therefore V_z(r) = \frac{1}{4\mu}\frac{dp}{dz} r^2 + A \ln r + B$

using condition (ii) in (8),

$$V_z(r) = 0 = \frac{1}{4\mu}\frac{dp}{dz} R^2 + 0 + B$$

$$0 = \frac{1}{4\mu}\frac{dp}{dz} R^2 + B$$

$$\boxed{B = -\frac{1}{4\mu}\frac{dp}{dz} R^2}$$

Sub A and B in (8)

$$V_z(r) = \frac{1}{4\mu}\frac{dp}{dz} r^2 - \frac{1}{4\mu}\frac{dp}{dz} R^2$$

$$= \frac{1}{4\mu}\frac{dp}{dz}(r^2 - R^2)$$

$$\therefore \boxed{V_z(r) = \frac{-R^2}{4\mu}\frac{dp}{dz}\left[1 - \left(\frac{r}{R}\right)^2\right]} \rightarrow (9)$$

which has the form of paraboloid of revolution

(i) Maximum and average velocities:

The Maximum velocity occurs in the center of the pipe where $r = 0$ then equation (9), reduces to

$$\boxed{(V_z)_{max} = -\frac{R^2}{4\mu}\frac{dp}{dz}}$$

where $\frac{dp}{dz} < 0$

velocity in the hagon

$$V_{zav} = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{R} V_z \, r \, dr \, d\theta$$

$$\therefore (V_z)_{av} = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{R} -\frac{R^2}{4\mu} \frac{dp}{dz} \left[1 - \left(\frac{r}{R}\right)^2\right] r \, dr \, d\theta$$

(or) $(V_z)_{max} \left[1 - \left(\frac{r}{R}\right)^2\right]$

$\left(r - \frac{r^3}{R^2}\right)$

$$= \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{R} -\frac{R^2}{4\mu} \frac{dp}{dz} \left(\frac{r^2 - R^2}{}\right) dr \, d\theta$$

$$= \frac{-1}{\pi R^2} \frac{R^2}{4\mu} \frac{dp}{dz} \int_0^{2\pi} \frac{r^2}{2} - \frac{r^{3}}{\cancel{4}R^2} \Big|_0^R \, d\theta$$

$$= \frac{-1}{\pi R^2} \frac{R^2}{4\mu} \frac{dp}{dz} \left(\frac{R^2}{2} - \frac{R^{3}R}{4R^2}\right) \theta \Big|_0^{2\pi}$$

$$= \frac{-R^2}{\cancel{12}\,4\mu} \frac{dp}{dz} \frac{1}{\cancel{\pi} R^2} \frac{R^2}{\cancel{4}\,2} \cancel{2\pi}$$

$$= -\frac{R^2}{8\mu} \frac{dp}{dz}$$

$$\boxed{(V_z)_{av} = -\frac{R^2}{8\mu} \frac{dp}{dz}}$$

$2 V_{zav} = V_{zmax}$

The Volumetric Flow, $Q = \pi R^2 V_{zav}$

$$Q = \pi R^2 \left(-\frac{R^2}{8\mu} \frac{dp}{dz}\right)$$

$$Q = -\frac{\pi}{8\mu} R^4 \frac{dp}{dz}$$

$$Q = \frac{\pi}{8\mu} R^4 \left(-\frac{dp}{dz}\right)$$

(ii) <u>Shearing</u> <u>Stress</u>:

at R

$$(\tau)_R = \left(-\mu \frac{dV_z}{dr}\right)_R$$

$$= -\mu \left(\frac{1}{2\mu} \frac{dp}{dz} r + \frac{A}{r}\right)_R$$

$$= -\frac{1}{2} \frac{dp}{dz} R + A$$

$$\frac{r}{2}\frac{dp}{dz} \qquad (\because A = 0)$$

$$(V_z)_{max} = -\frac{R^2}{4\mu}\frac{dp}{dz}$$

$$\frac{dp}{dz} = -(V_z)_{max}\frac{4\mu}{R^2}$$

$$(\tau)_R = -\frac{\cancel{R}}{\cancel{2}}\left(-V_z\right)_{max}\frac{4\mu}{R^{\cancel{2}}}$$

$$= (V_z)_{max}\frac{2\mu}{R}$$

$$\boxed{(\tau)_R = 2\mu\frac{V_{zmax}}{R}}$$

where $(\tau)_R$ is the shearing stress at the wall.

The Local Frictional coefficient C_f For Laminar Flow through a circular pipe can be expressed as $(\tau)_R$,

$$(\tau)_R = C_f\,\rho\,\frac{V_{av}^2}{2}$$

$$\Rightarrow C_f = \frac{2(\tau)_R}{\rho V_{zav}^2}$$

$$= 2 \times 2\mu\,\frac{V_{z\,max}}{R} \times \frac{1}{\rho V_{zav}^2}$$

$$= \frac{\frac{2\mu}{R}\,2V_{z\cancel{av}}}{\rho\left(\frac{V_{zav}^{\cancel{2}}}{2}\right)} \qquad \left(\because V_{zmax} = 2V_{zav}\right)$$

$$= \frac{8\mu}{R\rho V_{zav}}$$

$$= \frac{8\nu}{R V_{zav}} \times \frac{2}{2}$$

$$= \frac{16\nu}{2R V_{zav}}$$

$$\boxed{C_f = \frac{16}{Re}}$$

where $Re = \dfrac{2R V_{zav}}{\nu}$

The resistant coefficient of pipe flow is defined by the non-dimensional pressure gradient,

$$f = -\frac{d\bar{p}}{d\bar{z}}$$

where $\bar{p} = \dfrac{p}{\left(\dfrac{\rho V_{zav}^2}{2}\right)}$, $\bar{z} = \dfrac{z}{2R}$

$$\therefore f = -d\left[\frac{p}{\left(\dfrac{\rho V_{zav}^2}{2}\right)}\right] \Big/ d\left(\frac{z}{2R}\right)$$

$$= \frac{2R}{\left(\dfrac{\rho V_{zav}^2}{2}\right)}\left(-\frac{dp}{dz}\right)$$

$$= \frac{2R}{\left(\dfrac{\rho V_{zav}^2}{2}\right)}\left(\frac{8\mu}{R^2} V_{zav}\right)$$

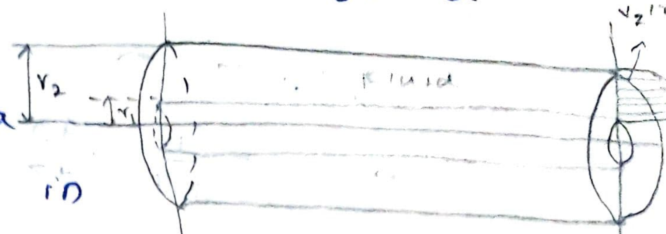
$$= \frac{32\mu}{\rho V_{zav} R} = \frac{32\nu}{R V_{zav}} \times \frac{2}{2}$$

$$= \frac{64\nu}{2R V_{zav}}$$

$$\boxed{f = \frac{64}{Re}}$$ where $Re = \dfrac{2R V_{zav}}{\nu}$

b) Flow between 2 coaxial cylinders :

The velocity distribution of a viscous Fluid which Flows steadyliy para-llel to the axis in the angular space between 2 coaxial cylinders of radii 'r_1' and 'r_2', is given by,



$$V_z(r) = \frac{1}{4\mu}\frac{dp}{dz} r^2 + A\ln r + B \rightarrow (1)$$

where A and B are constants to be determine by using the boundary conditions, $V_z = 0$ at $r = r_1$ and $r = r_2$ $\rightarrow (2)$

using (2) in (1),

$$0 = \frac{1}{4\mu}\frac{dp}{dz} r_1^2 + A\ln r_1 + B$$

$$\Rightarrow A\ln r_1 + B = -\frac{1}{4\mu}\frac{dp}{dz} r_1^2 \rightarrow (3)$$

lll'y

$$A\ln r_2 + B = -\frac{1}{4\mu}\frac{dp}{dz} r_2^2 \rightarrow (4)$$

$$A(\ln r_1 - \ln r_2) = \frac{1}{4\mu}\frac{dp}{dz}(r_2^2 - r_1^2)$$

$$\Rightarrow A\ln\left(\frac{r_1}{r_2}\right) = \frac{1}{4\mu}\frac{dp}{dz}(r_2^2 - r_1^2)$$

$$\Rightarrow A\ln\left(\frac{r_1}{r_2}\right) = -\frac{1}{4\mu}\frac{dp}{dz} r_1^2\left(1 - \frac{r_2^2}{r_1^2}\right)$$

$$\Rightarrow \cancel{-} A\ln\left(\frac{r_2}{r_1}\right) = \cancel{-}\frac{1}{4\mu}\frac{dp}{dz} r_1^2\left[1 - \left(\frac{r_2}{r_1}\right)^2\right]$$

Let $n = \frac{r_2}{r_1}$

$$\Rightarrow A\ln n = \frac{1}{4\mu}\frac{dp}{dz} r_1^2(1 - n^2)$$

$$A = \frac{1}{4\mu\ln n}\frac{dp}{dz} r_1^2(1 - n^2)$$

$$\therefore \boxed{A = \frac{-1}{4\mu \ln n} \frac{dp}{dz} r_1^2 (n^2-1)}$$

sub 'A' in ③,

$$B = -\frac{1}{4\mu}\frac{dp}{dz} r_1^2 - A \ln r_1$$

$$= -\frac{1}{4\mu}\frac{dp}{dz} r_1^2 + \frac{1}{4\mu \ln n}\frac{dp}{dz} r_1^2 (n^2-1) \ln r_1$$

$$= \frac{1}{4\mu}\frac{dp}{dz} r_1^2 \left[\frac{\ln r_1}{\ln n}(n^2-1) - 1\right]$$

$$\boxed{B = -\frac{1}{4\mu}\frac{dp}{dz} r_1^2 \left[1 - \frac{\ln r_1}{\ln n}(n^2-1)\right]}$$

∴ ① becomes,

$$V_z(r) = \frac{1}{4\mu}\frac{dp}{dz} r^2 + A \ln r + B$$

$$= \frac{1}{4\mu}\frac{dp}{dz} r^2 - \frac{\ln r}{4\mu \ln n}\frac{dp}{dz} r_1^2 (n^2-1)$$

$$- \frac{1}{4\mu}\frac{dp}{dz} r_1^2 \left[1 - \frac{\ln r_1}{\ln n}(n^2-1)\right]$$

$$= -\frac{1}{4\mu}\frac{dp}{dz}\Big\{-r^2 + \frac{r_1^2 (n^2-1)\ln r}{\ln n}$$

$$+ r_1^2 \cdot 1 - \left[\frac{\ln r_1}{\ln n} r_1^2 (n^2-1)\right]$$

$$= -\frac{1}{4\mu}\frac{dp}{dz}\Big\{-r^2 + r_1^2 + \frac{(n^2-1) r_1^2}{\ln n}(\ln r - \ln r_1)\Big\}$$

$$\boxed{V_z(r) = -\frac{1}{4\mu}\frac{dp}{dz}\left[r_1^2 - r^2 + \frac{(n^2-1) r_1^2}{\ln n} \ln\left(\frac{r}{r_1}\right)\right]}$$

The rate of Volumetric Flow is given by

$$Q = \int_0^{2\pi}\int_{r_1}^{r_2} V_z(r)\, dr\, d\theta$$

$$n = \frac{r_2}{r_1} \Rightarrow r_2 = n r_1$$

$$\therefore Q = \int_0^{2\pi}\int_{r_1}^{nr_1} v_z(r)\,dr\,d\theta$$

$$Q = 2\pi \int_{r_1}^{nr_1} v_z\, r\, dr$$

$$= 2\pi \int_{r_1}^{nr_1} -\frac{1}{4\mu}\frac{dp}{dz}\left[r_1^2 - r^2 + \frac{(n^2-1)r_1^2}{\ln n}\ln\left(\frac{r}{r_1}\right)\right] r\,dr$$

$$= 2\pi\left(-\frac{1}{4\mu}\right)\frac{dp}{dz}\int_{r_1}^{nr_1}\left[r_1^2 - r^2 + \frac{(n^2-1)r_1^2}{\ln n}\ln\left(\frac{r}{r_1}\right)\right] r\,dr$$

$(\ln r - \ln r_1)$

$u = \ln\left(\frac{r}{r_1}\right)$ $\quad dv = r\,dr$

$du = \frac{r_1}{r}\cdot\frac{1}{r_1}\,dr = \frac{1}{r}\,dr$ $\quad v = r^2/2$

$$= -\frac{1}{4\mu}\, 2\pi\frac{dp}{dz}\left[r_1^2\frac{r^2}{2}\Big|_{r_1}^{nr_1} - \frac{r^3\cdot r}{4}\Big|_{r_1}^{nr_1} + \frac{(n^2-1)r_1^2}{\ln n}\left(\frac{r^2}{2}\ln\left(\frac{r}{r_1}\right) - \frac{1}{4}(n^2r_1^2 - r_1^2)\right)\right]$$

$\int u\,dv = uv - \int v\,du$

$= \ln\left(\frac{r}{r_1}\right)\frac{r^2}{2} - \int \frac{r^2}{2}\frac{1}{r}\,dr$

$= \frac{r^2}{2}\ln\left(\frac{r}{r_1}\right) - \frac{1}{2}\left(\frac{r^2}{2}\right)\Big|_{r_1}^{nr_1}$

$= \frac{r^2}{2}\ln\left(\frac{r}{r_1}\right) - \frac{1}{4}\left((nr_1)^2 - r_1^2\right)$

$= \frac{r^2}{2}\ln\left(\frac{r}{r_1}\right) - \frac{1}{4}(n^2r_1^2 - r_1^2)$

$$Q = -\frac{\pi}{2\mu}\frac{dp}{dz}\left[\frac{r_1^2}{2}(n^2r_1^2 - r_1^2) - \frac{r_1}{4}(n^4r_1^3 - r_1^3) + \frac{(n^2-1)r_1^2}{\ln n}\,\frac{r^2}{2}\ln\left(\frac{r}{r_1}\right)\Big|_{r_1}^{nr_1} - \frac{1}{4}\frac{(n^2-1)r_1^2}{\ln n}(n^2r_1^2 - r_1^2)\right]$$

$$= -\frac{\pi}{2\mu}\frac{dp}{dz}\left\{r_1^4\left(\frac{n^2-1}{2} - \frac{1}{4}(n^4-1) + \left(-\frac{1}{4}\right)\frac{(n^2-1)^2}{\ln n}\right) + \frac{(n^2-1)r_1^2}{\ln n}\left[\frac{n^2r_1^2}{2}\ln\left(\frac{nr_1}{r_1}\right) - \underbrace{\frac{r_1^2}{2}\ln\frac{r_1}{r_1}}_{=0}\right]\right.$$

$$= -\frac{\pi}{2\mu}\frac{dp}{dz}\left(r_1^4\left\{\frac{n^2-1}{2} - \frac{n^4-1}{4} - \frac{1}{4}\frac{(n^2-1)^2}{\ln n}\right.\right.$$

$$+ \frac{(n^2-1)r_1^2}{\ldots}\,\frac{n^2r_1^2}{\ldots}\ln n \Big)$$

$Q = \frac{-\pi r_1^4}{8\mu}\frac{dp}{dz}$

$(n^4-1) - \frac{(n^2-1)^2}{\ln n}$

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$$Q = -\frac{\pi}{2\mu}\frac{dp}{dz} r_1^4 \left[\frac{n^2-1}{2} - \frac{n^4-1}{4} - \frac{1}{4}\frac{(n^2-1)^2}{\ln n} + \frac{n^2(n^2-1)}{2} \right]$$

$$= -\frac{\pi}{2\mu}\frac{dp}{dz} r_1^4 \left[\frac{(n^2-1)(n^2+1)}{2} - \frac{n^4-1}{4} - \frac{1}{4}\frac{(n^2-1)^2}{\ln n} \right]$$

$$= -\frac{\pi}{2\mu}\frac{dp}{dz} r_1^4 \left[\frac{2 n^4-1}{2 \cdot 2} - \frac{n^4-1}{4} - \frac{1}{4}\frac{(n^2-1)^2}{\ln n} \right]$$

$$= -\frac{\pi}{2\mu}\frac{dp}{dz} r_1^4 \left[\frac{n^4-1}{4} - \frac{1}{4}\frac{(n^2-1)^2}{\ln n} \right]$$

$$\boxed{Q = -\frac{\pi r_1^4}{8\mu}\frac{dp}{dz}\left[(n^4-1) - \frac{(n^2-1)^2}{\ln n} \right]}$$

The avarage velocity in the annulus can be determined by,

$$V_{zav} = \frac{Q}{\pi (n^2-1) r_1^2}$$

$$V_{zav} = \frac{-1}{\cancel{\pi}(n^2-1)\cancel{r_1^2}} \frac{\cancel{\pi} r_1^{\cancel{4}\,2}}{8\mu}\frac{dp}{dz}\left[(n^4-1) - \frac{(n^2-1)^2}{\ln n} \right]$$

$$\boxed{V_{zav} = -\frac{r_1^2}{8\mu}\frac{dp}{dz}\left[(n^2+1) - \frac{(n^2-1)}{\ln n} \right]}$$

Shearing Stress :

The shearing stress at the wall of the inner cylinder and outer cylinder are

$$(\tau_{rz})_{r_1} = \left(\mu \frac{dv_z}{dr}\right)_{r_1}$$

$$= \cancel{\mu}\frac{d}{dr}\left\{ -\frac{1}{4\cancel{\mu}}\frac{dp}{dz}\left[r_1^2 - r^2 + \frac{(n^2-1)r_1^2}{\ln n} \ln\left(\frac{r}{r_1}\right) \right] \right\}_{r_1}$$

$$= -\frac{1}{4}\frac{dp}{dz}\left[-2r + (n^2-1)r_1^2 \frac{1}{\ldots} \right]$$

5.10 Show that the velocity field

$V_r = 0$, $V_\theta = C_1 r + \frac{C_2}{r}$, $V_z = 0$

satisfies the equation of mothotion

$$\frac{d^2 V_\theta}{dr^2} + \frac{d}{dr}\left(\frac{V_\theta}{r}\right) = 0$$

where C_1 and C_2 are constants.

$$(T_{rz})_{r_1} = -\frac{1}{4}\frac{dp}{dz}\left[-2r_1 + \frac{(n^2-1)r_1^{\cancel{2}}}{\ln n}\frac{1}{\cancel{r_1}}\right]$$

$$\boxed{(T_{rz})_{r_1} = -\frac{r_1}{4}\frac{dp}{dz}\left[-2 + \frac{(n^2-1)}{\ln n}\right]}$$

continue

solid

note

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