Unit-5

Boundary Layer Theory:

Properties of the navior stokes Equations:-

Boundary layer concept :

The solution of the potential flow cannot satisfy both the boundary conditions for a viscous flow.

The solution of the Navier stokes equation for very large Reynolds numbers are important only in the immediate vicinity of a solid wall.

Hence beyond the immediate vicinity of the solid wall the solution of the Navier-stokes equations is replaced by the solution of the Euler's eqn.

Now we shall illustrate some properties of the Navier stokes eqns which will lead to the basic concept of boundary layer theory.

* Viscous compressible fluid with constant viscosity
* Viscous incompressible fluid.

UNIT - V

PROPERTIES OF THE NAVIER-STOKES EQUATIONS -

Boundary layer Concept :-

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Hence, beyond the immediate vicinity of the solid wall the soln of the Navier-stokes equations is replaced by the solution of the Euler's equation.

Now, we shall illustrate some properties of the Navier-stokes equations which will lead to the basic concept of boundary layer theory.

If the curl of the Navier-stokes equation.

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\bar{q} = X - \frac{\nabla P}{\rho} + \nu \nabla^2 \bar{q}$$

the body force is neglected. and no pressure then we have,

$$\boxed{\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\bar{q} = \nu \nabla^2 \bar{q}} \longrightarrow (1)$$

both side multiple by (∇) in eqn (1).

$$\nabla \times \frac{\partial \bar{q}}{\partial t} + \nabla \times [(\bar{q}\cdot\nabla)\bar{q}] = \nabla \times (\nu \nabla^2 \bar{q})$$

$$\frac{\partial}{\partial t}(\nabla \times \bar{q}) + \nabla \times \left[\frac{1}{2}\nabla\cdot(\bar{q}) - \bar{q} \times \underset{\sim}{\nabla \times \bar{q}}\right] = \nu \nabla^2 (\nabla \times \bar{q})$$

$$\frac{\partial \bar{\Omega}}{\partial t} + \left[\frac{1}{2}\nabla \times \nabla\cdot\bar{q} - \nabla \times (\bar{q} \times \bar{\Omega})\right] = \nu \nabla^2 \bar{\Omega}.$$

$$\frac{\partial \bar{\Omega}}{\partial t} - [\nabla \times (\bar{q} \times \bar{\Omega})] = \nu \nabla^2 \bar{\Omega}.$$

$\because \bar{\Omega} = \nabla \times \bar{q}$

$\because \nabla \times \nabla\cdot\bar{q} = 0$

$\because \nabla \cdot \nabla \times \bar{q} = 0$

$\because \nabla \times (a \times b) = a.(\nabla . b) - (a . \nabla) b - b(\nabla . a) + (b . \nabla) a$

$$\frac{\partial \bar{\Omega}}{\partial t} - \left[-\bar{\Omega}(\nabla\cdot\bar{q}) + (\bar{\Omega}\cdot\nabla)\bar{q} - (\bar{q}\cdot\nabla)\bar{\Omega} + \bar{q}(\nabla\cdot\bar{\Omega})\right] = \nu \nabla^2 \bar{\Omega}$$

$\because \nabla\cdot\bar{q} = 0$

$$\frac{\partial \bar{\Omega}}{\partial t} + \left[\bar{\Omega}(\nabla\cdot\bar{q}) - (\bar{\Omega}\cdot\nabla)\bar{q} + (\bar{q}\cdot\nabla)\bar{\Omega} - \bar{q}(\nabla\cdot\bar{\Omega})\right] = \nu \nabla^2 \bar{\Omega}$$

$$\frac{\partial \bar{\Omega}}{\partial t} + \left[-(\bar{\Omega}\cdot\nabla)\bar{q} + (\bar{q}\cdot\nabla)\bar{\Omega} - \bar{q}(\nabla\cdot\nabla\times\bar{q})\right] = \nu \nabla^2 \bar{\Omega}$$

$$\frac{\partial \bar{\Omega}}{\partial t} - (\bar{\Omega}\cdot\nabla)\bar{q} + (\bar{q}\cdot\nabla)\bar{\Omega} = \nu \nabla^2 \bar{\Omega}.$$

$$\frac{\partial \bar{\Omega}}{\partial t} + (\bar{q}\cdot\nabla)\bar{\Omega} - (\bar{\Omega}\cdot\nabla)\bar{q} = \nu \nabla^2 \bar{\Omega}$$

$$\frac{D\bar{\Omega}}{Dt} - (\bar{\Omega}\cdot\nabla)\bar{q} = \nu \nabla^2 \bar{\Omega}$$

$$\boxed{\frac{D\Omega}{Dt} - (\bar{\Omega}\cdot\nabla)\bar{q} = \nu \nabla^2 \bar{\Omega}} \longrightarrow (2)$$

Eqn (2) is referred to as the vorticity transport equation.

In the case of 2-dimensional flow in the x-y plane the vorticity transport equation becomes,

$$\bar{\Omega} = \nabla \times \bar{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

where,

$$\therefore \bar{\Omega}_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

$$\bar{\Omega}_y = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$$

$$\bar{\Omega}_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$\Rightarrow$ 2-dimensional flow in the x-y plane.

$$\bar{\Omega}_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$(2) \Rightarrow \boxed{\frac{D\bar{\Omega}_z}{Dt} = \nu \nabla^2 \bar{\Omega}_z} \longrightarrow (3)$$

A system of two equations for the two unknowns u and v.

In non-dimensional form of eqn (3).

we know that,

$$t^* = t/t_0, \quad \nabla^* = l\nabla, \quad q^* = \frac{q}{q_\infty}, \quad x^* = x/x_0, \quad p^* = p/p_\infty$$

$$\bar{\Omega}_z^* = \bar{\Omega}_z/\bar{\Omega}_0 .$$

$$(3) \Rightarrow \quad \frac{D\bar{\Omega}_z}{Dt} = \nu \nabla^2 \bar{\Omega}_z$$

$$\Rightarrow \quad \frac{D(\bar{\Omega}_z^* \bar{\Omega}_0)}{D(t^* t_0)} = \nu \frac{\nabla^{*2}}{l^2}(\bar{\Omega}_z^* \bar{\Omega}_0)$$

$$\frac{D\bar{\Omega}_z^*}{Dt^*}\left(\frac{\bar{\Omega}_0}{t_0}\right) = \frac{\nu \bar{\Omega}_0}{l^2}(\nabla^{*2} \bar{\Omega}_z^*)$$

$$\frac{D\bar{\Omega}_z^*}{Dt^*} = \frac{\nu \bar{\Omega}_0 t_0}{\bar{\Omega}_0 l^2}(\nabla^{*2} \bar{\Omega}_z^*)$$

$$\frac{D\bar{\Omega}_z^*}{Dt^*} = \frac{1}{Re} \nabla^{*2} \bar{\Omega}_z^* \qquad \because \frac{\nu t_0}{l^2} = \frac{1}{Re}$$

$$\therefore \boxed{\frac{D\bar{\Omega}_z^*}{Dt^*} = \frac{1}{Re} \nabla^{*2} \bar{\Omega}_z^*} \rightarrow (4)$$

If the frictional heat generated in the fluid is neglected the energy equation and the Non viscous fluid $\bar{\Phi} = 0$.

$$\boxed{k\nabla^2 T - \Phi_q = \rho C_\tau \frac{\partial T}{\partial t} + \rho C_\tau q.(\nabla T)} \rightarrow (5)$$

$\because$ Non viscous fluid

$\Phi = 0.$

$$(5) \Rightarrow \quad k\nabla^2 T = \rho C_\tau \frac{\partial T}{\partial t}$$

$\because \theta = T - T_\infty.$

$\because T = \theta + T_\infty.$

$$\Rightarrow k\nabla^2(\theta + T_\infty) = \rho C_\tau \frac{\partial(\theta + T_\infty)}{\partial t}$$

$$\Rightarrow k\nabla^2\theta + k\nabla^2 T_\infty = \rho C_\tau \frac{\partial \theta}{\partial t} + \rho C_\tau \frac{\partial T_\infty}{\partial t}$$

$$\Rightarrow k\nabla^2\theta - \rho C_\tau \frac{\partial \theta}{\partial t} = \rho C_\tau \frac{\partial T_\infty}{\partial t} - k\nabla^2 T_\infty$$

$\because T_\infty$ is a small velocity.

$$\Rightarrow k\nabla^2\theta - \rho C_\tau \frac{\partial \theta}{\partial t} = 0$$

$$k\nabla^2\theta = \rho C_\tau \frac{\partial \theta}{\partial t}$$

$$\therefore \boxed{k\nabla^2\theta = \rho C_\tau \frac{\partial \theta}{\partial t}} \rightarrow (6)$$

In non-dimensional form of (6), becomes,

we know that,

$$t^* = t/t_0, \quad \nabla^* = l\nabla, \quad \bar{\Omega}_z^* = \bar{\Omega}_z/\bar{\Omega}_0,$$

$$\theta^* = (T - T_\infty)/(T_0 - T_\infty), \quad \theta = T - T_\infty.$$

$(6) \Rightarrow$

$$\rho C_\tau \frac{\partial (T-T_\infty)}{\partial (t^* t_0)} = k \frac{\nabla^{*2}}{l^2} (T-T_\infty)$$

$$\frac{\partial (T-T_\infty)}{\partial (t^* t_0)} = \frac{k \frac{\nabla^{*2}}{l^2} (T-T_\infty)}{\rho C_\tau}$$

$$\frac{\partial (\theta^* (T_0 - T_\infty))}{\partial (t^* t_0)} = \frac{k \frac{\nabla^{*2}}{l^2} (\theta^* (T_0 - T_\infty))}{\rho C_\tau}$$

$$\frac{\partial \theta^*}{\partial t^*} \frac{(T_0 - T_\infty)}{t_0} = \frac{(T_0 - T_\infty)}{\rho C_\tau} k \nabla^{*2} \theta^*$$

$$\frac{\partial \theta^*}{\partial t^*} = \frac{(T_0 - \cancel{T_\infty}) t_0}{\rho C_\tau (T_0 - \cancel{T_\infty})} k \nabla^{*2} \theta^*$$

$$\frac{\partial \theta^*}{\partial t^*} = \frac{t_0}{\rho C_\tau} k \nabla^{*2} \theta^* \qquad \therefore k = \frac{C_p}{C_\tau}$$

~~kk = Cp~~

$$= \frac{t_0}{\rho \left(\frac{C_p}{k}\right)} k \nabla^{*2} \theta^* \qquad C_\tau = \frac{C_p}{k}$$

$$= \frac{1}{Re\, P_r} k \nabla^{*2} \theta^* \qquad \therefore \frac{C_p}{k} = P_r$$

$$\boxed{\frac{\partial \theta^*}{\partial t^*} = \frac{k}{Re\, P_r} \nabla^{*2} \theta^*} \longrightarrow (7)$$

Eqn (6) may be used to describe the temperature distribution around a heated body at a temperature T_0 immersed in a fluid stream with a temperature T_∞.

However, from the experience of heat conduction we know that if the surrounding fluid is moving slowly around a heated body, the heat would be transferred from the body outwards in all directions, far into the fluid.

If the fluid is flowing at high velocities only a relatively thin layer of the fluid near the body and a narrow wake will be influenced by the heated body.

Now let us compare the energy equation (4) & (7). These two differential equations are exactly the same except the constant.

$$\frac{D\theta^*}{Dt^*} = \frac{k}{Re\,Pr} \nabla^{*2} \theta^*$$

$$\frac{D\bar{\Omega}_z^*}{Dt^*} = \frac{1}{Re} \nabla^{*2} \Omega_z^*$$

the two energy equation differs by a factor $\frac{k}{Pr}$.

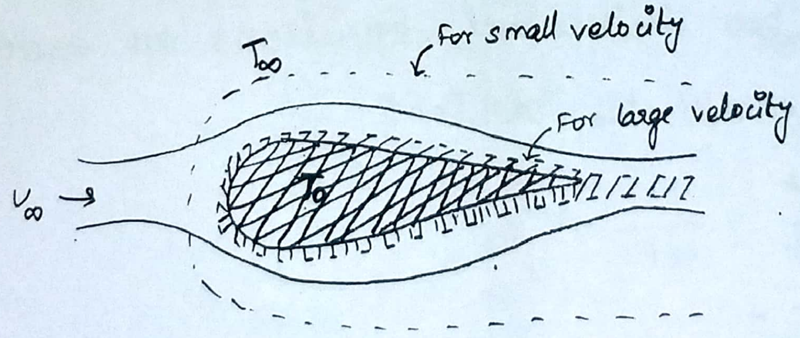
The values $\frac{k}{Pr}$ for most gases are in the order of magnitude of unity.

Thus for large Reynolds numbers, one except the following soln of Navier-stokes equations.

a) In the region of a thin boundary layer near the vicinity of a wall where $\bar{\Omega} \neq 0$, viscous forces are of the same order of magnitude as inertial forces.

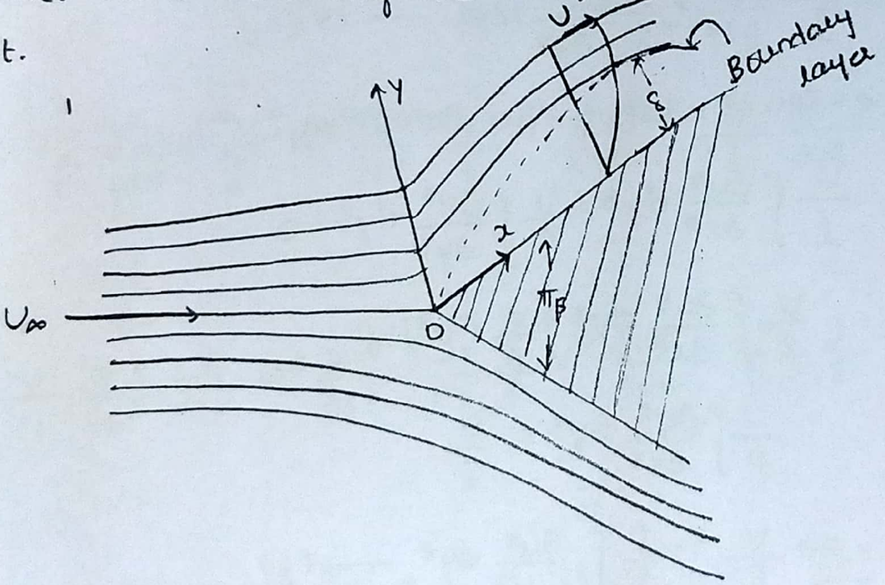
b) In the region outside of this thin boundary layer where $\bar{\Omega} = 0$, viscous forces can be completely neglected in comparison with the inertial forces:

Hence only potential flow exists.



The Boundary layer equations in two dimensional flow:-

Let us consider the flow around a wedge profile submerged in a fluid of very small viscosity. At the leading stagnation point O, the thickness of the boundary layer is zero and it grows slowly towards the rear of the wedge. The pattern of stream lines and the velocity distribution deviate only slightly from those in potential flow with the exception of the immediate vicinity of the wall. Within a very thin boundary layer of thickness δ of a large velocity gradient exists. (i.e) The velocity increases from zero at the wall to the value of potential flow at the edge of the boundary layer. If the x-axis of the co-ordinate system coincides with the wall of the wedge and the y-axis is perpendicular to it.



The Navier-Stoke's equ without body force for 2D flow are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{--- (1)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \text{--- (2)}$$

The continuity equ is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ --- (3)

From equ (1) & (2) in non-dimensional form by letting

$$\left.\begin{array}{l} x^* = \frac{x}{l}, \quad y^* = \frac{y}{\delta}, \quad u^* = \frac{u}{U}, \quad v^* = \frac{v}{V} \\ t^* = \frac{t}{l/U}, \quad p^* = \frac{p}{p_\infty} \end{array}\right\} \quad \text{--- (4)}$$

Where l, δ, U, V and p_∞ are certain reference values of the corresponding quantities x, y, u, v and p. The non-dimensional quantities are all of the order of unity.

From the continuity equ, equ (3) it follows that

$$\frac{\partial U u^*}{\partial x^* l} + \frac{\partial V v^*}{\partial y^* \delta} = 0$$

$$\Rightarrow \frac{U}{l}\frac{\partial u^*}{\partial x^*} + \frac{V}{\delta}\frac{\partial v^*}{\partial y^*} = 0 \quad \text{--- (5)}$$

Integrating equ (5) we of the condition $(v^*)_{y=1} = 1$.

$$\frac{U}{l}\int \frac{\partial u^*}{\partial x^*} dy^* + \frac{V}{\delta}\int \frac{\partial v^*}{\partial y^*} dy^* = 0$$

$$\frac{U}{l}\int \frac{\partial u^*}{\partial x^*} dy^* + \frac{V}{\delta}[v^*] = 0$$

$$\frac{U}{l}\int \frac{\partial u^*}{\partial x^*} dy^* = -\frac{V}{\delta}$$

$$\Rightarrow \frac{V}{U} = -\frac{\delta}{l}\int_0^1 \frac{\partial u^*}{\partial x^*} dy^* \quad \text{--- (6)}$$

Since the integral in equ (6) is the order of unity, the velocity ratio V/U is the order of δ/l. Hence $V \ll U$. The non-dimensional quantities form of equ (1) is obtained by inserting the non-dimensional quantities from (4) into equ (1).

$$\frac{\partial U u^*}{\partial l/U\, t^*} + U u^* \frac{\partial U u^*}{\partial x^* l} + V v^* \frac{\partial U u^*}{\partial y^* \delta} = \frac{-1}{\rho}\frac{\partial p_\infty p^*}{\partial x^* l} + \nu\left(\frac{\partial^2 u^* U}{\partial x^{*2} l^2} + \frac{\partial^2 U u^*}{\partial y^{*2}\delta^2}\right)$$

$$\frac{U^2}{l^2}\frac{\partial u^*}{\partial t^*}+\frac{U^2}{l}u^*\frac{\partial u^*}{\partial x^*}+\frac{VU}{\delta}v^*\frac{\partial v^*}{\partial y^*}=-\frac{P_\infty}{\rho l}\frac{\partial p^*}{\partial x^*}+\nu\left[\frac{U}{l^2}\left(\frac{\partial^2 u^*}{\partial x^*}\right)+\frac{U}{\delta^2}\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right)\right]$$

$$\times\frac{l}{U^2}$$

$$\Rightarrow \frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+\frac{V}{U}\frac{l}{\delta}v^*\frac{\partial v^*}{\partial y^*}=-\frac{P_\infty}{\rho U^2}\frac{\partial p^*}{\partial x^*}+\frac{\nu}{lU}\left[\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{l^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right]$$

$$\Rightarrow \frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+\frac{V}{U}\frac{l}{\delta}v^*\frac{\partial u^*}{\partial y^*}=-\frac{P_\infty}{\rho U^2}\frac{\partial p^*}{\partial x^*}+\frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{l^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right) \quad (7)$$

$1/\delta^2$

$1 \quad 1 \quad \delta \; 1/\delta \quad 1 \quad 1 \quad \delta^2 \quad 1$

Similarly, the non-dimensional form of (2) is

$$\frac{\partial v^* V}{\partial l/U\, t^*}+UV u^*\frac{\partial V v^*}{\partial x^* l}+VV^*\frac{\partial V v^*}{\partial y^*\delta}=-\frac{1}{\rho}\frac{\partial P_\infty p^*}{\partial y^*\delta}+\nu\left[\frac{\partial^2 v^* V}{\partial x^{*2} l^2}+\frac{\partial^2 v^* V}{\partial y^{*2}\delta^2}\right]$$

$$\frac{VU}{l}\frac{\partial v^*}{\partial t^*}+\frac{UV}{l}u^*\frac{\partial v^*}{\partial x^*}+\frac{V^2}{\delta}v^*\frac{\partial v^*}{\partial y^*}=-\frac{P_\infty}{\rho\delta}\frac{\partial p^*}{\partial y^*}+\nu\left[\frac{V}{l^2}\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{V}{\delta^2}\left(\frac{\partial^2 v^*}{\partial y^{*2}}\right)\right]$$

$$\times\frac{l}{U^2}$$

$$\frac{V}{U}\frac{\partial v^*}{\partial t^*}+\frac{V}{U}u^*\frac{\partial v^*}{\partial x^*}+\frac{V^2}{U^2}\frac{l}{\delta}v^*\frac{\partial v^*}{\partial y^*}=-\frac{P_\infty}{\rho U^2}\frac{l}{\delta}\frac{\partial p^*}{\partial y^*}+\nu\frac{V}{l^2}\left[\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{l^2}{\delta^2}\left(\frac{\partial^2 v^*}{\partial y^{*2}}\right)\right]$$

$$\Rightarrow \frac{V}{U}\frac{\partial v^*}{\partial t^*}+\frac{V}{U}u^*\frac{\partial v^*}{\partial x^*}+\frac{V^2}{U^2}\frac{l}{\delta}v^*\frac{\partial v^*}{\partial y^*}=-\frac{P_\infty}{\rho U^2}\frac{\partial p^*}{\partial y^*}\frac{l}{\delta}+\frac{1}{Re}\frac{V}{U}\left(\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{l^2}{\delta^2}\frac{\partial^2 v^*}{\partial y^{*2}}\right) \quad (8)$$

$\delta \quad \delta \quad \delta^2 \; 1/\delta$

$\delta^2 \quad \delta \quad 1 \quad 1/\delta^2$

Since the Reynolds number is inversely proportional to ν it is of the order $1/\delta^2$. By neglecting the terms of the order δ & smaller from equ (7) and (8) and reverting to dimensional variables, we obtain the equations.

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\ &\frac{\partial p}{\partial y} = 0 \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{aligned}\right\} \quad \text{—— (9)}$$

equ (9) are generally known as prandtl's boundary layer equations with boundary conditions.

$$y = 0 \; ; \quad u = v = 0$$

$$y = \infty \; ; \quad u = U(x, t).$$

Let U be the velocity in the potential flow (just outside the boundary layer) then the euler equation reduced from equ (9) becomes (since $\partial u/\partial y = v = 0$ at $y = 8$).

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx} \quad \text{—— (10)}$$

Introducing the expression above for dp/dx into equ (9), we have,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{—— (11)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{—— (12)}$$

The integration of equ (11) can be simplified if we can reduce the number of variables by introducing the stream function.

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{—— (13)}$$

The continuity equ, eq (12) is satisfied automatically.

equ (13). In terms of the variable ψ the boundary layer equ, equ (11) becomes

$$\frac{\partial^2 \psi}{\partial t \partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} + \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}$$

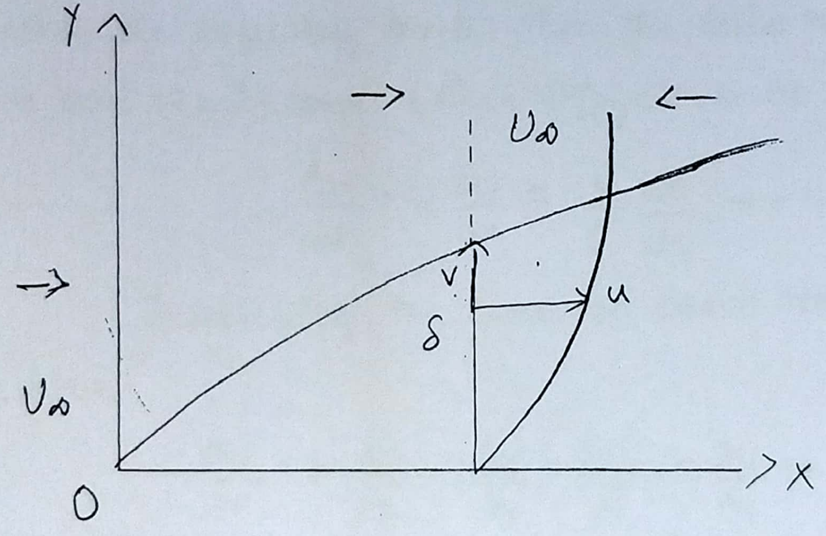
The boundary conditions are

$$y=0;\ \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0$$

$$y=\infty;\ \frac{\partial \psi}{\partial y} = U(x,t).$$

The Boundary Layer Along a flat plate & The Blasius solution:

Consider the flow of an incompressible viscous fluid past a thin flat plate which is placed in the direction of a uniform velocity U_∞. The plate is of infinite length. Hence the problem is one of two-dimensional motion which can be analyzed by using the prandtl boundary layer equations. Let the origin of the co-ordinates be at the leading edge of the plate, the x-axis be the direction of the uniform stream, and the y-axis, normal to the plate



By the prandtl boundary layer equations,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\because \frac{\partial p}{\partial x} = 0$$

It reduces to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{——— (1)}$$

By the Continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{——— (2)}$$

The boundary conditions are

$$\left.\begin{aligned} y = 0 : \quad & u = v = 0 \\ y = \infty : \quad & u = U_\infty \end{aligned}\right\} \qquad \text{——— (3)}$$

are to be satisfied by $u(x,y)$ and $v(x,y)$.

The characteristic parameters in these problems are U_∞, ν, x, y. (i.e) our flow problem is defined completely by these parameters.

By the law of similarity, the velocity profile may be assumed as ——— (4)

$$\frac{u}{U_\infty} = F(y, x, \nu, U_\infty) = F(\eta)$$

According to the exact solution of the unsteady motion of a flate plate, we found that

$$\delta \sim \sqrt{\nu t} \sim \frac{\sqrt{\nu x}}{\sqrt{U_\infty}} \qquad \text{——— (5)}$$

The time t is interpreted here as the time required for a fluid particles to travel a distance x with velocity U_∞.

Hence the non dimensional distance parameter may be expressed by

$$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x / U_\infty}} \qquad \text{——— (6)}$$

To find stream function:

$$\psi = \int u \, dy$$

$$= \int u \frac{dy}{d\eta} \, d\eta$$

$$= \frac{\sqrt{\gamma x}}{\sqrt{U_\infty}} \; U_\infty \int F(\eta)\, d\eta$$

$$\psi = \sqrt{U_\infty \gamma x} \; f(\eta) \quad \text{—— (7)}$$

$$\frac{dy}{d\eta} = \frac{d}{d\eta}\left(\eta\left(\frac{\sqrt{\gamma x}}{\sqrt{U_\infty}}\right)\right)$$

$$= \frac{\sqrt{\gamma x}}{\sqrt{U_\infty}}$$

The velocity components and their derivatives are

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \sqrt{U_\infty \gamma x} \; f'(\eta) \frac{\sqrt{U_\infty}}{\sqrt{x\gamma}}$$

$$= U_\infty f'(\eta) \quad \text{—— (8)}$$

$$\frac{\partial \psi}{\partial \eta} = \sqrt{U_\infty \gamma x} \; f'(\eta)$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{\frac{\gamma x}{U_\infty}}}$$

$$v = -\frac{\partial \psi}{\partial x}$$

$$= -\left[f(\eta) \frac{1}{2}\frac{U_\infty \gamma}{\sqrt{U_\infty \gamma x}} + \sqrt{U_\infty \gamma x}\; f'(\eta) \frac{\partial \eta}{\partial x} \right]$$

$$= -\left[f(\eta) \frac{\sqrt{U_\infty \gamma}}{2\sqrt{x}} + \sqrt{U_\infty \gamma x} + f'(\eta) \; -\frac{\eta}{2x} \right]$$

$$= \left[-f(\eta) \frac{\sqrt{U_\infty \gamma}}{2\sqrt{x}} + \sqrt{U_\infty \gamma x}\left(\frac{\eta}{2x}\right) f'(\eta) \right]$$

$$v = \frac{\sqrt{U_\infty \gamma}}{2\sqrt{x}} \left[\eta f'(\eta) - f(\eta) \right] \quad \text{—— (9)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x \partial y} = -U_\infty f''(\eta)\left(\frac{\eta}{2x}\right) \quad \text{—— (10)}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta)\sqrt{\frac{U_\infty}{\gamma x}} \quad \text{—— (11)}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty f'''(\eta)\sqrt{\frac{U_\infty}{\gamma x}} \cdot \sqrt{\frac{U_\infty}{\gamma x}}$$

$$= \frac{U_\infty^2}{\gamma x} f'''(\eta) \quad \text{—— (12)}$$

sub (8), (9), (10), (11), (12) in eq (1)

$$(U_\infty f'(\eta))\left(-U_\infty f''(\eta)\left(\frac{\eta}{2x}\right)\right) + \left(\frac{1}{2}\frac{\sqrt{\gamma U_\infty}}{\sqrt{x}}\left[\eta f'(\eta) - f(\eta)\right]\right)$$

$$\left(U_\infty \frac{\sqrt{U_\infty}}{\sqrt{\cancel{\gamma} x}} f''(\eta)\right) = \cancel{\gamma}\left(\frac{U_\infty^2}{\cancel{\gamma} x} f'''(\eta)\right)$$

$$-\frac{U_\infty^2}{2}\frac{\eta}{x} f'(\eta) f''(\eta) + \frac{U_\infty^2}{2x}\left[\eta f'(\eta) - f(\eta)\right] f''(\eta)$$

$$= \frac{U_\infty^2}{x} f'''(\eta)$$

$$-\frac{U_\infty^2}{2}\frac{\eta}{x}\cancel{f'(\eta) f''(\eta)} + \frac{U_\infty^2}{2x}\cancel{\eta f'(\eta) f''(\eta)} - \frac{U_\infty^2}{2x} f(\eta) f''(\eta)$$

$$= \frac{U_\infty^2}{x} f'''(\eta)$$

$$-\frac{\cancel{U_\infty^2}}{2\cancel{x}} f(\eta) f''(\eta) = \frac{\cancel{U_\infty^2}}{\cancel{x}} f'''(\eta)$$

$$-\frac{1}{2} f(\eta) f''(\eta) = f'''(\eta)$$

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0$$

$$2 f'''(\eta) + f(\eta) f''(\eta) = 0$$

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{\partial \eta^2} = 0 \quad \text{—— (13)}$$

The boundary condition from equation (3) in terms of f and η are 10

$$\eta = 0 \Rightarrow y = 0,\ u = v = 0 \Rightarrow U_\infty f'(\eta) = 0$$

$$f'(\eta) = 0 \text{ or } \frac{df}{d\eta} = 0$$

$$v = 0 \Rightarrow \eta f'(\eta) - f(\eta) = 0$$

$$\eta f'(\eta) = f(\eta) \qquad \text{— (14)}$$

$$f(\eta) = 0$$

$$\eta = \infty : \quad u = U_\infty \qquad U_\infty = U_\infty f'(\eta)$$

$$f'(\eta) = 1 \quad \text{or} \quad \frac{df}{d\eta} = 1$$

Equation (13) is a third-order nonlinear differential equation and no closed form solution has been found. Blasius in 1908 obtained the solution in the form of power series expansion about $\eta = 0$

$$f = A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \dots \quad \text{— (15)}$$

$$f' = A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \dots \quad \text{— (16)}$$

$$f'' = A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \dots \quad \text{— (17)}$$

$$f''' = A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \dots \quad \text{— (18)}$$

Applying the boundary conditions Eq (14) at $\eta = 0$ to Eqs 15, 16

$$15 \Rightarrow A_0 = 0.$$

$16 \Rightarrow \quad A_1 = 0 \}$ —— (19)

Substituting Eqs f, f'', f''' in eq (13), we have

$$2\left(A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \cdots\right) + \left(\frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3\right)$$

$$\left(A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \cdots\right) = 0$$

$$2A_3 + 2A_4\eta + 2\frac{A_5}{2!}\eta^2 + 2\frac{A_6}{3!}\eta^3 + \frac{A_2^2}{2!}\eta^2 + \frac{A_2A_3}{2!}\eta^3$$

$$+ \frac{A_2A_4}{2!2!}\eta^4 + \frac{A_2A_5}{3!2!}\eta^5 + \frac{A_2A_3}{3!}\eta^3 + \frac{A_3^2}{3!}\eta^4 + \frac{A_3A_4}{2!}\eta^5$$

$$+ \frac{A_3A_5}{3!3!}\eta^6 = 0.$$

$$2A_3 + (2A_4)\eta + (A_2 + 2A_5)\frac{\eta^2}{2!} + (4A_2A_3 + 2A_6)\frac{\eta^3}{3!} + \cdots = 0$$ —— (20)

In above equation (20) the co-efficients of the various powers of η must be identically equal to zero.

$$\left.\begin{aligned} A_3 = A_4 &= A_6 = A_7 = 0 \\ A_5 &= -\frac{A_2^2}{2} \\ A_8 &= 11/4 \; A_2^3 \end{aligned}\right\}$$ —— (21)

sub eq (19) and (21) into Eq (15)

$$f = \frac{A_2}{2!}\eta^2 - \frac{1}{2}\frac{A_2^2}{5!}\eta^5 + \frac{1}{4}\frac{11A_2^3}{8!} - \cdots$$ —— (22)

Eq (22) satisfies the two boundary conditions at $\eta = 0$

A_2 will determined from the boundary condition at $\eta = \infty$. (eq 14).

Now eq (22) can be rewritten in the form

$$f = A_2^{1/3}\left[\frac{(A_2^{1/3}\eta)^2}{2!} - \frac{1}{2}\frac{(A_2^{1/3}\eta)^5}{5!} + \frac{1}{4}\cdot 11\frac{(A_2^{1/3}\eta)^8}{8!}\cdots\right]$$

$$= A_2^{1/3}\, F(A_2^{1/3}\eta) \quad \text{——— (23)}$$

$$f'(\eta) = A_2^{1/3}\, A_2^{1/3}\, F'(A_2^{1/3}\eta)$$

$$= A_2^{2/3}\, F'(A_2^{1/3}\eta) \quad \text{——— (24)}$$

Applying the boundary condition (eq 14) at $\eta = \infty$ the above equation, we have

$$\lim_{\eta \to \infty}\left[A_2^{2/3}\, F'(A_2^{1/3}\eta)\right] = f'(\infty) = 1$$

$$A_2^{2/3} \lim_{\eta \to \infty} F'(A_2^{1/3}\eta) = 1$$

$$A_2^{2/3} \lim_{\eta \to \infty} F'(\eta) = 1$$

$$\therefore\ A_2^{1/3} = \left\{\lim_{\eta \to \infty} F'(\eta)\right\}^{-1/2}$$

$$\Rightarrow\ A_2^{2/3} = \left[\frac{1}{\lim_{\eta \to \infty} F'(\eta)}\right]$$

$$A_2 = \left[\frac{1}{\lim_{\eta \to \infty} F'(\eta)}\right]^{3/2}$$

The value of $A_2 = 0.33206$.

9.3

b) Shearing stress and boundary layer thickness: 12

The shearing stress on the surface of the plate can be easily calculated from the results of the Blasius solution

From eqns (1-7) & (9-29a) we have

$$\tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

(9-29a)

$$u(x,y) = U_\infty f'(\eta)$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta) \frac{\partial \eta}{\partial y}$$

Since $\left.\frac{\partial u}{\partial y} = U_\infty f''(\eta)\frac{\partial \eta}{\partial y}\right|_{\eta=0}$

$$\left[\because \eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right]$$

$$\tau_0 = \mu \times U_\infty f''(0) \frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(0) \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right)$$

$$= \mu \times U_\infty f''(0) \times \frac{1}{\sqrt{\frac{\nu x}{U_\infty}}}$$

$$\left[\because Re_x = \frac{U_\infty x}{\nu}\right]$$

$$= \frac{\rho U_\infty x}{Re_x} \times U_\infty \times 0.332 \times \frac{1}{\sqrt{\frac{\nu x}{U_\infty}}}$$

$$\nu = \frac{\mu}{\rho}$$

$$Re_x = \frac{U_\infty x}{\mu/\rho}$$

$$= \frac{\rho U_\infty^2 x \times 0.332}{Re_x \sqrt{\frac{\nu x}{U_\infty}}}$$

$$\boxed{Re_x = \frac{\rho U_\infty x}{\mu}}$$

$$\mu = \frac{\rho U_\infty x}{Re_x}$$

$$= \frac{\rho U_\infty^2 x \times 0.332}{\frac{\rho U_\infty x}{\mu} \times \sqrt{\frac{\nu x}{U_\infty}}}$$

$$\boxed{\mu = \rho\nu}$$

$$= \frac{\rho U_\infty^2 x \times 0.332}{\frac{\cancel{\rho} U_\infty x}{\cancel{\rho}\nu} \times \sqrt{\frac{\nu x}{U_\infty}}} = \frac{\rho U_\infty^2 x \times 0.332}{\frac{U_\infty x}{\nu} \times \sqrt{\frac{\nu x}{U_\infty}}}$$

$$= \frac{\rho U_\infty^2 x \times 0.332 \times \sqrt{U_\infty/\nu x}}{\frac{U_\infty x}{\nu}}$$

$$Re_x = \frac{U_\infty x}{\nu}$$

$$\sqrt{\frac{U_\infty}{\nu x}} = \sqrt{Re_x/x}$$

$$= \frac{\rho U_\infty^2 \cancel{x} \times 0.332 \times \sqrt{Re/\cancel{x}}}{Re_x}$$

$$= \frac{\rho U_\infty^2 \times 0.332 \times \sqrt{\cancel{Re_x}}}{\sqrt{Re_x}\,\sqrt{\cancel{Re_x}}}$$

$$\boxed{\tau_0 = \rho U_\infty^2 \times 0.332/\sqrt{Re_x}} \longrightarrow (1)$$

The local skin-friction coefficient (or) frictional drag coefficient is

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} \Rightarrow \frac{2\tau_0}{\rho U_\infty^2}$$

$$= \frac{2 \times \frac{0.332}{\sqrt{Re_x}}\,\rho \cancel{U_\infty^2}}{\rho \cancel{U_\infty^2}}$$

$$\boxed{C_f = \frac{0.664}{\sqrt{Re_x}}} \longrightarrow (2)$$

The total frictional force per unit width for one side of the plate of length l is given by

$$F = \int_0^l \tau_0\, dx$$

$$= \int_0^l \frac{0.332 \times \rho U_\infty^2}{\sqrt{Re_x}} \cdot dx$$

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$$= \int_0^l \frac{0.332 \times \rho U_\infty^2}{\sqrt{\frac{U_\infty x}{\nu}}} \cdot dx$$

$$= \int_0^l \frac{0.332\, \rho U_\infty^2}{\sqrt{\frac{U_\infty}{\nu}}} \cdot \frac{dx}{\sqrt{x}}$$

$x^{-1/2}$

$x^n = \frac{x^{n+1}}{n+1}$

$x^{-1/2} = \frac{x^{-1/2+1}}{-1/2+1}$

$= \frac{x^{1/2}}{1/2}$

$$= \left[\frac{0.332\, \rho U_\infty^2}{\sqrt{U_\infty/\nu}} \cdot \frac{x^{1/2}}{1/2} \right]_0^l \quad (l)^{1/2}$$

$$= \left[\frac{2 \times 0.332\, \rho U_\infty^2 \sqrt{l}}{\sqrt{\frac{U_\infty}{\nu}}} \right] = \frac{0.664\, \rho U_\infty^2}{\sqrt{\frac{U_\infty}{\nu l}}}$$

$$\boxed{F = 0.664\, \rho U_\infty^2 \sqrt{\nu l / U_\infty}} \rightarrow (3)$$

Eqn (3) indicates that the frictional force is proportional to the 3/2 power of the free-stream velocity.

The average skin friction coefficient in this case is

$$C_f = \frac{F}{\frac{1}{2}\rho U_\infty^2 l}$$

$$= \frac{0.664\, \rho U_\infty^2 \sqrt{\nu l / U_\infty}}{\frac{1}{2}\rho U_\infty^2 l}$$

$$= \frac{2 \times 0.664 \times \rho U_\infty^2 \sqrt{\nu l / U_\infty}}{\rho U_\infty^2 l}$$

$$= \frac{1.328 \sqrt{\nu / U_\infty} \cdot \cancel{\sqrt{l}}}{\sqrt{l}\ \cancel{\sqrt{l}}}$$

$$= \frac{1.328 \sqrt{\nu / U_\infty}}{\sqrt{l}}$$

$$C_f = 1.328 \times \sqrt{\frac{\nu}{U_\infty l}} = \frac{1.328}{\sqrt{\frac{U_\infty l}{\nu}}}$$

$$\boxed{C_f = \frac{1.328}{\sqrt{Re_l}}} \longrightarrow (4) \qquad \left[\because \sqrt{Re_l} = \sqrt{U_\infty l / \nu}\right]$$

In figure 9.7 a comparison between theoretical and experimental results for the local skin friction coefficient is shown. Again excellent agreement between theory and experiment is obtained.

$y = 0 ;\ u = v = 0$

$y = \infty ;\ u = U_\infty$

According to the boundary condition [eqn (9-24)] the velocity in the boundary layer does not reach the value of free stream velocity until $y \to \infty$. Hence, this theory does not give an exact prediction of the boundary layer thickness. However, at a certain finite value of η the velocity in the boundary layer asymptotically blends into the free-stream velocity. In an arbitrary limit of the boundary layer at $u = 0.9975$ is used, the thickness boundary layer can be found from

gure 9.6 to be

$[\, \delta(\eta) = 0.99$

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It is found at when $\eta \approx 5$

$$\delta = y\,(\eta = 5)$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} = y \Big/ \sqrt{\frac{\nu x}{U_\infty}}$$

$$\delta = 5 \cdot \sqrt{\frac{\nu x}{U_\infty}} \,]$$

$$\eta = \frac{y}{\delta}$$

$$\boxed{\delta = 5.64\sqrt{\nu x / U_\infty}} \rightarrow (5)$$

Since the definition of above of the boundary layer thickness is some what arbitrary a more physically meaningful definition of the thickness δ is introdues.

potential flow

$$U_\infty \delta^* = \int_0^\infty (U_\infty - u)\, dy \quad \rightarrow (6)\,(a)$$

$$\div\, U_\infty \Rightarrow \delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy \quad \rightarrow (6)\,(b)$$

Substituting the expression for u/U_∞ & η

$$[\because u = U_\infty f'(\eta),\ \eta = y \Big/ \sqrt{\nu x / U_\infty}\,]$$

$$\delta^* = \int_0^\infty [1 - f'(\eta)]\, dy$$

η-parameter

y-axis

$$[\because dy = \delta\, d\eta = \sqrt{\nu x / U_\infty}\, d\eta\,]$$

The right hand side of eqn ((6)-(a)) signifies the decrease in total flow rate caused by the action of the friction and the caused by the left hand side represents the potential flow that has been displaced from the wall (Fig 9.8). Hence to distance by which the free stream is displaced outwards due to the where δ^* is called the displacement thickness in a boundary layer, substituting the expressions for u/U_∞ and η from equations (9-29a) and

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \rightarrow (7)$$

$$\delta^* = \int_0^\infty \left[1 - f'(\eta)\right] dy \qquad \left[\because u = U_\infty f'(\eta)\right.$$

$$\eta = y / \sqrt{\nu x / U_\infty}$$

$$\left[\because dy = \delta d\eta = \sqrt{\nu x / U_\infty}\, d\eta\right]$$

$$\frac{dy}{d\eta} = \sqrt{\frac{\nu x}{U_\infty}}$$

$$\delta^* = \int_0^\infty \left[1 - f'(\eta)\right] \sqrt{\frac{\nu x}{U_\infty}}\, d\eta$$

$$\left. dy = \sqrt{\nu x / U_\infty}\, d\eta \right]$$

$$= \int_0^\infty \sqrt{\nu x / U_\infty}\, d\eta - \int_0^\infty f'(\eta) \sqrt{\nu x / U_\infty}\, d\eta$$

$$= \sqrt{\nu x / U_\infty} \int_0^\infty (1 - f'(\eta)) d\eta$$

$$= \sqrt{\nu x / U_\infty}\, \left[\eta - f(\eta)\right]$$

η at δ

[from the table we have $\eta_\delta = f(\eta)_\delta = 1.7208$

$\eta - f(\eta) = 1.7208$]

$$= \sqrt{\nu x / U_\infty}\; 1.7208$$

$$\boxed{\delta^* = 1.7208 \sqrt{\nu x / U_\infty}} \rightarrow (8)$$

$2\frac{d^3 f}{d\eta^3} + f \frac{d^2 f}{d\eta^2} = 0$

where the numerical value of $f(\eta)$ is obtained from the numerical solution of equation (9-30b) by Howarth.

In analogy to the displacement thickness a momentum thickness may be defined in accordance with the momentum law. This is established by equating the loss of momentum flow as a consequence of the wall friction in the boundary layer (R.H.S of the following eqn) to the boundary momentum flow in the absence of the boundary layer.

Momentum flow rate:

Momentum flow rate $\rho \theta U_\infty^2 = \rho \int_0^\infty u(U_\infty - u)\,dy \rightarrow (9)$

is the rate of transport of momentum across a unit area perpendicular to the direction of fluid flow

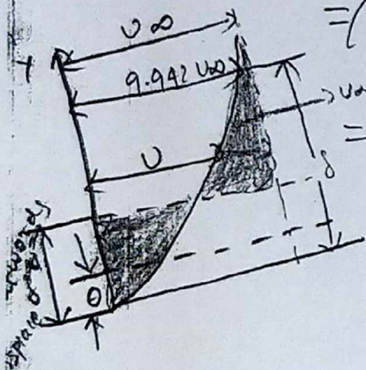
Or the momentum thickness is defined as

$$\theta = \int_0^\infty \left(\frac{uU_\infty}{U_\infty^2} - \frac{u^2}{U_\infty^2}\right)\sqrt{\frac{\nu x}{U_\infty}}\,d\eta \qquad [u = U_\infty f'(\eta)]$$

$$f'(\eta) = \frac{u}{U_\infty}$$

$$dy = \sqrt{\frac{\nu x}{U_\infty}}\,d\eta$$

$$= \sqrt{\frac{\nu x}{U_\infty}}\int_0^\infty$$



$$= \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)\sqrt{\frac{\nu x}{U_\infty}}\,d\eta$$

$$= \sqrt{\frac{\nu x}{U_\infty}}\int_0^\infty f'(\eta)\,(1 - f'(\eta))\,d\eta$$

with the expression of u/U_∞ and η in eqns (9-29a) and (9-27) we can evaluate numerically the value of θ for a flat plate as follows.

$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x/U_\infty}}$ $\qquad u = U_\infty f'(\eta)$

$$\boxed{\theta = 0.664\sqrt{\frac{\nu x}{U_\infty}}}$$

Fig 9.9 shows a comparison among various thicknesses of a boundary layer as discussed in this section.