UNIT - I

CRISP SETS AND FUZZY SETS

CRISP SETS AND FUZZY SETS:

The characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set. there by discriminating between members and non-members of the crisp set under consideration.

This function can be generalized, such that the values assigned to the elements of the Universal set fall within a specified range and indicate the membership grade of these elements in the set in question.

Larger Values:

Larger values denote higher degree of a set. such a function is called a membership function and the set is defined by a Fuzzy set

Defn:

The membership function of a Fuzzy Set A denoted by μ_A. i.e.,

$$\mu_A : X \longrightarrow [0,1]$$

i.e., for a given Universal set X, a Fuzzy set A is defined as a function

$$\mu_A : X \longrightarrow [0,1]$$

Note:

1. $\mu_A : X \longrightarrow [0,1]$ is simply denoted by

$$A : X \longrightarrow [0,1]$$

2. Degree 1 denotes the definite membership and degree 0 denotes the definite non-membership.

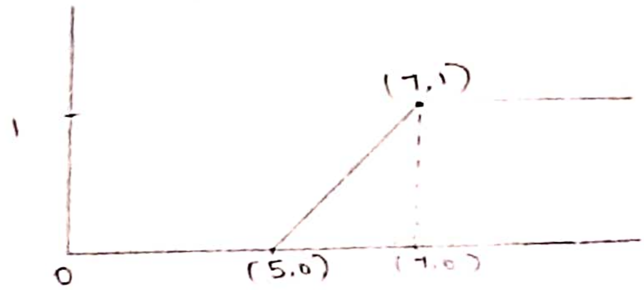
3. Fuzzy sets are used to represent vague concepts expressed in natural language.

Examples:

1. Suppose X is the set of all M.Sc. Students and A is the set of all tall Students defined over X. If more than 5 feet is considered minimum for tallness and

7 feet is considered as sufficient for tallness, then we have

$$A(x) = \begin{cases} 0, & x \text{ is a student with height } \leq 5 \text{ ft} \\ 1, & x \text{ is a student with height } \geq 7 \text{ ft} \\ \frac{|x-5|}{2}, & x \text{ is a student with height b/w 5 and 7 ft} \end{cases}$$



$(x_1, y_1) = (5, 0)$
$(x_2, y_2) = (7, 1)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1},$$

$$\frac{y-0}{1-0} = \frac{x-5}{7-5}$$

$$y = \frac{x-5}{2}$$

2. Suppose A, B, C are the sets of young, middle aged and old persons respectively, over the crisp set of human beings. Clearly, they are Fuzzy sets. They may be given as follows:

$$A(x)=\begin{cases} 1, & x \le 20 \\ \frac{35-x}{15}, & 20 < x < 35 \\ 0, & x \ge 35 \end{cases}$$

$[(20,1)\ (35,0)]$

$\frac{y-1}{0-1} = \frac{x-20}{35-20}$

$y = \frac{20-x}{15} + 1$

$y = \frac{35-x}{15}$

$$B(x)=\begin{cases} 0, & x \le 20 \text{ (or) } x \ge 60 \\ \frac{x-20}{15}, & 20 < x < 35 \quad (20,0)\ (35,1) \Rightarrow \frac{y-0}{1-0} = \frac{x-20}{35-20} \\ 1, & 35 \le x \le 45 \\ \frac{60-x}{15}, & 45 < x < 60 \quad (45,1), (60,0) \Rightarrow \frac{y-1}{0-1} = \frac{x-45}{60-45} \end{cases}$$

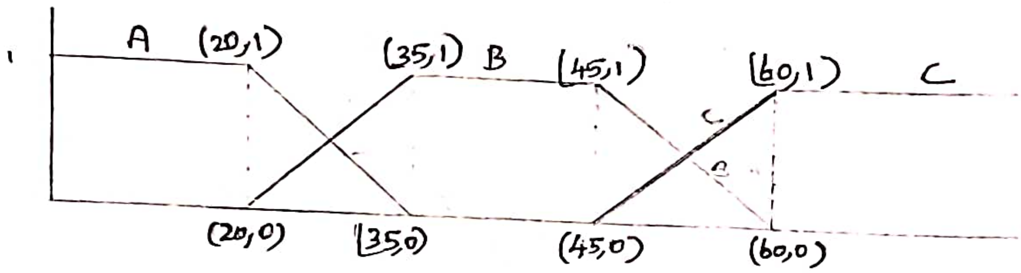
$y = \frac{x-20}{15}$

$y = \frac{45-x+15}{15}$

$y = \frac{60-x}{15}$

$$C(x)=\begin{cases} 0, & x \le 45 \\ \frac{x-45}{15}, & 45 < x \le 60 \quad (45,0).(60,1) \Rightarrow \frac{y-0}{1-0} = \frac{x-45}{60-45} \\ 1, & x \ge 60 \end{cases}$$

$y = \frac{x-45}{15}$



3. We define Fuzzy sets of real numbers which is closed to the value 2.

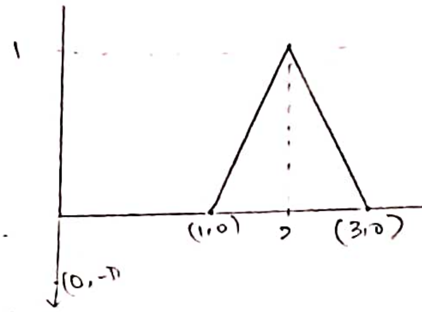
Let A_1 be a fuzzy set defined as

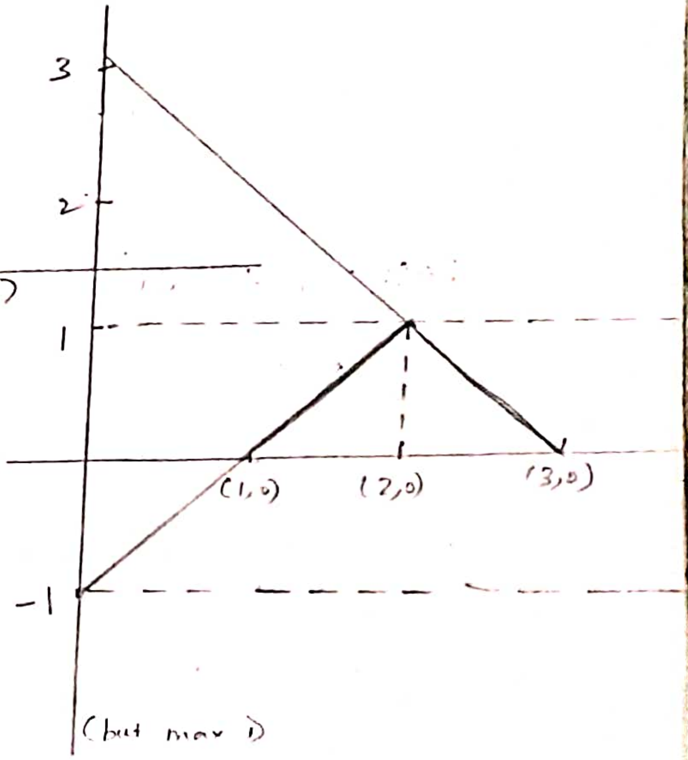
$$A_1 = \begin{cases} (x-2)+1, & \text{if } x \in [1,2] \\ (2-x)+1, & \text{if } x \in [2,3] \\ 0, & \text{otherwise} \end{cases}$$

[The general formula describing the membership function where, r denotes the real number and p_1 is a parameter that determines the rate at which each x, the function decreases with the increasing difference $|r-x|$

$$A_1(x) = \begin{cases} p_1(x-r)+1, & x \in [(r-1)/p_1, r] \\ p_1(r-x)+1, & x \in [r, (r+1)/p_1] \\ 0, & \text{otherwise} \end{cases}$$

The graph of this function is





$A_1(x) = \begin{cases} x-1, & [1,2] \\ 3-x, & [2,3] \end{cases}$

i) $y = x-1$

$x=0 \Rightarrow y=-1, \therefore (0,-1)$

$y=0 \Rightarrow x=1, \therefore (1,0)$

ii) $y = 3-x$

$x=0 \Rightarrow y=3 \quad (0,3)$

$y=0 \Rightarrow x=3, \quad (3,0)$

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4. Let the Universal Set X be a collection of ages $X = \{5, 10, 20, 30, 40, 50, 60, 70, 80\}$

We consider 4 fuzzy sets namely infant, young, adult and old

x	infant	young	adult	old
5	0	1	0	0
10	0	1	0	0
20	0	0.8	0.8	0.1
30	0	0.5	1	0.2
40	0	0.2	1	0.4
50	0	0.1	1	0.6
60	0	0	1	0.8
70	0	0	1	1
80	0	0	1	1

Defn: (α-cut) of a fuzzy set.

Let A be a fuzzy set defined on X, for any $\alpha \in [0,1]$, the sets denoted by A_α and $A_{\alpha+}$ called α-cut and strong α-cut respectively are given by

m.s.fn.

$$A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$$

$$A_{\alpha+} = \{x \in X \mid A(x) > \alpha\}$$

From example 4,
consider the fuzzy set A - young
and if $\alpha = 0.8$, then

$$A_{0.8} = \{5, 10, 20\}$$

if $\alpha = 1$, then

$$A_1 = \{5, 10\}$$

Level Set:

The set of all levels $\alpha \in [0,1]$, Let a fuzzy set A has distinct α - cuts is called level set of A and it is denoted by $\Delta(A)$

i.e., $\Delta(A) = \{\alpha \mid A(x) = \alpha$. for some $x \in X\}$

Example:

Consider a fuzzy set A - young. then

$\Delta(A) = \{1, 0.8, 0.5, 0.2, 0.1\}$

Support of a fuzzy set:

If A is a fuzzy set defined on X, then the support of A is denoted by Supp (A) or S(A) and is defined as

$$\text{Supp}(A) = \{x \mid A(x) > 0\}$$

Example:

Consider the fuzzy set from ex. 4

Supp (young) = $\{5, 10, 20, 30, 40, 50\}$

Supp (old) = $\{20, 30, 40, 50, 60, 70, 80\}$

Core of a fuzzy set.

If A is a fuzzy set defined on X, then the set A_1 (1-cut) is called the core of A

Example:

The core of a fuzzy set A is 1-cut of A

From (ex)(1),

i.e., for A - young

$$A_1 = \{5, 10\}$$

for A - adult

$$A_1 = \{30, 40, 50, 60, 70, 80\}$$

Height of A:

For a fuzzy set A, the value of largest membership is called height of A and it is denoted by $h(A)$

i.e.,
$$h(A) = \sup_{x \in X} A(x)$$

If $h(A) = 1$, then A is called Normal, Otherwise called Sub-Normal.

Cardinality of a set:

Scalar cardinality of a set:

Scalar cardinality of a fuzzy set A is defined as

$$|A| = \sum_{x \in X} A(x)$$

Example: From (ex)(4),

Consider a fuzzy set - A - young

then, the cardinality of the set is

$$|A| = 1 + 1 + 0.8 + 0.5 + 0.2 + 0.1$$

$$\therefore |A| = 3.6$$

Fuzzy cardinality of a fuzzy set:

Fuzzy cardinality of a fuzzy set A is defined by

$$|\bar{A}| = \sum \left[\frac{\alpha}{\text{no. of elements have the membership value} \geq \alpha} \right]$$

Example:

from ex: 4

$$|\text{old}| = \frac{0.1}{7} + \frac{0.2}{6} + \frac{0.4}{5} + \frac{0.6}{4} + \frac{0.8}{3} + \frac{1}{2}$$

Operations on a fuzzy set: (Standard Union, standard intersection, standard complement)

Standard Union:

If A and B be two fuzzy sets, then the standard union of A and B is defined by

$$(A \cup B)(x) = \text{Max}\,[A(x), B(x)]$$

Example: from ex: 4,

consider two fuzzy sets A - young, B - old

then $A \cup B$ = young $\cup$ old

doesn't mean addition

$$= \frac{1}{5} + \frac{1}{10} + \frac{0.8}{20} + \frac{0.5}{30} + \frac{0.4}{40} + \frac{0.6}{50} + \frac{0.8}{60} + \frac{1}{70} + \frac{1}{80}$$

$$\left\{\frac{1}{5}, \frac{1}{10}, \frac{0.8}{20}, \dots, \frac{1}{80}\right\}$$

Standard intersection:

If A and B be two fuzzy sets, then $A \cap B$ is defined by

$$(A \cap B)(x) = \text{Min}\,[A(x), B(x)]$$

Example: from ex - 4;

Consider two fuzzy sets A - young, B - old

then $A \cap B$ = young $\cap$ old

$$= \frac{0.1}{20} + \frac{0.2}{30} + \frac{0.1}{40} + \frac{0.2}{50}$$

Standard complement:

Given a fuzzy set A on X, the standard complement of A is denoted by $\bar{A}$ on X and defined as

$$\bar{A}(x) = 1 - A(x)$$

Example: from ex - 4,

$$\frac{0}{5} + \frac{0}{10} +$$

$$\text{old} = \frac{0.1}{20} + \frac{0.2}{30} + \frac{0.4}{40} + \frac{0.6}{50} + \frac{0.8}{60} + \frac{1}{70} + \frac{1}{80}$$

$$\overline{\text{old}} = \frac{1}{5} + \frac{1}{10} + \frac{0.9}{20} + \frac{0.8}{30} + \frac{0.6}{40} + \frac{0.4}{50} + \frac{0.2}{60}$$

Note:

The Law of contradiction $A \cap \bar{A} = \phi$ is valid only for Crisp sets. It is not true in the case of fuzzy sets

Example:

If $A(x) = 0.7$, then $\bar{A}(x) = 0.3$

then,
$$(A \cap \bar{A})(x) = \text{Min}\,[A(x), \bar{A}(x)]$$
$$= \text{Min}\,[0.7, 0.3]$$
$$= 0.3 \neq 0$$

Hence, the law of contradiction fails in the case of fuzzy sets.

Example - 5:

Consider two fuzzy sets A and B on X where, $X = \{a, b, c, d, e\}$ referred to as A and B

$$A = \left\{ \frac{1}{a}, \frac{0.3}{b}, \frac{0.2}{c}, \frac{0.8}{d}, \frac{0}{e} \right\}$$

and

$$B = \left\{ \frac{0.6}{a}, \frac{0.9}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0.2}{e} \right\}$$

Find Supp(A), Supp(B), core A, core B, cardinality of A [card(A)], card(B), $A \cup B$, $A \cap B$ and $\bar{A}$

Soln:

$$\text{Supp}(A) = \{x : A(x) > 0\}$$

$$\therefore \text{Supp}(A) = \{a, b, c, d\}$$

$$\text{Supp}(B) = \{a, b, c, d, e\}$$

$$\text{core}(A) = \{a\}$$

$$\text{core}(B) = \{\ \}$$

$$\text{Card}(A) = |A| = 1 + 0.3 + 0.2 + 0.8$$
$$= 2.3$$

$$\text{card}(B) = |B| = 0.6 + 0.9 + 0.1 + 0.3 + 0.2$$
$$= 2.1$$

$\underset{(x)}{A \cup B} = \text{Max}\{A(x), B(x)\}$

$\therefore A \cup B = \left\{\frac{1}{a}, \frac{0.9}{b}, \frac{0.2}{c}, \frac{0.8}{d}, \frac{0.2}{e}\right\}$

$A \cap B\,(x) = \min\,[A(x), B(x)]$

$A \cap B = \left\{\frac{0.6}{a}, \frac{0.3}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0}{e}\right\}$

$\bar{A}(x) = 1 - A(x)$

$= \left\{\frac{0}{a}, \frac{0.7}{b}, \frac{0.8}{c}, \frac{0.2}{d}, \frac{1}{e}\right\}$

Example - 6:

For $A = \left\{\frac{0.2}{a}, \frac{0.4}{b}, \frac{1}{c}, \frac{0.8}{d}, \frac{0}{e}\right\}$

and $B = \left\{\frac{0}{a}, \frac{0.9}{b}, \frac{0.3}{c}, \frac{0.2}{d}, \frac{0.1}{e}\right\}$

calculate the following

a) i) supp ii) core iii) card iv) $\bar{A}$ v) $\bar{B}$

b) $A \cup B$, $A \cap B$

c) The new set $C = A^2$, $D = 0.5B$, $E = A_{0.5}$

Soln:

Supp $(A) = \{a, b, c, d\}$

supp $(B) = \{b, c, d, e\}$

core $(A) = \{c\}$

core $(B) = \{\ \}$

$$\text{card}(A) = |A| = 0.2 + 0.4 + 1 + 0.8$$

$$= 2.4$$

$$\text{card}(B) = |B| = 0.9 + 0.3 + 0.2 + 0.1$$

$$= 1.5$$

$$\bar{A}(x) = 1 - A(x)$$

$$\therefore \quad \bar{A} = \left\{\frac{0.8}{a}, \frac{0.6}{b}, \frac{0}{c}, \frac{0.2}{d}, \frac{1}{e}\right\}$$

$$\bar{B} = \left\{\frac{1}{a}, \frac{0.1}{b}, \frac{0.7}{c}, \frac{0.8}{d}, \frac{0.9}{e}\right\}$$

b)

$$A \cup B = \left\{\frac{0.2}{a}, \frac{0.9}{b}, \frac{1}{c}, \frac{0.8}{d}, \frac{0.1}{e}\right\}$$

$$A \cap B = \left\{\frac{0}{a}, \frac{0.4}{b}, \frac{0.3}{c}, \frac{0.2}{d}, \frac{0}{e}\right\}$$

c)

$$C = A^2 = \left\{\frac{0.04}{a}, \frac{0.16}{b}, \frac{1}{c}, \frac{0.64}{d}, \frac{0}{e}\right\}$$

$$D = 0.5\,B = \left\{\frac{0}{a}, \frac{0.45}{b}, \frac{0.15}{c}, \frac{0.1}{d}, \frac{0.05}{e}\right\}$$

$$E = A_{0.5} = \{c, d\}$$

Example

Assume $X = \{-2, -1, 0, 1, 2, 3, 4\}$

and $A = \frac{0.0}{-2} + \frac{0.3}{-1} + \frac{0.6}{0}, \frac{1.0}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0.0}{4}$

Find α-cut for $0 \le \alpha \le 0.3$, $0.3 < \alpha \le 0.6$

and $0.6 < \alpha \leq 1$.

Soln:

$$A_\alpha = \begin{cases} \{-2,-1,3,4\}, & 0 \leq \alpha \leq 0.3 \\ \{0, 2\}, & 0.3 < \alpha \leq 0.6 \\ \{1\}, & 0.6 < \alpha \leq 1 \end{cases}$$

Defn:

A Fuzzy set A of a Universal set X is called Normal if $\exists$ $x \in X$ $\ni$ $A(x) = 1$

Otherwise A is sub-normal.

Defn:

A Fuzzy set $A = \phi$, i.e., empty, if its membership function is zero everywhere in its universe of discourse.

i.e., $A = \phi$ if $\mu_A(x) = 0$, for every $x \in X$

Defn: (Convexity of a fuzzy set)

A set A in $\mathbb{R}^n$ is called convex iff for every pair of points

$r = \{r_i : i = \{1, 2, \dots n\}\}$

and $s = \{s_i : i = \{1, 2, \dots n\}\}$

in A and $\forall$ real number $\lambda \in [0,1]$, the point $t = \{\lambda r_i + (1-\lambda) s_i : i = \{1, 2, \dots\}\}$ is also in A

Result:

A Fuzzy set defined on R is convex iff all its α-cuts, $\alpha > 0$ are convex.

THEOREM:

A Fuzzy set A on set of all real numbers is convex iff

$$A(\lambda x_1 + (1-\lambda) x_2) \geq \min (A(x_1), A(x_2)), \quad \forall x_1, x_2 \in R$$
$$\& \ \lambda \in [0,1]$$

Proof:

Suppose A is convex and

$$A(x_2) \geq A(x_1)$$

Let $\alpha = \min \{A(x_1), A(x_2)\}$

clearly, $x_1, x_2 \in A_\alpha$

and A_α is convex as A is convex

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in A_\alpha$

$\Rightarrow A(\lambda x_1 + (1-\lambda) x_2) \geq \alpha$

$\Rightarrow A(\lambda x_1 + (1-\lambda) x_2) \geq \min (A(x_1), A(x_2))$

Conversely, assume that

$$A(\lambda x_1 + (1-\lambda) x_2) \geq \min (A(x_1), A(x_2))$$

To prove A is convex,

Let $x_1, x_2 \in A_\alpha$

$\Rightarrow \quad A(x_1) \geq \alpha \quad$ and $\quad A(x_2) \geq \alpha$

$$\therefore A(\lambda x_1 + (1-\lambda) x_2) \geq \min(A(x_1), A(x_2))$$
$$\geq \min(\alpha, \alpha)$$
$$\geq \alpha$$

$\therefore \lambda x_1 + (1-\lambda) x_2 \in A_\alpha$

$\Rightarrow A_\alpha$ is convex

Hence A is convex.

Properties of α- cuts:

THEOREM:

set of all fuzzy sets

If A and B $\in \mathcal{P}(X)$, then for all

$\alpha, \beta \in [0,1]$, then

i) $A_{\alpha+} \subseteq A_\alpha$

ii) $\alpha \leq \beta \Rightarrow A_\beta \subseteq A_\alpha$ and $A_{\beta+} \subseteq A_{\alpha+}$

iii) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$ and $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$

iv) $(\bar{A})_\alpha = \bar{A}_{(1-\alpha)+}$

Proof:

i) Let $x \in A_{\alpha+}$

$\Rightarrow A(x) > \alpha$

$\Rightarrow A(x) \geq \alpha$

$\Rightarrow x \in A_\alpha$

$\Rightarrow$ Hence $A_{\alpha+} \subseteq A_\alpha$

ii) $\alpha \leq \beta$

Let $x \in A_\beta \Rightarrow A(x) \geq \beta$

$\Rightarrow A(x) \geq \beta \geq \alpha$

$\Rightarrow A(x) \geq \alpha$

$\Rightarrow x \in A_\alpha$

Hence $A_\beta \subseteq A_\alpha$

and

Let $x \in A_{\beta+} \Rightarrow A(x) > \beta$

$\Rightarrow A(x) > \beta > \alpha$

$\Rightarrow A(x) > \alpha$

$\Rightarrow x \in A_{\alpha+}$

Hence $A_{\beta+} \subseteq A_{\alpha+}$

iii) Let $x \in (A \cap B)_\alpha \Leftrightarrow (A \cap B)(x) \geq \alpha$

$\Leftrightarrow \min[A(x), B(x)] \geq \alpha$

$\Leftrightarrow A(x) \geq \alpha$ and $B(x) \geq \alpha$

$\Leftrightarrow x \in A_\alpha$ and $x \in B_\alpha$

$\Leftrightarrow x \in A_\alpha \cap B_\alpha$

Hence, $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$

and Let $x \in (A \cup B)_\alpha = (A \cup B)x \geq \alpha$

$= \max[A(x), B(x)] \geq \alpha$

Let $\max[A(x), B(x)] = A(x)$

$\Rightarrow \quad A(x) \geq \alpha$

$\Rightarrow \quad x \in A_\alpha$

$\Rightarrow \quad x \in A_\alpha \cup B_\alpha$

$|||^{ly}$, when $\max[A(x), B(x)] = B(x)$

$\Rightarrow \quad B(x) \geq \alpha$

$\Rightarrow \quad x \in B_\alpha$

$\Rightarrow \quad x \in A_\alpha \cup B_\alpha$

Hence, $(A \cup B)_\alpha \subseteq A_\alpha \cup B_\alpha$

conversely,

Let $x \in A_\alpha \cup B_\alpha \Rightarrow x \in A_\alpha$ or $x \in B_\alpha$

$\Rightarrow A(x) \geq \alpha$ (or) $B(x) \geq \alpha$

$\Rightarrow \text{Max}[A(x), B(x)] \geq \alpha$

$\Rightarrow (A \cup B)(x) \geq \alpha$

$\Rightarrow x \in (A \cup B)_\alpha$

Hence, $A_\alpha \cup B_\alpha \subseteq (A \cup B)_\alpha$

Hence, $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$

iv) To prove : $\bar{A}_\alpha = \bar{A}_{(1-\alpha)+}$

Let $x \in \bar{A}_\alpha \Rightarrow \bar{A}(x) \geq \alpha$

$\Rightarrow 1 - A(x) \geq \alpha$

$\Rightarrow 1 - \alpha \geq A(x)$

$\Rightarrow A(x) \leq 1 - \alpha$

$\Rightarrow \bar{A}(x) > 1 - \alpha$

$\Rightarrow x \in \bar{A}_{(1-\alpha)+}$

Hence, $\bar{A}_\alpha \subseteq \bar{A}_{(1-\alpha)+}$

Conversely,

Let $x \in \bar{A}_{(1-\alpha)+}$

$$\Rightarrow \bar{A}(x) > 1-\alpha$$

$$\Rightarrow 1 - A(x) > 1-\alpha$$

$$\Rightarrow -A(x) > -\alpha$$

$$\Rightarrow A(x) < \alpha$$

$$\Rightarrow \bar{A}(x) \geq \alpha$$

$$\Rightarrow x \in \bar{A}_\alpha$$

Hence $\bar{A}_{(1-\alpha)+} \subseteq \bar{A}_\alpha$

$$\therefore \bar{A}_\alpha = \bar{A}_{(1-\alpha)+}$$

Fuzzy complement:

A complement of a Fuzzy set A is specified by a function

$$c : [0,1] \longrightarrow [0,1]$$

which assigns a value $c(\mu_A(x))$ to each membership grade $\mu_A(x)$.

A fuzzy complement function c must satisfy atleast the following two axioms

Axiom - 1:

$c(0) = 1$ and $c(1) = 0$ [Boundary conditions]

Axiom - 2:

For all $a, b \in [0, 1]$, if $a < b$, then

$c(a) \geq c(b)$

i.e., c is monotonic non-increasing function

Axiom - 3:

c is a continuous function

Axiom - 4:

c is involutive.

i.e., $c(c(a)) = a$, $\forall a \in [0, 1]$,

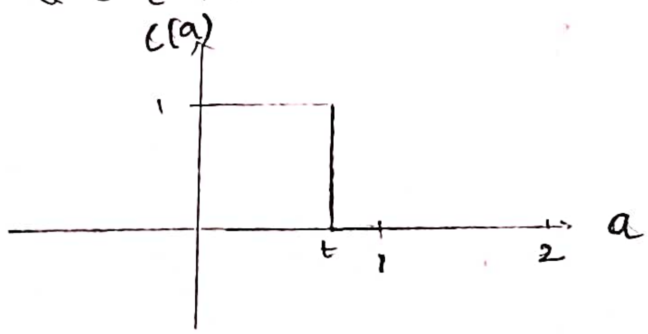
[the inverse of c is itself]

Examples of Fuzzy complement:

1.

$$c(a) = \begin{cases} 1 & ; a \leq t \\ 0 & ; a > t \end{cases}$$

where $a \in [0, 1]$ and $t \in [0, 1)$



$c(0) = 1$ and $c(1) = 0$

Hence axiom 1 satisfies.

Let $a_1 < a_2 < t$

$c(a_1) = 1$

$c(a_2) = 1$

$\therefore c(a_1) = c(a_2)$

If $a_1 = 0, \; a_2 = 1$, then $a_1 < a_2$

$c(a_1) = c(0) = 1$

$c(a_2) = c(1) = 0$

$c(a_1) > c(a_2)$

Hence axiom 2 is satisfied.

$\therefore$ the given function is a fuzzy complement.

2. $$C(a) = \frac{1}{2}\left[1 + \cos \pi a\right], \quad a \in [0,1].$$

Soln:

axiom - 1:

$c(0) = \frac{1}{2}\left[1 + \cos 0\right] = \frac{1}{2}(2) = 1$

$c(1) = \frac{1}{2}\left[1 + \cos \pi\right] = \frac{1}{2}(0) = 0$

axiom - 2:

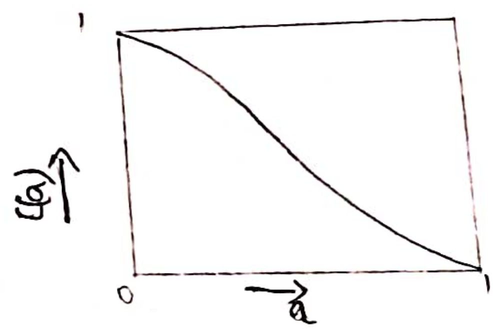
if $a = 0, \; b = 1$, i.e., $0 < 1 \; (a < b)$

then, $c(a) = c(0) = 1$

$c(b) = c(1) = 0$

$1 > 0$

i.e., $c(a) > c(b)$



axiom-3:

c is a continuous function

axiom-4:

Take $a = \frac{1}{3}$

$$c\left(\tfrac{1}{3}\right) = \frac{1}{2}\left[1 + \cos\frac{\pi}{3}\right] = \frac{1}{2}\left[1 + \frac{1}{2}\right] = \frac{3}{4}$$

$$c\left(c\left(\tfrac{1}{3}\right)\right) = \frac{1}{2}\left[1 + \cos\frac{3\pi}{4}\right] = \frac{1}{2}\left[1 - 0.707\right]$$

$c(3/4)$

$$= 0.146$$

$$\simeq 0.15 \neq 0.33 \left(\tfrac{1}{3}\right)$$

c is not involutive.

3. Sugeno class of fuzzy complements.

$$c_\lambda(a) = \frac{1-a}{1+\lambda a}, \quad \lambda \in (-1, \infty)$$

is known as sugeno class of fuzzy complement.

Axiom -1:

$$c_\lambda(0) = \frac{1-0}{1+0} = 1$$

$$c_\lambda(1) = \frac{1-1}{1+\lambda} = 0$$

$a, b \in [0,1]$

$\lambda \in (-1, \infty)$

$0.1 < 0.2$

$\lambda = 2$

$c_2(0.1) = \frac{1-0.1}{1+2(0.1)} = \frac{0.9}{1.2} = 0.75$

$c_2(0.2) = \frac{1-0.2}{1+2(0.2)} = \frac{0.8}{1.4} = 0.57$

$c_2(0.1) > c_2(0.2)$

Axiom-2:

Whenever $a < b$, $c_\lambda(a) > c_\lambda(b)$

Axiom - 3:

C_λ is a continuous function.

Axiom - 4:

$$C_\lambda(a) = \frac{1-a}{1+\lambda a}$$

$$C_\lambda(C_\lambda(a)) = \frac{1 - C_\lambda(a)}{1+\lambda . C_\lambda(a)}$$

$$= \frac{1 - \left(\frac{1-a}{1+\lambda a}\right)}{1+\lambda\left(\frac{1-a}{1+\lambda a}\right)}$$

$$= \frac{\cancel{1}+\lambda a - \cancel{1} + a}{\cancel{1+\lambda a}} \times \frac{\cancel{1+\lambda a}}{1+\cancel{\lambda a}+\lambda-\cancel{\lambda a}}$$

$$= \frac{\cancel{(\lambda+1)}\, a}{\cancel{(\lambda+1)}}$$

$\therefore C_\lambda(C_\lambda(a)) = a$

This C_λ is involutive.

This complement satisfies all the four axioms

4. Yager class of fuzzy complements:

$$C_\omega(a) = (1 - a^\omega)^{1/\omega}, \quad \omega \in (0, \infty)$$

is known as Yager class of fuzzy complement.

Axiom-1: Assume $w=1$

$$C_w(a) = c_1(a) = 1-a$$

$$\therefore c_1(0) = 1$$

$$c_1(1) = 0$$

assume $w=2$

$$c_2(a) = (1-a^2)^{1/2} = \sqrt{1-a^2}$$

$$\therefore c_2(0) = 1$$

$$c_2(1) = 0$$

Axiom-2:

Whenever $a < b$, then $c_2(a) \geq c_2(b)$

Hence c_2 is an non-increasing monotonic function.

Axiom-3:

$C_w(\cdot)$ is a continuous function

Axiom-4: $C_1(a) = 1-a$

$$C_1(C_1(a)) = C_1(1-a) = 1-(1-a) = a$$

$$c_2(a) = \sqrt{1-a^2}$$

$$c_2(c_2(a)) = \sqrt{1-[c_2(a)]^2}$$

$$= \sqrt{1-\left(\sqrt{1-a^2}\right)^2}$$

$$= \sqrt{1-1+a^2}$$

$$= a$$

Hence, c_2 is involutive

$\therefore$ This complement satisfies all the four axioms.

Equilibrium of a fuzzy complement: C:

Equilibrium of a fuzzy complement C is defined as $\underset{\text{equilibrium}}{C(a)} = a$, for $a \in [0,1]$

THEOREM - 1:

Every fuzzy complement has atmost one equilibrium.

Proof:

Let C be an arbitrary fuzzy complement.

An equilibrium of C is the soln. of the eqn.

$$C(a) = a, \quad a \in [0,1]$$

$$\Rightarrow \quad C(a) - a = 0, \quad a \in [0,1]$$

Let $C(a) - a = b$, where b is a real constant, must have atmost one soln.

Let a_1 and a_2 be two different soln.s of the equations $c(a) - a = b$

i.e., $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$

$$\Rightarrow \quad c(a_1) - a_1 = c(a_2) - a_2$$

$$\Rightarrow \quad c(a_1) - c(a_2) = a_1 - a_2$$

assume $a_1 < a_2$

$$\Rightarrow \quad a_1 - a_2 < 0$$

$$\Rightarrow \quad c(a_1) - c(a_2) < 0$$

$$\Rightarrow \quad c(a_1) < c(a_2)$$

which is a contradiction to axiom 2

[Since, c is a fuzzy complement].

$\therefore$ $c(a) - a = b$ must have atmost one soln.

In particular, $b = 0$ the eqn.

$c(a) - a = 0$ must have atmost one soln.

THEOREM_2:

Assume that a given fuzzy complement c has an equilibrium e_c, then

i) $a \leq c(a)$ iff $a \leq e_c$

ii) $a \geq c(a)$ iff $a \geq e_c$

Proof:

Given a fuzzy complement c has an equilibrium e_c.

$$\text{i.e., } c(e_c) = e_c$$

Since, c is a fuzzy complement,

case (i):

assume $a < e_c$

i.e., $c(a) \geq c(e_c)$ [from axiom - 2

$$= e_c$$

$$> a$$

$$\therefore c(a) > a$$

case (ii):

assume $a = e_c$

then, $c(a) = c(e_c)$

$$= e_c$$

$$= a$$

$$\therefore c(a) = a$$

case (iii):

assume $a > e_c$

then $c(a) \leq c(e_c)$

$$= e_c$$

$$< a$$

$$\therefore c(a) < a$$

from case (i), (ii), (iii)

$$a \leq e_c \Rightarrow a \leq c(a)$$

$$a \geq e_c \Rightarrow a \geq c(a)$$

converse can be solved in the similar manner.

THEOREM- 3:

If c is a continuous fuzzy complement, then c has an unique equilibrium

Proof:

Let e_c be the equilibrium of the fuzzy complement c. $[c(e_c) = e_c]$

i.e., e_c is the soln. of the eqn.

$$c(a) - a = 0$$

[This is the special case of the general eqn. $c(a) - a = b$, where $b \in [-1, 1]$]

by axiom - 1,

when $a = 0$

$$c(0) - 0 = 1 - 0 = 1$$

when $a = 1$,

$$c(1) - 1 = 0 - 1 = -1$$

Since, c is a continuous fuzzy complement, by intermediate value theorem

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– for continuous function, for each $b \in [-1, 1]$

$\exists$ atleast one 'a' $\ni$ C(a)

$$c(a) - a = b$$

in particular $b = 0$, $c(a) = a$ has atmost one soln.

$\therefore$ c has an unique equilibrium. [from thm 1]

Example:

Equilibrium of Sugeno class:

The Sugeno class of fuzzy complement is defined by

$$c_\lambda(a) = \frac{1-a}{1+\lambda a}, \quad \lambda \in (-1, \infty)$$

The Prove that the equilibrium of Sugeno class is

$$e_{c_\lambda} = \begin{cases} \dfrac{\sqrt{1+\lambda}-1}{\lambda}, & \lambda \neq 0 \\ 1/2, & \lambda = 0 \end{cases}$$

Proof:

From the defn. of equilibrium

$$c(a) = a$$

where, a is an equilibrium

Here, prove that

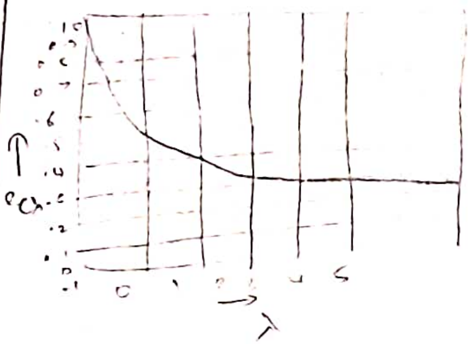
$$c_\lambda(ec_\lambda) = ec_\lambda$$

for $\lambda \neq 0$

$$c_\lambda\left(\frac{\sqrt{1+\lambda}-1}{\lambda}\right) = \frac{1-\left[\frac{\sqrt{1+\lambda}-1}{\lambda}\right]}{1+\cancel{\lambda}\left[\frac{\sqrt{1+\lambda}-1}{\cancel{\lambda}}\right]}$$

$$= \frac{\lambda - \sqrt{1+\lambda} + 1}{\lambda\sqrt{1+\lambda}}$$

$$= \frac{\frac{(1+\lambda) - \sqrt{1+\lambda}}{\sqrt{1+\lambda}}}{\lambda}$$



$$= \frac{\frac{1+\lambda}{\sqrt{1+\lambda}} - 1}{\lambda}$$

$$= \frac{\sqrt{1+\lambda}-1}{\lambda}$$

$$= ec_\lambda, \quad \text{for } \lambda \neq 0$$

for $\lambda = 0$

$$c_\lambda(ec_\lambda) = ec_\lambda$$

$$\Rightarrow c_0(ec_\lambda) = ec_\lambda$$

$$\Rightarrow 1 - ec_\lambda = ec_\lambda \quad \left[\because c_\lambda(ec_\lambda) = \frac{1-ec_\lambda}{1+\lambda ec_\lambda}\right]$$

$$\Rightarrow 2ec_\lambda = 1$$

$$\Rightarrow ec_\lambda = 1/2$$

$$\therefore \; e_{c_\lambda} = \begin{cases} \dfrac{\sqrt{1+\lambda} - 1}{\lambda}, & \lambda \neq 0 \\ 1/2, & \lambda = 0 \end{cases}$$

Dual point:

Let c be a given fuzzy complement. Let $a \in [0,1]$ be a membership grade value. The real number $^d a \in [0,1]$ such that

$$c(^d a) - {}^d a = a - c(a)$$

is called a Dual point of 'a' with respect to c.

THEOREM – 4:

If a complement c has an equilibrium e_c, then $^d e_c = e_c$

Proof:

If $a = e_c$, then by the defn. of equilibrium $c(a) = a$

$$\Rightarrow a - c(a) = 0$$

If $^d a = e_c$, then from the defn. of equilibrium $c(^d a) = {}^d a$

$$\Rightarrow \quad c(d_a) - d_a = 0$$

$$\therefore \quad c(d_a) - d_a = a - c(a) \quad \text{--- (1)}$$

(1) satisfies the condition of dual point

$$\therefore \quad a = d_a$$

$$\Rightarrow \quad a = d_a = e_c$$

$$\Rightarrow \quad d_a = e_c$$

$$\Rightarrow \quad d_{e_c} = e_c \quad [\because a = e_c].$$

THEOREM – 5:

For each $a \in [0, 1]$, $d_a = c(a)$ iff $c(c(a)) = a$

Proof:

Assume that $d_a = c(a)$

Prove that $c(c(a)) = a$

consider, the dual equation

$$c(d_a) - d_a = a - c(a) \quad \text{--- (1)}$$

$$\Rightarrow \quad c(c(a)) - \cancel{c(a)} = a - \cancel{c(a)}$$

$$\Rightarrow \quad c(c(a)) = a$$

conversely,

A.T $c(c(a)) = a$

P.T. $d_a = C(a)$

(1) $\Rightarrow$

$$c(\delta) - d_a = c(c(a)) - c(a)$$

$$\Rightarrow d_a = c(a)$$

Fuzzy union:

The union of two fuzzy sets A and B is specified by a function of the form

$$u : [0,1] \times [0,1] \longrightarrow [0,1]$$

such that

$$\mu_{A \cup B}(x) \text{ (or) } A \cup B(x) = u[A(x), B(x)]$$

The function u must satisfy the following axioms:

Axiom - u_1: (Boundary conditions)

$$u(0,0) = 0,$$

$$u(0,1) = u(1,0) = u(1,1) = 1$$

Axiom - u_2:

$$u(a,b) = u(b,a) \quad \text{[commutative}$$

Axiom - u_3:

If $a \leq a'$ and $b \leq b'$, then

$u(a,b) \le u(a', b')$ [monotonicity property

Axiom - U_4:

$$u(u(a,b), c) = u(a, u(b,c))$$

[Associative

These four axioms are called axiomatic skeleton for fuzzy set union.

Axiom - U_5:

u is a continuous function.

Axiom - U_6:

$$u(a, a) = a$$ [Idempotent

Example:

For Yager class, Fuzzy union is defined by the function

$$u_\omega(a,b) = \min\left[1, (a^\omega + b^\omega)^{1/\omega}\right]$$

where, $\omega \in (0, \infty)$

THEOREM - 6:

Prove that $\lim_{\omega \to \infty} u_\omega(a,b) = \lim_{\omega\to\infty} \min\left[1, (a^\omega + b^\omega)^{1/\omega}\right]$

$= \max(a,b)$

Proof:

case (i):

Assume $a \neq 0$ and $b = 0$ also $a < 1$

consider,

$$\lim_{\omega \to \infty} u_\omega(a,0) = \lim_{\omega \to \infty} \min\left[1, (a^\omega + 0)^{1/\omega}\right]$$

$$= \lim_{\omega \to \infty} \min[1, a]$$

$$= \lim_{\omega \to \infty} a$$

$$= a$$

$$= \max(a, 0)$$

$$= \max(a, b)$$

case (ii):

Assume $a = b$ and $a < 1$

consider,

$$\lim_{\omega \to \infty} u_\omega(a,b) = \lim_{\omega \to \infty} \min\left(1, (a^\omega + a^\omega)^{1/\omega}\right)$$

$$= \lim_{\omega \to \infty} \min\left(1, (2a^\omega)^{1/\omega}\right)$$

$$= \lim_{\omega \to \infty} \min\left(1, 2^{1/\omega} \cdot a\right)$$

$$= \min[1, a]$$

$$= a = \max(a, a)$$

$$= \max(a, b)$$

case (iii):

Assume $a \neq b$ and $a < b$

Let $\min\left[1, (a^\omega + b^\omega)^{1/\omega}\right] = (a^\omega + b^\omega)^{1/\omega} = Q$

consider,

$$\lim_{\omega \to \infty} u_\omega(a,b) = \lim_{\omega \to \infty} Q$$

$$\underset{\omega\to\infty}{lt} \log Q = \underset{\omega\to\infty}{lt} \log\left[(a^\omega + b^\omega)^{1/\omega}\right]$$

$$= \underset{\omega\to\infty}{lt} \frac{a^\omega \log a + b^\omega \log b}{a^\omega + b^\omega} \quad = \underset{\omega\to\infty}{lt} \frac{b^\omega\left(\log b + (\frac{a}{b})^\omega [illegible]\right)}{b^\omega\left(\frac{a^\omega}{b^\omega} + 1\right)}$$

$$= \underset{\omega\to\infty}{lt} \frac{\left(\frac{a}{b}\right)^\omega \log a + \log b}{1 + \left(\frac{a}{b}\right)^\omega}$$

$$\underset{\omega\to\infty}{lt} \log Q = \log b$$

$$\Rightarrow \underset{\omega\to\infty}{lt} Q = b$$

$$= \max(a,b)$$

$$\underset{\omega\to\infty}{lt} \alpha_\omega(a,b) = \underset{\omega\to\infty}{lt} \min\left[1, (a^\omega + b^\omega)^{1/\omega}\right]$$

$$= \underset{\omega\to\infty}{lt} Q$$

$$= b$$

$$= \max(a,b)$$

Case (iv):

Let $\min\left(1, (a^\omega + b^\omega)^{1/\omega}\right) = 1$

$\Rightarrow \quad (a^\omega + b^\omega)^{1/\omega} \geq 1$

$\Rightarrow \quad a^\omega + b^\omega \geq 1$

$\Rightarrow$ either $a \geq 1$ or $b \geq 1$

but $a, b \in [0, 1]$

i.e., $a \geq 1$ or $b \geq 1$

$\Rightarrow \quad a = 1$ or $b = 1$

$\Rightarrow \quad \max(a,b) = 1$

$$\lim_{\omega \to \infty} u_\omega(a,b) = \lim_{\omega \to \infty} \min\left(1, (a^\omega + b^\omega)^{1/\omega}\right) = 1 = a \text{ (or } b)$$

$$= \max(a,b)$$

Fuzzy intersection:

A fuzzy intersection of two fuzzy sets A and B is defined as the function

$$i : [0,1] \times [0,1] \longrightarrow [0,1]$$

This function i must satisfy the following axioms.

Axiom – i_1: (Boundary conditions)

$$i(1,1) = 1$$

$$i(0,1) = i(1,0) = i(0,0) = 0$$

Axiom – i_2: (commutative)

$$i(a,b) = i(b,a)$$

(i is commutative)

Axiom – i_4 (associative)

$$i\left(i(a,b), c\right) = i\left(a, i(b,c)\right)$$

(i is associative)

Axiom – i_3 (monotonicity)

If $a \le a'$ and $b \le b'$, then

$i(a,b) \le i(a',b')$ (i is monotonic)

These 4 axioms are called axiomatic Skeleton for fuzzy set intersection.

Axiom - i_5 : (continuous)

i is a continuous function.

Axiom - i_6 :

$$i(a, a) = a$$

(i is idempotent)

Example:

The fuzzy set intersection i for Yager class is defined as

$$i_\omega(a, b) = 1 - \min\left[1, \left[(1-a)^\omega + (1-b)^\omega\right]^{1/\omega}\right]$$

where, $\omega \in (0, \infty)$

THEOREM - 7 :

Prove that

$$\lim_{\omega \to \infty} i_\omega(a, b) = \underset{\omega \to \infty}{lt}\left[1 - \min\left[1, \left[(1-a)^\omega + (1-b)^\omega\right]^{1/\omega}\right]\right]$$

$$= \min(a, b)$$

Proof:

From theorem - 6, we know that

$$\underset{\omega \to \infty}{lt} \min\left[1, \left((1-a)^\omega + (1-b)^\omega\right)^{1/\omega}\right] = \max(1-a, 1-b)$$

consider,

$$\lim_{\omega \to \infty} i_\omega(a,b) = \lim_{\omega \to \infty} [1 - \max(1-a, 1-b)]$$

$$= 1 - \max(1-a, 1-b) \quad \text{—— (1)}$$

Let $a \leq b$

$\Rightarrow 1-a \geq 1-b$

$\therefore$ (1) $\Rightarrow$

$$\lim_{\omega \to \infty} i_\omega(a,b) = 1-(1-a)$$

$$= a$$

$$= \min(a,b)$$

Combination of Operators:

We define

$$u_{max}(a,b) = \begin{cases} a, & \text{if } b=0 \\ b, & \text{if } a=0 \\ 1, & \text{otherwise} \end{cases}$$

and

$$i_{min}(a,b) = \begin{cases} a, & \text{if } b=1 \\ b, & \text{if } a=1 \\ 0, & \text{otherwise} \end{cases}$$

THEOREM - 8

For all $a,b \in [0,1]$, $u(a,b) \geq \max(a,b)$

Proof:

From axiom - u_4,

$$u(u(a,b), c) = u(a, u(b,c))$$

If $b, c = 0$

$$u(u(a,0), 0) = u(a, u(0,0)) \quad \text{--- } (1)$$

from axiom - U_6,

$$u(a,a) = a \Rightarrow u(0,0) = 0$$

(1) $\Rightarrow$

$$u(a,0) = u(u(a,0), 0) \quad \text{--- } (2)$$

Assume the soln. of $u(a,0) = \alpha \neq a$ — (3)

substitute (3) in (2)

$$\alpha = u(\alpha, 0)$$

which is a contradiction to (3)

$$\therefore u(a,0) = a$$

If $\underline{\underline{b \geq 0}}$, from axiom - U_3,

$$u(a,b) \geq u(a,0)$$

$$\Rightarrow \boxed{u(a,b) \geq a} \quad \text{--- } (4)$$

from axiom - U_2,

$$u(a,b) = u(b,a)$$

If $a \geq 0$, $u(b,a) \geq u(b,0)$

$$\Rightarrow u(a,b) = u(b,a) \geq u(b,0) = b \quad \text{, if } a \geq c$$

$$\Rightarrow \boxed{u(a,b) \geq b} \quad \text{--- } (5)$$

from (4) and (5),

$$u(a,b) \geq \max(a,b)$$

THEOREM-9:

For all $a, b \in [0,1]$, $u(a,b) \leq u_{max}(a,b)$

Proof:

case (i):

When $b = 0$,

$u(a,b) = a$ [from thrm - 8]

From the defn, $u_{max}(a,b) = u_{max}(a,0) = a$

and the theorem holds $(u(a,b) = u_{max}(a,b))$

case (ii):

When $a = 0$,

$u(a,b) = b$ [from thrm - 8]

From the defn, $u_{max}(a,b) = u_{max}(0,b) = b$

and the theorem holds $(u(a,b) = u_{max}(a,b))$

case (iii):

If $b = 1$, from thrm - 8,

$u(a,1) \geq max(a,1) = 1$

If $a = 1$,

$u(1,b) \geq max(1,b) = 1$

$\therefore u(1,b) = u(b,1) = 1$ [from axiom u_2]

If $b \leq 1$, from axiom u_3, If $a \leq 1$,

$u(a,b) \leq u(a,1)$ $u(a,b)$

$= 1$ $\leq u(1,a)$

$= u(b,1)$ $= 1$

$= 1 \qquad = u_{max}(a,b)$ (from the defn)

$\therefore\ u(a,b) \leq u_{max}(a,b)$

Note: $max(a,b) \leq u(a,b) \leq u_{max}(a,b)$

THEOREM - 10:

For all $a,b \in [0,1]$, $i(a,b) \leq min(a,b)$

Proof:

from axiom - i_4,

$$i(a, i(b,c)) = i(i(a,b), c) \quad \text{--- (1)}$$

if $b = c = 1$,

(1) $\Rightarrow$

$$i(a, i(1,1)) = i(i(a,1), 1) \quad \text{--- (2)}$$

from axiom - i_6,

$$i(1,1) = 1$$

(2) $\Rightarrow$

$$i(a, 1) = i(i(a,1), 1)$$

$$\Rightarrow i(a,1) = a \quad \left[\because i(a,1) \neq a = \alpha \quad \alpha = i(\alpha, 1 \ldots \right]$$

If $b \leq 1$, from axiom - i_3,

$$i(a,b) \leq i(a,1) = a$$

$$\Rightarrow i(a,b) \leq a \quad \text{--- (3)}$$

If $a \leq 1$

from axiom - i_2, i_3,

$$i(a,b) = i(b,a) \leq i(b,1) = b$$

$$\therefore i(a,b) \leq b \quad \text{--- (4)}$$

from (3), (4)

$$i(a,b) \leq min(a,b)$$

THEOREM-11:

For all $a, b \in [0,1]$, $i(a,b) \geq i_{min}(a,b)$

Proof:

case-(i):

When $b = 1$,

$$i(a,b) = a \quad [\text{from thrm-10}]$$

and the theorem holds.

case-(ii):

When $a = 1$,

$$i(a,b) = b \quad [\text{from thrm-10}$$

and the theorem holds.

case-(iii):

If $b = 0$, from thrm-10,

$$i(a,0) \leq \min(a,0) = 0$$

If $a = 0$, from thrm-10,

$$i(0,b) \leq \min(0,b) = 0$$

$$\Rightarrow \quad i(0,b) = i(b,a) = 0$$

If $a \geq 0$, from i_3,

$$\begin{aligned} i(a,b) &\geq i(0,b) \\ &= i(a,0) \\ &= 0 \\ &= i_{min}(a,b). \end{aligned}$$

$\Rightarrow i(a,b) \geq i_{min}(a,b)$

<u>Note:</u> $i_{min}(a,b) \leq i(a,b) \leq min(a,b)$

<u>THEOREM - 12:</u>

$u(a,b) = max(a,b)$ is the only continuous and idempotent fuzzy set union.

Proof:

By Associativity,

$$u(a, u(a,b)) = u(u(a,a), b)$$

but, $u(a,a) = a$

$$\therefore u(a, u(a,b)) = u(a,b) \quad \text{--- (1)}$$

IIIrly,

$$u(u(a,b), b) = u(a, u(b,b)) = u(a,b) \quad \text{--- (2)}$$

from (1), (2)

$$u(a, u(a,b)) = u(u(a,b), b) \quad \text{--- (3)}$$
$$= u(a,b)$$

Case (i): Let $a=b$

(1) $\Rightarrow$ $u(a, u(a,a)) = u(a,a) = a$

(2) $\Rightarrow$

$$u(u(a,a), a) = u(a,a) = a$$
$$u(a,a) = max(a,a)$$

$\therefore u(a,b) = max(a,b)$, for $a=b$

Case (ii): $a < b$

Assume $u(a,b) = \alpha \neq a, b$

$\Rightarrow \quad u(a,\alpha) = \underset{=\alpha}{u(b,\alpha)}$ —— (4)

Since u is continuous and monotonic

non - increasing function

$u(0,\alpha) = \alpha$

and $u(1,\alpha) = 1$

$\therefore \exists$ a pair $a, b \in [0,1] \ni$

$u(a,\alpha) < u(b,\alpha)$

which is a $\Rightarrow\Leftarrow$ to (4)

Suppose $u(a,b) = a = \min(a,b)$ $[\because a<b]$ —— (5)

when $a=0$, $b=1$

$u(0,1) = 0$ [from (5)

but $u(0,1) = 1$ (by fact $u(a,b) = \max(a,b) = 1$

which is a $\Rightarrow\Leftarrow$

$\therefore$ $u(a,b) = \min(a,b)$ is not acceptable.

$\therefore$ $u(a,b) = \max(a,b)$, if $a<b$

i.e., $u(a,b) = b$.

when $a = 0$, $b = 1$,

$$u(0, 1) = 1$$

Finally, consider

$$u(a,b) = b = \max(a,b)$$

now all the boundary conditions are satisfied and eqn. 3 becomes

$$u(a, b) = u(b, b)$$

As $a < b$ and $u(a, b) = u(b, a) = b$
$= \max(a,b)$

case (iii): Let $a > b$

Here, $u(a,b) = \max(a,b) = a$

$$u(a,b) = u(b,a) = a = \max(a,b)$$

Hence, maximum is the only function which satisfies all the axioms u_1 to u_6.

$$\therefore u(a,b) = \max(a,b)$$

is the only fuzzy set union which is continuous and idempotent.

Example:

THEOREM - 13:

Fuzzy set operations of union, intersection and continuous complement that satisfy the law of excluded middle and law of contradiction $A \cap \bar{A} = \phi$ are neither idempotent nor distributive.

Proof:

W.K.T.

Standard fuzzy union and intersection are the only idempotent operations and they do not satisfy the law of excluded middle and the law of contradiction (refer ex)

Hence, it is enough to show that the fuzzy operations that satisfy these laws cannot be distributive.

Suppose that,

$$u(a, c(a)) = 1$$

and

$$i(a, c(a)) = 0$$

Let 'e' be an equilibrium point

then $e \neq 0$ as $c(0) = 1$ and
$c(e) = e \Rightarrow c(0) = 0$ but $c(0) = 1$ by bound.

$e \neq 1$ as $c(1) = 0$

now,

$$u(e, \overset{i(a,c(a))}{i(e,e)}) = u(e,0) = e$$

and

$$i[\overset{u(a,c(a))}{u(e,e)}, \overset{u(a,c(a))}{u(e,e)}] = i(1,1) = 1$$

$$\therefore\ u(e, i(e,e)) \neq i[u(e,e), u(e,e)]$$

again,

$$i(e, \overset{u(a,c(a))}{u(e,e)}) = i(e,1) = e$$

and

$$u[\overset{i(a,c(a))}{i(e,e)}, \overset{i(a,c(a))}{i(e,e)}] = u(0,0) = 0$$

$$\therefore\ i(e, u(e,e)) \neq u[i(e,e), i(e,e)]$$

Thus, the distributive law fails in the case of fuzzy set operations. (unions and intersection)

Note:

u_{max}, i_{min} satisfy the law of excluded middle and law of contradiction under standard complement

$$u_{max}(a, 1-a) = 1$$

$$i_{min}(a, 1-a) = 0$$

General aggregation operations :

Aggregation operation on fuzzy sets are used to combine several fuzzy sets to produce a single fuzzy set.

Defn:

An aggregation operation of n fuzzy sets $(n \geq 2)$ is represented by a function

$$h : [0, 1]^n \longrightarrow [0, 1]$$

Satisfying the following axioms.

axiom - 1: (Boundary conditions)

$$h(0, 0, \dots 0) = 0$$

$$h(1, 1, \dots 1) = 1$$

axiom - 2: (Monotonicity)

for $a_i \leq b_i$, $\forall i$

$$h(a_1, a_2, \dots a_n) \leq h(b_1, b_2, \dots b_n)$$

Note :

axiom - 3 :

h is a continuous function.

axiom - 4:

If h is such that

$$h(a_1, a_2 \dots a_n) = h[p(a_1, a_2 \dots a_n)]$$

where, p is any permutation on $a_1, a_2 \dots a_n$

then, h is said to be symmetric

THEOREM_14:

Let $h_\alpha(a_1, a_2, \dots a_n) = \left[\frac{a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha}{n}\right]^{1/\alpha}$

then, P.T. $\lim_{\alpha \to 0} h_\alpha = (a_1 a_2 \dots a_n)^{1/n}$

Proof:

Consider

$$\lim_{\alpha \to 0} \log h_\alpha = \lim_{\alpha \to 0} \log \left[\frac{a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha}{n}\right]^{1/\alpha}$$

$$= \lim_{\alpha \to 0} \frac{1}{\alpha}\left[\log(a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha) - \log n\right]$$

using L' hospital rule

$\frac{d}{dx}(\log x) = \frac{1}{x}$

$\frac{d}{dx}(a^x) = a^x \log a$

$$= \lim_{\alpha \to 0} \frac{a_1^\alpha \log a_1 + a_2^\alpha \log a_2 + \dots + a_n^\alpha \log a_n}{a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha}$$

$$= \frac{\log a_1 + \log a_2 + \dots + \log a_n}{n}$$

$$= \frac{\log(a_1 a_2 \dots a_n)}{n}$$

$$= \frac{1}{n} \log(a_1 a_2 \dots a_n)$$

$$= \log(a_1 a_2 \dots a_n)^{1/n}$$

taking exp. on both sides

$$\lim_{\alpha \to 0} h_\alpha = (a_1 a_2 \dots a_n)^{1/n}$$

Fuzzy set operations union and intersection satisfies Demorgan's law under the complement c.

i.e, $c(u(a,b)) = i(c(a), c(b))$

$$c(i(a,b)) = u(c(a), c(b))$$

Proof:

W.K.T.

$$u(a,b) = \max(a,b)$$

$$i(a,b) = \min(a,b)$$

Let $c(a) = 1-a$

now,

$$c(u(a,b)) = c(\max(a,b))$$

Let $a<b$, then

$$c(u(a,b)) = c(b)$$

$$= 1-b \quad \text{—— (1)}$$

$$i(c(a), c(b)) = i(1-a, 1-b)$$

$$= \min(1-a, 1-b)$$

$$= 1-b \quad \text{—— (2)}$$

from (1), (2)

$$c(u(a,b)) = i(c(a), c(b))$$

Let $a < b$.

$\therefore$ P.T $c(i(a,b)) = u(c(a), c(b))$

$$c(i(a,b)) = c(\min(a,b))$$
$$= c(a)$$
$$= 1-a \quad \text{—— } (3)$$

$$u(c(a), c(b)) = u(1-a, 1-b)$$
$$= \max(1-a, 1-b)$$
$$= 1-a \quad \text{—— } (4)$$

from (3), (4)

$$c(i(a,b)) = u(c(a), c(b))$$