UNIT – II

FUZZY RELATIONS

Defn:

If X, Y are any two crisp sets, then their cartesian product denoted by

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

$$X_1 \times X_2 \times \dots \times X_n = \{(x_1, x_2, \dots x_n) \mid x_i \in X_i,\ i = 1, 2, \dots n\}$$

Relation on Crisp Set:

Any subset of $X_1 \times X_2 \times \dots \times X_n$ is called a relation along the crisp sets $X_1, X_2, \dots X_n$.

A relation is called Binary if the no. of sets involved is 2.

~~A~~ The relation is called Ternary if the no. of sets involved is 3.

A relation is called Quartenary if the no. of sets involved is 4.

A relation is called Quinary if the no. of sets involved is 5.

Defn:

A Fuzzy set defined on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is called the Fuzzy relation if

$$R(x_1, x_2 \dots x_n) \subset X_1 \times X_2 \times \cdots \times X_n$$

Example: 1

Let R be a fuzzy relation between two sets of cities $X = \{A, B\}$, $Y = \{C, D, E\}$ that represent the relational concept very far. The relation is given as

$$R(X, Y) = \begin{array}{c|cc} & A & B \\ \hline C & 1 & 0.9 \\ D & 0 & 0.7 \\ E & 0.6 & 0.3 \end{array}$$

$$\simeq \frac{1}{A,C} + \frac{0.9}{B,C} + \frac{0}{A,D} + \frac{0.7}{B,D} + \frac{0.6}{A,E} + \frac{0.3}{B,E}$$

Defn:

Given a relation $R(x_1, x_2, \dots x_n)$. Let $[R \downarrow Y]$ denotes the projection of R, Thus Disregards all variables in X.

i.e., $Y = \{x_j \mid j \in J \subset N_n\}$

then, $[R \downarrow Y]$ is a fuzzy set relation whose membership is defined on Y as

$$\mu_{[R \downarrow Y]}(y) = \max \mu_R(x)$$

Example: 2

Let $X_1 = \{x, y\}$

$X_2 = \{a, b\}$

$X_3 = \{*, \$\}$

$$R(X_1, X_2, X_3) = \frac{0.9}{x,a,*} + \frac{0.4}{x,b,*} + \frac{1}{y,a,*} + \frac{0.7}{y,a,\$} + \frac{0.8}{y,b,\$}$$

Consider the projection,

$$R_{ij} = [R \downarrow (x_i, x_j)]$$

$$R_i = [R \downarrow x_i], \quad i,j \in N_3$$

now,

$$R_{1,2} = \frac{0.9}{x,a} + \frac{0.4}{x,b} + \frac{1}{y,a} + \frac{0.8}{y,b}$$

$$R_{1,3} = \frac{0.9}{x,*} + \frac{0}{x,\$} + \frac{1}{y,*} + \frac{0.8}{y,\$}$$

$$R_{2,3} = \frac{1}{a,*} + \frac{0.7}{a,\$} + \frac{0.4}{b,*} + \frac{0.8}{b,\$}$$

and

$$R_1 = \frac{0.9}{x} + \frac{1}{y}$$

$$R_2 = \frac{1}{a} + \frac{0.8}{b}$$

$$R_3 = \frac{1}{*} + \frac{0.8}{\$}$$

Cylindric extensions and Cylindric closure:

Defn: (Cylindric extension)

As a inverse process to the projection the cylindric extension of R into the sets X_i, there are in X but not in Y. denoted by $R \uparrow X-Y$ is represented by the membership function $\mu_{[R \uparrow X-Y]}(x) = \mu_R(y)$, $x > y$

Cylindric closure:

Relations that are reconstructed from one of their projection never ~~go~~ coincide with the original one. However, the reconstruction can be made with the more precession by taking the intersections of the cylindrical extensions of the projections. The resulting relation is called cylindrical closure. It is represented by means of the membership function as follows.

$$\mu_{cyl(R_i)}(x) = \text{Min}\ \mu_{(R \uparrow (x-y))}(x)$$

Example: 3

For the projections determined in the example 2, the cylindrical extensions are represented by means of membership values as below.

$R(x_1, x_2, x_3)$	R_{12}	R_{13}	R_{23}	R_1	R_2	R_3
(x, a, *)	0.9	0.9	1	0.9	1	1
(x, a, #)	0.9	0	0.7	0.9	1	0.8
(x, b, *)	0.4	0.9	0.4	0.9	0.8	1
(x, b, #)	0.4	0	0.8	0.9	0.8	0.8
(y, a, *)	1	1	1	1	1	1
(y, a, #)	1	0.8	0.7	1	1	0.8
(y, b, *)	0.8	1	0.4	1	0.8	1
(y, b, #)	0.8	0.8	0.8	1	0.8	0.8

The cylindrical closure $cyl(R_{12}, R_{13}, R_{23})$, $cyl(R_1, R_2, R_3)$ and $cyl(R_{12}, R_3)$ are as follows:

$R(x_1, x_2, x_3)$	$cyl(R_{12}, R_{13}, R_{23})$	$cyl(R_1, R_2, R_3)$	$cyl(R_{12}, R_3)$
$(x, a, *)$	0.9	0.9	0.9
$(x, a, \$)$	0	0.8	0.8
$(x, b, *)$	0.4	0.8	0.4
$(x, b, \$)$	0	0.8	0.4
$(y, a, *)$	1	1	1
$(y, a, \$)$	0.7	0.8	0.8
$(y, b, *)$	0.4	0.8	0.8
$(y, b, \$)$	0.8	0.8	0.8

Binary Relation:

A relation defined for two sets or ~~for~~ two elements is called a binary relation and it is denoted by $R(X, Y)$.

If $X = Y$, then R is called <u>Digraph</u>. If $X \neq Y$, then R is called <u>Bipartite Graph</u>.

A Binary relation can be represented through membership matrix or Sagittal diagram.

Example: 4

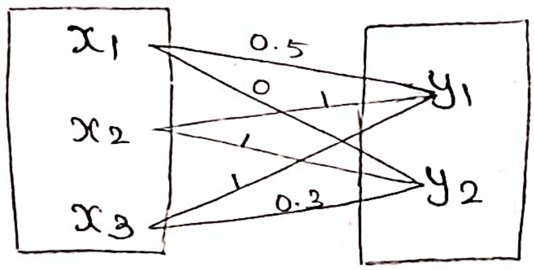
Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$

$$R(X,Y) = \frac{0.5}{x_1,y_1} + \frac{0}{x_1,y_2} + \frac{1}{x_2,y_1} + \frac{1}{x_2,y_2} + \frac{1}{x_3,y_1} + \frac{0.3}{x_3,y_2}$$

Matrix Representation:

$$M_{R(X,Y)} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0 \\ 1 & 1 \\ 1 & 0.3 \end{bmatrix} \end{array}$$

Sagittal Diagram:



Domain of a Crisp Binary Relation:

If $R(X,Y)$ is a fuzzy relation, then its domain is the Fuzzy set

$$\text{dom } R(X,Y) = \left\{ x \in X \;/\; (x,y) \in R(X,Y), \text{ for some } y \in Y \right\}$$

with membership function

$$\mu_{\text{dom } R} = \max_{y \in Y} \mu_R(x,y)$$

Range :

$$\text{ran } R(X,Y) = \left\{ y \in Y \;|\; (x,y) \in R(X,Y), \text{ for } \cancel{\text{some}}\ \text{all } x \in X \right\}$$

with membership function

$$\mu_{\text{ran } R} = \max_{x \in X} \mu_{R(x,y)}$$

Resolution of a fuzzy relation :

~~Res~~ Every fuzzy relation $R(X,Y)$ can be represented as

$$R = \bigcup_{\alpha \in \Lambda(R)} \alpha R_\alpha .$$

where, αR_α is defined by

$$\mu_{\alpha R_\alpha}(x,y) = \alpha \, \mu_{R_\alpha(x,y)}$$

Example : ~~5~~

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0.7 & 0.4 & 0 \\ 0.9 & 1 & 0.4 \\ 0 & 0.7 & 1 \\ 0.7 & 0.9 & 0 \end{bmatrix} \end{array}$$

$\Lambda_R = \{0, 0.4, 0.7, 0.9, 1\}$

when $\alpha = 0$

$$0 \cdot R_0 = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

α cut defn
if the values $\in$ mat
$\alpha \geq 0.4$
put 1
if < 0.4
put 0

When $\alpha = 0.4$

$$R_{0.4} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$0.4 \cdot R_{0.4} = \begin{bmatrix} 0.4 & 0.4 & 0 \\ 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0 \end{bmatrix} \rightarrow$$

find min b/w
0.4 & $R_{0.4}$
mat.

When $\alpha = 0.7$

$$R_{0.7} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$0.7 \; R_{0.7} = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.7 & 0.7 & 0 \\ 0 & 0.7 & 0.7 \\ 0.7 & 0.7 & 0 \end{bmatrix}$$

when $\alpha = 0.9$

$$R_{0.9} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$0.9\, R_{0.9} = \begin{bmatrix} 0 & 0 & 0 \\ 0.9 & 0.9 & 0 \\ 0 & 0 & 0.9 \\ 0 & 0.9 & 0 \end{bmatrix}$$

when $\alpha = 1$

$$R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$1 \cdot R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R = 0\, R_0 \cup 0.4\, R_{0.4} \cup 0.7\, R_{0.7} \cup 0.9\, R_{0.9} \cup 1\, R_1$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0.4 & 0.4 & 0 \\ 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0 \end{bmatrix} \cup \begin{bmatrix} 0.7 & 0 & 0 \\ 0.7 & 0.7 & 0 \\ 0 & 0.7 & 0.7 \\ 0.7 & 0.7 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 \\ 0.9 & 0.9 & 0 \\ 0 & 0 & 0.9 \\ 0 & 0.9 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.4 & 0 \\ 0.9 & 1 & 0.4 \\ 0 & 0.7 & 1 \\ 0.7 & 0.9 & 0 \end{bmatrix}$$

Inverse Fuzzy relation: $(R^{-1}(x,y))$

$$R^{-1}(x,y) = \mu_{R^{-1}}(y,x)$$

$R^{-1}(x,y)$ is the transpose of the matrix M_R for the relation $R(x,y)$.

Example:

Let $X = \{x, y, z\}$

$Y = \{a, b\}$

Let $R(X, Y)$ be a fuzzy relation such that

$$M_R = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.3 & 0.2 \\ 0 & 1 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

$$\therefore M_R^{-1} = M_R^{T} = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0.3 & 0 & 0.6 \\ 0.2 & 1 & 0.4 \end{bmatrix} \end{array}$$

Composition of two fuzzy relations:

If $P(X, Y)$ and $Q(X, Z)$ are any two fuzzy relations, then the composition $P.Q$ denoted by $R(X, Z)$ is given by

$$R(X, Z) = P(X, Y) \cdot Q(Y, Z)$$

Example:

Let $P(X, Y)$ and $Q(Y, Z)$ ~~are~~ be two fuzzy relations given by

$$M_P = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0.2 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix}$$

and $$M_Q = \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0. & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix}$$

We define ~~two~~ composition operators

(i) $\odot$ $\circ$

$$R(x,z) = P(x,y) \circ Q(y,z)$$
$$= \max\min [P(x,y), Q(y,z)]$$

$$M_{P \circ Q} = M_P \cdot M_Q$$

$$= \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}$$

min, min, min → max

Note:

$$P(x,y) \circ Q(y,z) \neq Q(y,z) \circ P(x,y)$$

Defn: (Relational Join of $P(x,y)$ and $Q(y,z)$ is $P * Q$.

$$P * Q = \{(x,y,z) \mid (x,y) \in P, (y,z) \in Q\}$$

Note:

(The relational join $P * Q$ corresponding to the standard maximin composition is a Ternary relation $R(x,y,z)$ defined by

$$R(x,y,z) = [P*Q](x,y,z)$$
$$= \min[P(x,y), Q(y,z)]) \quad \forall \; x \in X, \; y \in Y, \; z \in Z$$

The fact that the relational join produces a ternary relation from two binary relations is a major difference from the composition, which results in another binary relation. In fact, the maximin composition is obtained by aggregating appropriate elements of the corresponding join by the maximum operator.

Formally,

$$[P \circ Q](x,z) = \max_{y \in Y} [P * Q](x,y,z), \quad \forall \, x \in X, \; z \in Z$$

Example:

The relations $P(X,Y)$ and $Q(Y,Z)$ is given by

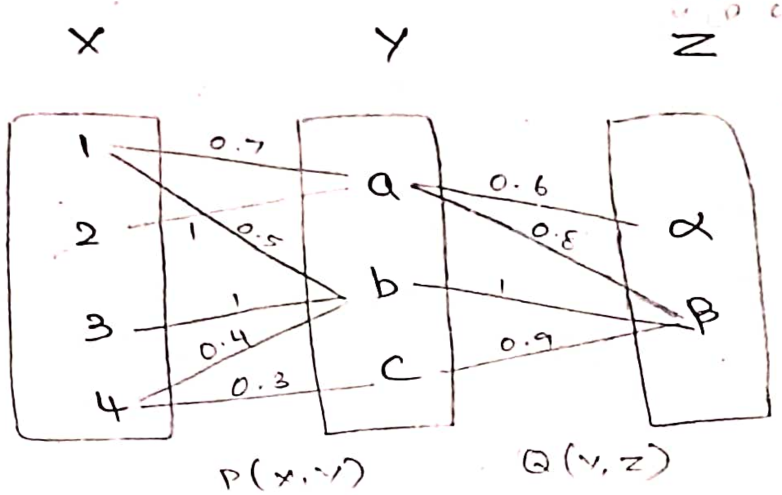
$$M_P = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0.7 & 0.5 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0.4 & 0.3 \end{array}$$

$$M_Q = \begin{array}{c|cc} & \alpha & \beta \\ \hline a & 0.6 & 0.8 \\ b & 0 & 1 \\ c & 0 & 0.9 \end{array}$$

$$M_{P \circ Q} = \begin{bmatrix} 0.7 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 \\ 0 & 1 \\ 0 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.7 \\ 0.6 & 0.8 \\ 0 & 1 \\ 0 & 0.4 \end{bmatrix}_{4\times 2}$$

Note:



Join $P * Q$

x	y	z	μ(x,y,z)
1	a	α	0.6
1	a	β	0.7
1	b	β	0.5
2	a	α	0.6
2	a	β	0.8
3	b	β	1
4	b	β	0.4
4	c	β	0.3

composition $P \circ Q$

x	z	μ(x,z)
1	α	0.6
1	β	0.7
2	α	0.6
2	β	0.8
3	β	1
4	β	0.4

Binary relations: on a single set X:

Defns:

Reflexive:

The fuzzy relation $R(x,x)$ is reflexive iff $\mu_R(x,x) = 1$, for every $x \in X$

Otherwise it is irreflexive.

If it is not satisfied for all $x \in X$, then it is called anti-reflexive.

$\in$ - Reflexive:

(A weaker form of reflexivity)

$\in$ - Reflexivity is defined by

$$\mu_R(x,x) \geq \in, \quad 0 < \in < 1$$

Symmetric:

A fuzzy relation is symmetric iff

$$\mu_R(x,y) = \mu_R(y,x), \quad \forall\, x,y \in X$$

If this equality is not satisfied for some $(x,y) \in X$, the relation is called asymmetric.

Further when $\mu_R(x,y) > 0$ and $\mu_R(y,x) > 0$ implies $x = y$, $\forall\, x,y \in X$

Then the relation is called anti-symmetry.

Defn

A fuzzy relation $R(x, y)$ is transitive iff $\mu_R(x,z) \geq \max\min[\mu_R(x,y), \mu_R(y,z)]$, $\forall\, x, y, z \in X$

A relation is non-transitive if $\mu_R(x,z) < \max\min[\mu_R(x,y), \mu_R(y,z)]$, $\forall\, x, y, z \in X$

Example.

Let R be a fuzzy relation defined on the set of cities and representing the concept very near.

	A	B	C
A	1	0.7	0.5
B	0.7	1	0.8
C	0.5	0.8	1

Reflexive:

$\mu_R(A, A) = 1$

$\mu_R(B, B) = 1$

$\mu_R(C, C) = 1$

Hence, R is reflexive.

Symmetric:

$\mu_R(A, B) = \mu(B, A) = 0.7$

$$\mu_R(A, C) = \mu_R(C, A) = 0.5$$

$$\mu_R(B, C) = \mu_R(C, B) = 0.8$$

Hence, R is symmetric.

Transitive:

$R = P \circ Q$

$$\mu_R(A, B) = 0.7$$

$$\mu_R(B, C) = 0.8$$

$$\mu_R(A, C) = 0.5$$

$\mu_R(1, \beta) = \max [\min(0.7, 0.8), \min(0.5, 1)]$

$= \max(0.7, 0.5)$

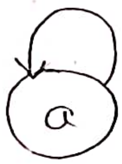
$\mu_R(A, C) < \max \min [\mu_R(A, B), \mu_R(B, C)] = 0.7$

$0.5 < \max [0.7, 0.8]$

$0.5 < 0.8$

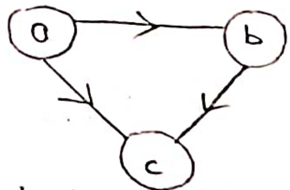
$\therefore$ R is non-transitive.

Note:



Reflexive

Symmetric



transitive

Transitive closure:

For a fuzzy relation $R(x, x)$, a transitive closure $R_T(x, x)$ can be determined by a simple algorithm that consisting of three steps:

Step -1:

$$R' = R \cup (R \circ R)$$

Step -2:

If $R' \neq R$, make $R = R'$ and go to Step -1;

Step -3:

Stop when $R' = R_T$ (when $R' = R$)

Example:

Determine the transitive closure for a fuzzy relation $R(x, x)$ given by

$$M_R = \begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$$

$$M_{R \circ R} = \begin{bmatrix} 0.7 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.4 \\ 0 & 0.4 & 0 & 0 \end{bmatrix}$$

$$M_R' = M_R \cup (R \circ R) = \begin{bmatrix} 0.7 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.8 & 0 \end{bmatrix}$$

Take $R' = R = \begin{bmatrix} 0.7 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.8 & 0 \end{bmatrix}$

$$M_{R\circ R} = \begin{pmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \end{pmatrix}$$

$$M_{R \cup (R\circ R)} = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 1 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.8 & 0.4 \end{bmatrix} = M_R'$$

again, take $R' = R = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 1 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.8 & 0.4 \end{bmatrix}$

$$M_{R\circ R} = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \end{bmatrix}$$

$$M_{R \cup (R\circ R)} = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 1 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.8 & 0.4 \end{bmatrix} = M_R$$

Hence $R' = R$

Equivalence and similarity relations:

A Crisp binary relation $R(x, X)$ which is reflexive, symmetric and transitive is called an equivalence relation.

Example:

$$X = \{1, 2, \ldots, 10\}$$

$$R(X, X) = \{(x, y) \mid x \text{ and } y \text{ have the same remainder when divided by } 3\}$$

$A_1 = \{1, 4, 7, 10\}$, remainder 1

$A_2 = \{2, 5, 8\}$, remainder 2

$A_3 = \{3, 6, 9\}$, remainder 0

So we have 3 equivalence classes.

Similarity relation:

A fuzzy binary relation, that is reflexive, symmetric and transitive is known as a similarity relation.

Any fuzzy relation can be represented by a resolution form

$$R = \bigcup_{\alpha \in \Lambda_\alpha} \alpha R_\alpha$$

where, Λ_R is the level set.

If the relation represented in this way is a similarity relation, then each α-cut R_α is an equivalence relation.

Let πR_α denote the partition corresponding to the equivalence relation R_α

If $x, y \in \Pi R_\beta \Leftrightarrow \mu R(x,y) = \beta$

ΠR_α is an refinement of ΠR_β iff $\alpha \geq \beta$.

Example:

The fuzzy relation $R(X, X)$ on $X = \{a, b, c, d, e, f, g\}$ represented by the membership matrix

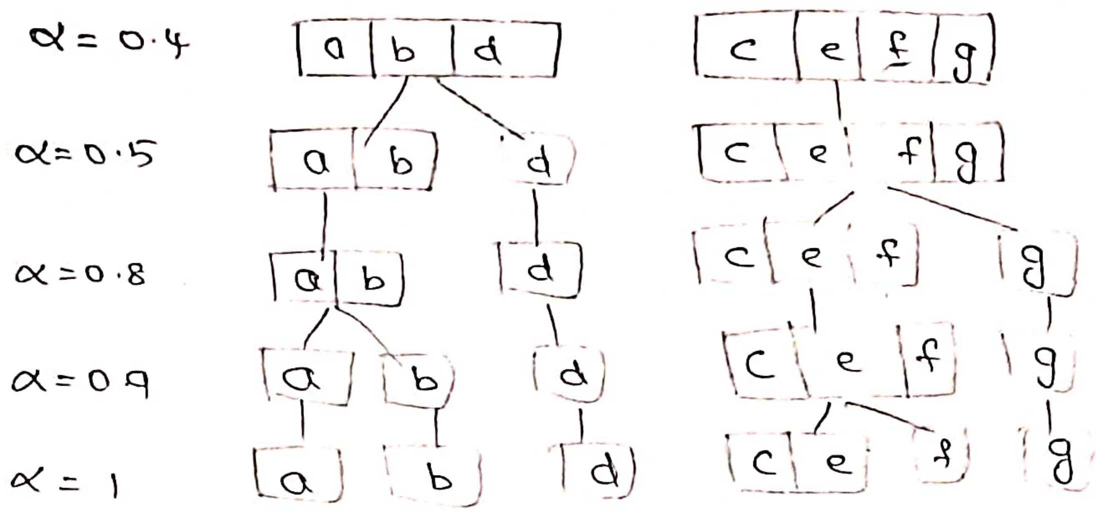
$$
\begin{array}{c c}
 & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} &
\left[\begin{array}{ccccccc}
1 & 0.8 & 0 & 0.4 & 0 & 0 & 0 \\
0.8 & 1 & 0 & 0.4 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0.9 & 0.5 \\
0.4 & 0.4 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0.9 & 0.5 \\
0 & 0 & 0.9 & 0 & 0.9 & 1 & 0.5 \\
0 & 0 & 0.5 & 0 & 0.5 & 0.5 & 1
\end{array}\right]
\end{array}
$$

The Level set of the relation

{membership values}

$$\Lambda_R = \{0, 0.4, 0.5, 0.8, 0.9, 1\}$$

The refinement relationship can be conveniently diagrammed by a partition tree.



similarity class of a is given by

$$\mu_a = \frac{1}{a} + \frac{0.8}{b} + \frac{0}{c} + \frac{0.4}{d} + \frac{0}{e} + \frac{0}{f} + \frac{0}{g}$$

III'ly, the other similarity classes may be given.

Compatibility: (Tolerence relations)

Defn-1 A binary relation $R(x, x)$ is called compatible if it is reflexive and symmetric. A fuzzy compatible relation is also called <u>proximity relation</u>.

Defn:-2:

Given a Crisp compatible relation $R(x, x)$, a subset A of X is called compatibility class if

$$\{x, y \in A \Rightarrow (x, y) \in R$$

i.e., $\{x R y \mid \forall x, y \in A\}$

Defn. Maximum compatibility class:

A compatibility class which is not properly contained in any other compatibility class is called maximal compatibility class or maximum comptabile.

Defn. 4 Complete Cover:

The family of all maximal compatibiles induced by R on X is called complete cover of X with respect to R.

Defn. 5 α - compatibility class:

If R is a fuzzy compatibility relation, then α- compatibility class denoted by A_α and is defined as

$$x, y \in A_\alpha \Rightarrow \mu_R(x, y) \geq \alpha)$$

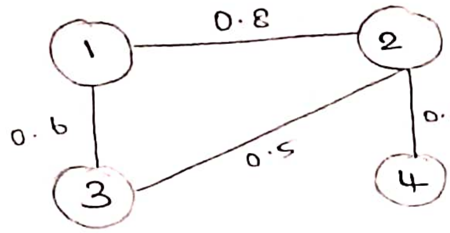
Remark:

1. Compatibility relations are represented by reflexive undirected graphs

Reflexive – Each node is assumed to have loops.

Example:

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 0.8 & 0.6 & 0 \\
2 & 0.8 & 1 & 0.5 & 0.4 \\
3 & 0.6 & 0.5 & 1 & 0 \\
4 & 0 & 0.4 & 0 & 1
\end{array}
$$



2. The complete α-covers of $R(X,X)$ in general need not be a partition of X.

3. The transitive closure of compatibility relations are similarity relations.

Fuzzy ordering relations:

A Binary relation $R(X,X)$ is called Partial ordering iff it is reflexive, anti-symmetric and transitive.

A fuzzy binary relation $R(X,X)$ is called fuzzy partial ordering iff it is reflexive, anti-symmetric and transitive.

Some basic concepts of Partial ordering.

Relation:

i) A partial order is denoted by "$\leq$".

ii) A set with partially ordered relation is called PO set

iii) If $x \leq y$, x is called <u>predecessor</u> of y, y is called <u>successor</u> of x.

If $x \leq y$ and there exist a number $z \ni x \leq z \leq y$, then x is called <u>immediate predecessor</u> of y and y is called <u>immediate sucessor</u> of x.

iv) If neither $x \leq y$ nor $y \leq x$, then x and y are said to be <u>non-comparable</u>. Otherwise comparable.

If for any two x, y either $x \leq y$ or $y \leq x$, then the partial ordering is said to be <u>connected</u>. In such cases, X is called a <u>chain</u> or <u>linearly ordered set</u>.

v) If x is such that $x \leq y$, for every y, then x is called the first member (minimum or least).

If x is such that $y \leq x$, for every y, then x is called the last member (maximum, greatest)

vi) If x has no predecessor, then x is called minimal member.

If x has no successor, then x is called maximal member.

vi) If $A \subseteq X$, then x is such that $x \leq y$, $\forall\, y \in A$ is called a lower bound of A. Similarly x is such that $y \leq x$, $\forall\, y \in A$ is called an upper bound of A.

If x is a lower bound and $x' \leq x$, for every lower bound x' of A, then x is called least upper bound of A.

vii) A set X with a partial order '$\leq$' is called a Lattice if every subset of two elements of X has a glb and lub.

viii) POSET's are represented by Hasse diagram. $x \leq y$ is denoted by arrow between the nodes x and y

Example : 1:

Let $X = R$ and $A = (0,1)$

$\therefore A \subseteq R$

$lub(A) = 1$

$glb(A) = 0$

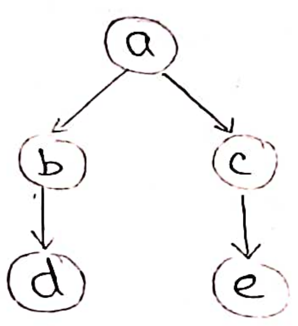
Ex - 2:

Let P, Q, R be three partial ordering on X consisting of five elements

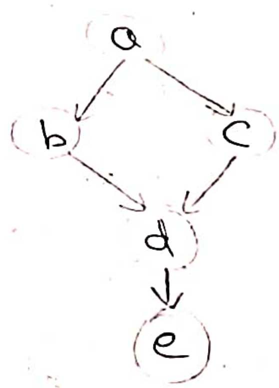
$X = \{a, b, c, d, e\}$

Hasse diagrams of P, Q and R

P



Q



$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

R

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 0 & 0 & 0 & 0 \\ b & 1 & 1 & 0 & 0 & 0 \\ c & 1 & 1 & 1 & 0 & 0 \\ d & 1 & 1 & 1 & 1 & 0 \\ e & 1 & 1 & 1 & 1 & 1 \end{array}$$

Dominating class:

If R is a fuzzy partial ordering on X, then the fuzzy set denoted by $R \geq \{x\}$ and defined as

$$\mu_{R \geq \{x\}}(y) = \mu_R(\overset{\text{row}}{x}, y)$$

is called dominating class of X.

The class dominated by its X is denoted by $R \cdot \leq \{x\}$ and defined as

$$\mu_{R \leq \{x\}}(y) = \mu_R(y, \overset{\text{col}}{x})$$

Undominated :

An element $x \in X$ is called Undominated iff $\overset{\text{m.v}}{R(x, y)} = 0$, $\forall\ y \in X$ & $y \neq x$

Undominating:

An element $x \in X$ is called Undominating iff $R(y, x) = 0$, $\forall\ y \in X$, $y \neq x$

Fuzzy Upper bound:

Suppose X is a set with a partial Ordering R for a crisp subset A of X $A(x)$, the fuzzy upper bound denoted by $U(R, A)$ and is defined as

$$U(R, A) = \bigcap_{x \in A} R \geq \{x\}$$

Example:

Let $X = \{a, b, c, d, e\}$

	a	b	c	d	e
a	1	0.7	0	1	0.7
b	0	1	0	0.9	0
c	0.5	0.7	1	1	0.8
d	0	0	0	1	0
e	0	0.1	0	0.9	1

m-v

like equiv class

$[a] = \{b \mid (a, b) \in R\}$

i) Dominating class of $a = \{1, 0.7, 0, 1, 0.7\}$

ii) Dominated class of $a = \{1, 0, 0.5, 0, 0\}$

iii) Consider the element d from the d^{th} row,

$$\mu_R(d, a) = \mu_R(d, b) = \mu_R(d, c) = \mu_R(d, e) = 0$$

but, $\mu_R(d, d) = 1 \neq 0$

Hence, the element d is undominated.

iv) Consider the element c from the c^{th} column,

$$\mu_R(a,c) = \mu_R(b,c) = \mu_R(d,c) = \mu_R(e,c) = 0$$

but $\mu_R(c,c) = 1 \neq 0$

Hence, the element c is undominating.

Note:

Dominating class of a

$$= \frac{1}{a} + \frac{0.7}{b} + \frac{0}{c} + \frac{1}{d} + \frac{0.7}{e}$$

$|||^{rly}$,

Dominating class of b

$$= \frac{0}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0.9}{d} + \frac{0}{e}$$

v) Example of fuzzy upper bound

Let $A = \{a, b\}$.

$X = \{a, b, c, d, e\}$

and $A \subset X$

The fuzzy upper bound of a fuzzy set is produced by intersection of the dominating class of a and b

From the above note,

$$U(R, A) = \frac{0.7}{b} + \frac{0.9}{d}$$

Fuzzy morphisms:

Suppose $R(X, X)$ and $Q(Y, Y)$ are two crisp relations on X and Y respectively, a function $h: X \to Y$ is said to be a homomorphism from $(X, R) \to (Y, Q)$ iff $(x_1, x_2) \in R \Rightarrow (h(x_1), h(x_2)) \in Q$, $\forall x_1, x_2 \in X$

If R and Q are fuzzy, then h is said to be a morphism if

m.v. of $\mu_R(x_1, x_2) \leq \mu_Q(h(x_1), h(x_2))$

Note:

When h is an homomorphism, the following need not be true

$(h(x_1), h(x_2)) \in Q \Rightarrow (x_1, x_2) \in R$

Strong homomorphism:

1) $h: X \to Y$ is said to be strong homomorphism if

for crisp

i) $(x_1, x_2) \in R \Rightarrow (h(x_1), h(x_2)) \in Q$, $\forall x_1, x_2 \in X$

ii) $(h(x_1), h(x_2)) \in Q \Rightarrow (x_1, x_2) \in R$, $\forall h(x_1), h(x_2) \in Y$

2) If R and Q are fuzzy, then

$h: X \rightarrow Y$ is said to be strong homo-morphism if

i) $\mu_R(x_1, x_2) \leq \mu_Q(h(x_1), h(x_2))$

ii) For every block of partition

$$\mu_Q(h(x_1), h(x_2)) = \max_{i,j} \mu_R(x_i, x_j)$$

Example : (of ...... homomorphism)

Suppose, $M_R =$

$$M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0.5 & 0 & 0 \\ b & 0 & 0 & 0.9 & 0 \\ c & 1 & 0 & 0 & 0.5 \\ d & 0 & 0.6 & 0 & 0 \end{array}$$

$$M_Q = \begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline \alpha & 0.5 & 0.9 & 0 \\ \beta & 1 & 0 & 0.9 \\ \gamma & 1 & 0.9 & 0 \end{array}$$

If $h: X \rightarrow Y$, such that

$h(a) = h(b) = \alpha$, $h(c) = \beta$ and $h(d) = \gamma$

i) $\mu_R(x_1, x_2) \leq \mu_Q(h(x_1), h(x_2))$

$\mu_R(a, b) = 0.5$

$\mu_Q(h(a), h(b)) = \mu_Q(\alpha, \alpha) = 0.5$

$\therefore \mu_R(a, b) \leq \mu_Q(h(a), h(b))$

and $\mu_R(b, c) = 0.9$

$$\mu_Q(h(b), h(c)) = \mu_Q(\alpha, \beta) = 0.9$$

$$\therefore \mu_R(b, c) \leq \mu_Q(h(b), h(c))$$

also, $\mu_R(a, c) = 0$

$$\mu_Q(h(a), h(c)) = \mu_Q)(\alpha, \beta) = 0.9$$

$$\therefore \mu_R(a, c) \leq \mu_Q(h(a), h(c))$$

III^{rly},

$$\mu_R(b, a) \leq \mu_Q(h(b), h(a)) \text{ etc,} \ldots$$

ii) $$\mu_Q(h(x_1), h(x_2)) = \max_{i,j} \mu_R(x_i, x_j)$$

$$\mu_Q(h(a), h(c)) = 0.9$$

i.e., $$\mu_Q(\alpha, \beta) = 0.9$$

$$\max_{i,j} \mu_R(x_i, x_j) = \mu_Q(\alpha, \beta)$$

$$\therefore \text{Max} \left\{ \begin{array}{l} \mu_R(a, c) = 0 \\ \mu_R(b, c) = 0.9 \end{array} \right\} = 0.9$$

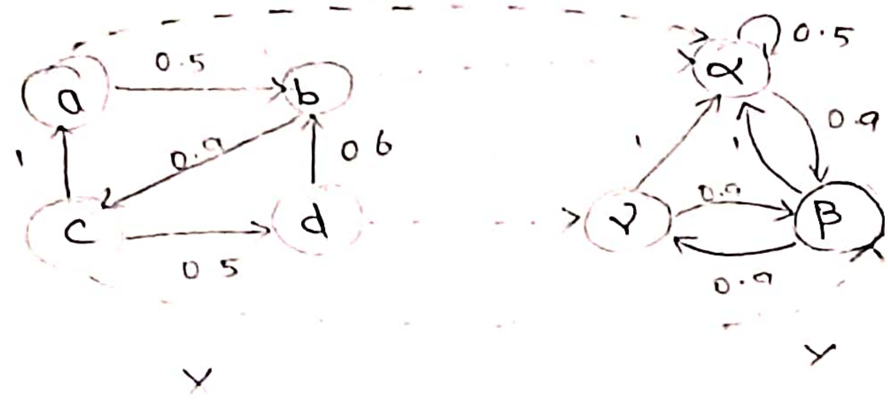
$$\therefore LHS = RHS$$

$$\text{Max} \{ \mu_R(c, d) \} = 0.5$$

$$\mu_Q(h(c), h(d)) = 0.9$$

$$L.H.S \neq R.H.S$$

Hence H is the homomorphism but not strong homomorphism.



$h : X \longrightarrow Y$

Example - 2 : (h is a strong homomorphism)

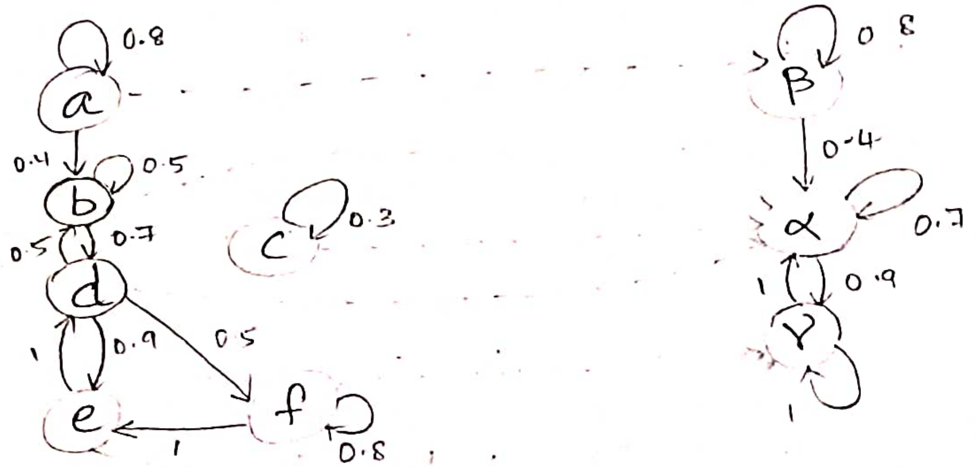
Suppose

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.9 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.8 \end{bmatrix} \end{array}$$

$$M_Q = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \end{array} \begin{array}{c} \begin{array}{ccc} \alpha & \beta & \gamma \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0.9 \\ 0.4 & 0.8 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$h : X \rightarrow Y$, such that $h(a) = \beta$

$h(b) = h(c) = h(d) = \alpha$, $h(e) = h(f) = \gamma$



i) $\mu_R(x_1, x_2) \le \mu_Q(h(x_1), h(x_2))$

$\mu_R(a, b) = 0.4$

$\mu_Q(h(a), h(b)) = \mu_Q(\beta, \alpha) = 0.4$

$\mu_R(a, c) = 0$

$\mu_Q(h(a), h(c)) = \mu_Q(\beta, \alpha) = 0.4$

$\mu_R(a, d) = 0$

$\mu_Q(h(a), h(d)) = \mu_Q(\beta, \alpha) = 0.4$

$\mu_R(a, e) = 0$

$\mu_Q(h(a), h(e)) = \mu_Q(\beta, \gamma) = 0$

$\mu_R(a, f) = 0$

$\mu_Q(h(a), h(f)) = \mu_Q(\beta, \gamma) = 0$

$\mu_R(b, a) = 0$

$\mu_Q(h(b), h(a)) = \mu_Q(\cancel{\beta}\alpha, \beta) = 0$

$\mu_R(b, c) = 0$

$\mu_Q(h(b), h(c)) = \mu_Q(\alpha, \alpha) = 0.7$

$\mu_R(b, d) = 0.7$

$\mu_Q(h(b), h(d)) = \mu_Q(\alpha, \alpha) = 0.7$

$$\mu_R(b, e) = 0$$

$$\mu_Q(h(b), h(e)) = \mu_Q(\alpha, \gamma) = 0.9$$

$$\mu_R(b, f) = 0$$

$$\mu_Q(h(b), h(f)) = \mu_Q(\alpha, \gamma) = 0.9$$

$$\mu_R(c, a) = 0$$

$$\mu_Q(h(c), h(a)) = \mu_Q(\alpha, \beta) = 0$$

$$\mu_R(c, b) = 0$$

$$\mu_Q(h(c), h(b)) = \mu_Q(\alpha, \alpha) = 0.7$$

$$\mu_R(c, d) = 0$$

$$\mu_Q(h(c), h(d)) = \mu_Q(\alpha, \alpha) = 0.7$$

$$\mu_R(c, e) = 0$$

$$\mu_Q(h(c), h(e)) = \mu_Q(\alpha, \gamma) = 0.9$$

$$\mu_R(c, f) = 0$$

$$\mu_Q(h(c), h(f)) = \mu_Q(\alpha, \gamma) = 0.9$$

$$\mu_R(d, a) = 0$$

$$\mu_Q(h(d), h(a)) = \mu_Q(\alpha, \beta) = 0$$

$$\mu_R(d, b) = 0.5$$

$$\mu_Q(h(d), h(b)) = \mu_Q(\alpha, \alpha) = 0.7$$

$$\mu_R(d, c) = 0$$

$$\mu_Q(h(d), h(c)) = \mu_Q(\alpha, \alpha) = 0.7$$

$$\mu_R(d,e) = 0.9$$

$$\mu_Q(h(d), h(e)) = \mu_Q(\alpha, \gamma) = 0.9$$

$$\mu_R(d,f) = 0.5$$

$$\mu_Q(h(d), h(f)) = \mu_Q(\alpha, \gamma) = 0.9$$

$$\mu_R(e,a) = 0$$

$$\mu_Q(h(e), h(a)) = \mu_Q(\gamma, \beta) = 0$$

$$\mu_R(e,b) = 0$$

$$\mu_Q(h(e), h(b)) = \mu_Q(\gamma, \alpha) = 1$$

$$\mu_R(e,c) = 0$$

$$\mu_Q(h(e), h(c)) = \mu_Q(\gamma, \alpha) = 1$$

$$\mu_R(e,d) = 1$$

$$\mu_Q(h(e), h(d)) = \mu_Q(\gamma, \alpha) = 1$$

$$\mu_R(e,f) = 0$$

$$\mu_Q(h(e), h(f)) = \mu_Q(\gamma, \gamma) = 1$$

$$\mu_R(f,a) = 0$$

$$\mu_Q(h(f), h(a)) = \mu_Q(\gamma, \beta) = 0$$

$$\mu_R(f,b) = 0$$

$$\mu_Q(h(f), h(b)) = \mu_Q(\gamma, \alpha) = 1$$

$$\mu_R(f,c) = 0$$

$$\mu_Q(h(f), h(c)) = \mu_Q(\gamma, \alpha) = 1$$

$$\mu_R(f, d) = 0$$

$$\mu_Q(h(f), h(d)) = \mu_Q(\gamma, \alpha) = 1$$

$$\mu_R(f, e) = 0.8$$

$$\mu_Q(h(f), h(e)) = \mu_Q(\gamma, \gamma) = 1$$

$$\therefore \mu_R(x_1, x_2) \leq \mu_Q(h(x_1), h(x_2))$$

ii) $$\mu_Q(h(x_1), h(x_2)) = \max_{i,j} \mu_R(x_i, x_j)$$

$$\mu_Q(h(a), h(b)) = \mu_Q(\beta, \alpha) = 0.4$$

$$\mu_Q(\alpha\beta, \alpha) = \max_{i,j} \mu_R(x_i, x_j)$$

$$\therefore \max \left\{ \begin{array}{l} (a,b) = 0.4 \\ (a,c) = 0 \\ (a,d) = 0 \end{array} \right\} = 0.4$$

$$\mu_Q(h(a), h(c)) = \mu_Q(\beta, \alpha) = 0.4$$

$$\max \left\{ \begin{array}{l} (a,b) = 0.4 \\ (a,c) = 0 \\ (a,d) = 0 \end{array} \right\} = 0.4$$

$$\mu_Q(h(a), h(e)) = \mu_Q(\beta, \gamma) = 0$$

$$\max \left\{ \begin{array}{l} (a,e) = 0 \\ (a,f) = 0 \end{array} \right\} = 0$$

$$\mu_Q(h(b), h(a)) = \mu_Q(\alpha, \beta) = 0$$

$$\max \left\{ \begin{array}{l} (b,a) = 0 \\ (c,a) = 0 \\ (d,a) = 0 \end{array} \right\} = 0$$

Similarly, check the remaining equations.

Fuzzy relation equation.

Let R be a fuzzy relation between 2 sets X and Y that represents the relational concepts very far between the cities.

The relation can be represented by the following membership matrix

$$M_R = \begin{array}{c|ccc} & \text{Chennai} & \text{mumbai} & \text{Delhi} \\ \hline \text{Trivandrum} & 0.3 & 0.9 & 1 \\ \text{chennai} & 0 & 0.6 & 0.7 \end{array}$$

The composition of two binary relatione $A(x,y)$ and $B(y,z)$ are defined interms of maximin operation denoted as $A \circ B$.

The composition of Two binary relations

The maximin composition of matrices resembles matrix multiplication, where multiplication and addition are replaced by min & max operations respectively.

The maximin form of composition is one of the most fundamental composition of fuzzy relation that has been utilized in many applications in the real world problem.

Let us consider a fuzzy binary relation $A(x,y)$, $B(y,z)$ and $R(x,z)$ defined on the Sets.

$$X = \{x_i \mid i \in N_s\}$$
$$Y = \{y_j \mid j \in N_m\}$$
$$Z = \{z_k \mid k \in N_n\}$$

where, N_n, N_m, N_s is the set of all positive integers from one to m, one to s and one to n respectively.

Let the corresponding membership matrices represented as

$$A = (a_{ij}), \quad B = (b_{jk}), \quad R = (r_{ik})$$

respectively.

Here,

$$a_{ij} = \mu_A (x_i, y_j)$$
$$b_{jk} = \mu_B (y_j, z_k)$$
$$r_{ik} = \mu_R (x_i, z_k)$$

are all fuzzy matrices.

The composition $R(A,B)$ is given by the fuzzy matrix eqn.

i.e., $A B = R$ —— (1)

$$\max_{j \in N_m} \min (a_{ij}, b_{jk}) = r_{ik}, \quad \forall \, i \in N_s, k \in N_n \quad \text{—— (2)}$$

The fuzzy matrix eqn (1) encompasses $s \times n$ simultaneous eqn's of the form of eqn. (2) when 2 of the components in each of the eqn's given and one is unknown, these equations are called fuzzy relation equations.

In eqn. ①, when fuzzy matrices A & B are given, then R can be determined uniquely by performing the maximin operation on A and B.

When one of the two matrices on the left hand side of ① is unknown, then the soln. is neither guaranteed to exist not to be unique.

Since R in ① is obtained by composing A and B, the problem of determining A from R and B from R and A can be viewed as the decomposition of R with respect to B (or) with respect to A respectively.

In the following discussion, let us assume that the pair of matrices R and B are given.

We wish to determine the set of particular matrices of the form A and satisfy eqn. ①

Let each particular matrix A that satisfies ①. we call its soln g, the set

$$\Omega(B, R) = \{A \mid AB = R\}$$

denotes the set of all solutions.

The fuzzy matrix eqn (1) can be expressed by matrix eqn's

$$A_{i*} B = R_{i*} \quad , \quad \forall \; i \in N_s \quad \text{—— (3)}$$

where,

$$A_{i*} = [a_{ij} \mid j \in N_m]$$

and

$$R_{i*} = [r_{ik} \mid k \in N_n]$$

are i^{th} row of A and R respectively.

Since (1) can be decomposed into (3), we need only methods for solving eqns of the latter form to arrive at the soln.

Let us restrict our further discussion to fuzzy matrix eqn. of the form

$$xA = b$$

with

$$x = [x_j \mid j \in N_m]$$

$$b = [b_k \mid k \in N_n]$$

$A \in P_{mn}$ of order $m \times n$

and determine the soln. set $\Omega(A, b)$

Lemma:

Let $xA = b$ be a fuzzy relation equation.

If $\max_j (a_{jk}) < b_k$ for some $k \in N_n$

then $\Omega(A, b) = \phi$.

Example:

Let $A = \begin{bmatrix} 0.8 & 0.5 \\ 0.6 & 0.7 \\ 0.1 & 0.3 \end{bmatrix}_{3\times 2}$ and $b = [0.3 \quad 1]_{1\times 2}$

Soln:

$i = 1, \quad j = 3, \quad k = 2$

$\max_j (a_{j2}) < b_2$

$\max \{0.5, 0.7, 0.3\} = 0.7 < 1 \; (= b_2)$

$\therefore \Omega(A, b) = \phi$

The fuzzy relation eqn $xA = b$ has no soln.

THEOREM:

For the eqn. $xA = b$, $\Omega(A, b) \neq \phi$ iff

$\hat{x} = \{\hat{x}_j \mid j \in N_m\}$ defined as $\hat{x}_j = \min \sigma(a_{jk}, b_k)$

where, $\sigma(a_{jk}, b_k) = \begin{cases} b_k, & \text{if } a_{jk} \geq b_k \\ 1, & \text{otherwise} \end{cases}$

is the maximum soln. of the eqn. $xA = b$

Example:

Given $A = \begin{bmatrix} 0.5 & 0.3 & 0.5 & 0.2 \\ 0.8 & 0.6 & 0.3 & 0 \\ 0.8 & 0.9 & 0.5 & 0 \\ 0.4 & 0.7 & 0.6 & 0 \end{bmatrix}$

$b = [0.7 \quad 0.6 \quad 0.5 \quad 0]$

Find out all the solns. of the eqn.

$xA = b$

Soln:

First let us find out the max soln

$$\max_j (a_{jk}) \nless b_k \quad \forall k$$

$$\Omega(A, b) \neq \phi$$

$$\hat{x}_1 = \min_k \sigma(a_{1k}, b_k) = \min\{1, 1, 1, 0\} = 0$$

$$\hat{x}_2 = \min_k \sigma(a_{2k}, b_k) = \min\{0.7, 1, 1, 1\} = 0.7$$

$$\hat{x}_3 = \min_k \sigma(a_{3k}, b_k) = \min\{0.7, 0.6, 1, 1\} = 0.6$$

$$\hat{x}_4 = \min_k \sigma(a_{4k}, b_k) = \min\{1, 0.6, 0.5, 1\} = 0.5$$

$x = [0 \quad 0.7 \quad 0.6 \quad 0.5]$ is the max soln.