UNIT – III

FUZZY MEASURES

Defn:

Given a set X, a function

$$g : \mathcal{P}(X) \rightarrow [0,1]$$

satisfying the following axioms is called a fuzzy measure

Axiom – 1:

$$g(\phi) = 0 \quad \text{and} \quad g(X) = 1$$

(Boundary conditions)

Axiom – 2:

$$\forall \; A, B \in \mathcal{P}(X),$$

$$\text{if } A \subseteq B \text{ then } g(A) \leq g(B)$$

(Monotonicity property)

Axiom – 3:

If $\{A_i\}$ is a monotone sequence of fuzzy sets, where $A_i \in \mathcal{P}(X)$; $(i \in N)$

then,

$$\lim_{i \to \infty} g(A_i) = g\left(\lim_{i \to \infty} A_i\right)$$

Note:

Monotone Sequence:

i) Either $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n$ (or)

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_n$

ii) $g(A)$ is referred as the degree of evidence, that a given element of X is in A

iii) $\wp(X)$ in the defn. of fuzzy measure may be replaced by the family ① which is called Borel field or σ- field.

iv) A family $\mathcal{B}$ is called a Borel field if

$$④ \begin{cases} 1) & \phi, X \in \mathcal{B} \\ 2) & A \in \mathcal{B} \Rightarrow \bar{A} \in \mathcal{B} \\ 3) & A, B \in \mathcal{B} \Rightarrow A \cup B \in \mathcal{B} \end{cases}$$

Defn: (Borel field) (or) σ- field

A fuzzy measure is often defined as

$$g : \mathcal{B} \to [0, 1] \quad \ni ⊗$$

where, $\mathcal{B} \subset \mathcal{P}(X)$ is a family of subsets of X σ- field.

The set $\mathcal{B}$ is usually called a Borel field (or) σ- field.

Since $A \cup B \supseteq A$ and $A \cup B \supseteq B$

we have

$$\max [g(A), g(B)] \leq g(A \cup B)$$

III^{rly}, $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then

$$\min [g(A), g(B)] \geq g(A \cap B)$$

Belief measures.

A fuzzy measure

$$Bel : \mathcal{P}(X) \rightarrow [0,1]$$

is called a Belief measure if it satisfies the following condition.

$$Bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_i Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) + \cdots (-1)^{n+1} Bel(A_1 \cap A_2 \cap \cdots \cap A_n)$$

for every collection of subset of X and for all $n \in N$

Remark:

i) For $n = 2$

$$Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2)$$

ii) For $n = 3$

$$Bel(A_1 \cup A_2 \cup A_3) \geq Bel(A_1) + Bel(A_2) + Bel(A_3) - Bel(A_1 \cap A_2) - Bel(A_2 \cap A_3) - Bel(A_1 \cap A_3) + Bel(A_1 \cap A_2 \cap A_3)$$

iii) For $A \in \mathcal{P}(X)$, $Bel(A)$ is referred as the degree of Belief based on evidence that a given element in X belongs to A.

iv) $Bel(A) + Bel(\bar{A}) \leq 1$

THEOREM - 1:

If $A \subset B$, then Prove that $Bel(A) \leq Bel(B)$

Proof:

$A = \{1, 2, 3\}$

Given $A \subset B$

$B = \{1, 2, 3, 4, 5\}$

Let $C = B - A$

$C = \{4, 5\}$

$\therefore A \cup C = B$ and $A \cap C = \phi$

$A \cup C = B$

$Bel(A \cup C) = Bel(B)$ & $Bel(A \cap C) = Bel(\phi) = 0$

$Bel(A \cup C) \geq Bel(A) + Bel(C) - Bel(A \cap C)$

$\Rightarrow Bel(A \cup C) \geq Bel(A) + Bel(C)$

$\Rightarrow Bel(B) \geq Bel(A) + Bel(C)$

and consequently $Bel(A) \leq Bel(B)$

Hence the proof

THEOREM - 2:

Prove that $Bel(A) + Bel(\bar{A}) \leq 1$

Proof:

Let $A \cup \bar{A} = X$ & $A \cap \bar{A} = \phi$

$Bel(A \cup \bar{A}) = Bel(X)$

$= 1$

$Bel(A \cup \bar{A}) \geq Bel(A) + Bel(\bar{A}) - Bel(A \cap \bar{A})$

$\geq Bel(A) + Bel(\bar{A}) - Bel(\phi)$

$\geq Bel(A) + Bel(\bar{A}) - 0$

$\therefore Bel(A) + Bel(\bar{A}) \leq 1$

Hence the proof.

Plausibility Measures:

If $Bel : \wp(X) \to [0,1]$ is a Belief measure $\forall A \in \wp(X)$, the measure

$$Pl : \wp(X) \to [0,1]$$

defined by

$$Pl(A) = 1 - Bel(\bar{A})$$

is called Plausibility measure.

Remark :

Belief measures and Plausibility measures are referred as mutually Dual measures.

i.e., $$Pl(A) = 1 - Bel(\bar{A})$$

$$\Rightarrow Bel(\bar{A}) = 1 - Pl(A)$$

(or)

$$Bel(A) = 1 - Pl(\bar{A})$$

Defn:

$Pl : \wp(X) \to [0,1]$ satisfies the three axioms of fuzzy measure and the following additional axiom

$$Pl(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_i Pl(A_i) - \sum_{i<j} Pl(A_i \cup A_j) \cdots + (-1)^{n+1} Pl(A_1 \cup A_2 \cup \cdots \cup A_n)$$

THEOREM - 3 :

Prove that $Pl(A) + Pl(\bar{A}) \geq 1$

Proof :

$$Pl(X) = 1 - Bel(\bar{X})$$
$$= 1 - Bel(\phi)$$
$$= 1 - 0$$
$$= 1$$

and

$$Pl(\phi) = 1 - Bel(\bar{\phi})$$
$$= 1 - Bel(X)$$
$$= 1 - 1$$
$$= 0$$

now,

$$A \cap \bar{A} = \phi \quad \& \quad A \cup \bar{A} = X$$

$$Pl(A \cap \bar{A}) = Pl(\phi) = 0$$

$$Pl(A \cap \bar{A}) \leq Pl(A) + Pl(\bar{A}) - Pl(A \cup \bar{A})$$

$$Pl(\phi) \leq Pl(A) + Pl(\bar{A}) - Pl(X)$$

$$0 \leq Pl(A) + Pl(\bar{A}) - 1$$

$$\Rightarrow \quad Pl(A) + Pl(\bar{A}) \geq 1$$

Defn :

A function $m : \mathcal{P}(X) \rightarrow [0, 1]$ such that $m(\phi) = 0$ and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$ is called a Basic Assignment (Basic Probability Assignment)

Note :

1. $m(A)$ is interpretted as degree of evidence

supporting the claim that a specific element of X, whose characterisation is deficient belongs to A but not to any special subset of A

2. $m(\phi) = 0$ is optional. such basic assignment are said to be normal.

Result:

Even though a basic assignment is not a fuzzy measure, a belief measure and a plausibility measure can be uniquely constructed by defining

$$Bel(A) = \sum_{B \subseteq A} m(B)$$

and $$Pl(A) = \sum_{B \cap A \neq \phi} m(B) \quad , \forall A \in \mathcal{P}(X)$$

Clearly, $\boxed{Pl(A) \geq Bel(A)}$

Focal element:

A set $A \in \mathcal{P}(X)$, such that $m(A)>0$ is called focal element of m.

Body of evidence:

If $\mathcal{F}$ is a family of all focal elements, then $(\mathcal{F}, m)$ is called body of evidence

Total ignorance:

If $m(X) = 1$ and $m(A) = 0$, $\forall A \neq X$ is referred as Total ignorance.

Simple Support function:

A basic assignment m is said to be a simple support function focussed at a set A if $m(A) = s$, $m(X) = 1 - s$ and $m(B) = 0$, $\forall B \neq A$

Result:

1. If a belief measure 'Bel' is given, then the basic assignment m can be constructed by setting

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} Bel(B)$$

where, $|A-B|$ denotes the cardinality of $A-B$.

2. If m_1, m_2 are two basic assignments on $\mathcal{P}(X)$, they can be appropriately combined to get another basic assignment (Joint Basic

Assignment $m_{1,2}$). The standard way of combining evidence is given by the Dempster's rule of combination which gives $m_{1,2}$

$$m_{1,2}(A) = \frac{\sum_{B \cap C = A} m_1(B) \cdot m_2(C)}{1-k}, \quad \text{for } A \neq \phi$$

where,

$$k = \sum_{B \cap C = \phi} m_1(B)\, m_2(C)$$

$$m_{1,2}(\phi) = 0 \quad , \text{for } A = \phi$$

Probability measures: (Bayesian Belief measure)

A Belief measure 'Bel' such that

$$Bel(A \cup B) = Bel(A) + Bel(B) \quad \text{—— } (*)$$

whenever, $A \cap B = \phi$ is called Probability measure

THEOREM-4:

A Belief measure 'Bel' on a finite power set $P(X)$ is a probability measure iff its basic assignment 'm' is given by $m(\{x\}) = Bel(\{x\})$ and $m(A) = 0$, for all subsets of X that are not singletons.

Proof:

Assume that Bel is a probability measure.

case(i):

For the empty set ϕ, the theorem trivially holds, since $m(\phi) = 0$, by the defn. of basic assignment 'm'

case(ii):

Let $A \neq \phi$ and assume that $A = \{x_1, x_2 \dots x_n\}$. then by repeated application of axiom, we obtained

$$Bel(A) = Bel(\{x_1, x_2 \dots x_n\})$$
$$= Bel(\{x_1\}) + Bel(\{x_2\}) + \dots + Bel(\{x_n\}) \quad \text{--- (1)}$$

Since $Bel(\{x\}) = m\{x\}$

(1) $\Rightarrow$

$$Bel(A) = m(\{x_1\}) + m(\{x_2\}) + \dots + m(\{x_n\})$$
$$= \sum_{i=1}^{n} m(\{x_i\})$$

Hence Bel is defined in terms of a basic assignment.

Conversely,

Assume that a basic assignment m is given such that $\sum_{x \in X} m(\{x\}) = 1$,

Then, for any sets $A, B \in \mathcal{P}(X)$ $\ni$ $A \cap B = \phi$, we have

$$Bel(A) + Bel(B) = \sum_{x \in A} m(\{x\}) + \sum_{x \in B} m(\{x\})$$
$$= \sum_{x \in A \cup B} m(\{x\})$$
$$= Bel(A \cup B)$$

Hence Belief measure is a probability measu

Note:

From the above theorem, [the probability measures on finite sets are represented by a function

$$P : X \longrightarrow [0,1] \ni$$

$$P(x) = m(\{x\})$$

P is called Probability distribution function.]

In such case, 'Bel' and 'Pl' merge.

Thus, $Bel(A) = Pl(A) = \sum_{x \in A} P(x)$.

We denote it by $P(A)$, when

$$Bel(A) = Pl(A)$$

[When a probability distribution P is defined on the Cartesian product $X \times Y$, it is called a joint probability distribution]

Projections:

Projections P_X and P_Y of P on X and Y respectively are called marginal Probability distribution, they are defined by the formulas

$$p_X(x) = \sum_{y \in Y} p(x,y) \quad , \quad \forall x \in X$$

$$p_Y(y) = \sum_{x \in X} p(x,y) \quad , \quad \forall y \in Y$$

Sets X and Y are called <u>non-interactive</u> with respect to p if

$$p(x,y) = p_X(x) \cdot p_Y(y) \quad , \quad \forall \begin{matrix} x \in X \\ y \in Y \end{matrix}$$

Two <u>conditional</u> <u>probability</u> <u>distributions</u> $p_{X|Y}(x|y)$ and $p_{Y|X}(y|x)$ are defined in terms of a joint distribution p by the formulas

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)} \quad , \quad \forall x \in X$$

and

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} \quad , \quad \forall y \in Y$$

Set X is called independent of Y, if

$$p_{X|Y}(x|y) = p_X(x) \quad , \quad \forall \begin{matrix} x \in X \\ y \in Y \end{matrix}$$

III^{rly}, set Y is called independent of X, if

$$p_{Y|X}(y|x) = p_Y(y) \quad , \quad \forall \begin{matrix} x \in X \\ y \in Y \end{matrix}$$

The sets X and Y are independent

of each other iff they are non interactive.

Defn: (Nested)

A family of subsets of a Universal set is nested if they can be ordered so that one is contained in another.

i.e., $A_1 \subset A_2 \subset \cdots \subset A_n$ are nested sets.

Defn: (Consonant)

If the focal elements of a body of evidence $(\mathcal{F}, m)$ are nested, the corresponding Belief and plausibility measures are said to be consonant.

<u>Necessity measure and Possibility measure:</u>

Defn:

A Consonant Belief measure is called necessity measure.

A consonant Plausibility measure is called Possibility measure.

THEOREM:

If $(\mathcal{F}, m)$ is consonant, then

i) $Bel(A \cap B) = \min[Bel(A), Bel(B)]$, $\forall A, B \in \mathcal{P}(X)$

ii) $Pl(A \cup B) = \max[Pl(A), Pl(B)]$, $\forall A, B \in \mathcal{P}(X)$

Proof:

Suppose that

$$\mathcal{F} = \{A_1, A_2, \ldots A_n\}$$

and $A_i \subset A_j$ for $i < j$

Consider, arbitrary subsets A and, B of X

Let i_1 be the largest integer such that $A_{i_1} \subset A$

Let i_2 be the largest integer such that $A_{i_2} \subset B$

now,

$A_i \subset A$ and $A_i \subset B$ $\Longleftrightarrow$ $i \leq i_1$ & i_2

also,

$A_i \subset A \cap B$ $\Longleftrightarrow$ $i \leq \min(i_1, i_2)$

$$Bel(A) = \sum_{B \subseteq A} m(B)$$

now,

$$Bel(A \cap B) = \sum_{i=1}^{\min(i_1, i_2)} m(A_i)$$

$$= \min\left[\sum_{i=1}^{i_1} m(A_i), \sum_{i=1}^{i_2} m(A_i)\right]$$

$$= \min[Bel(A), Bel(B)]$$

ii) $Pl(A \cup B) = 1 - Bel(\overline{A \cup B})$

$= 1 - Bel(\bar{A} \cap \bar{B})$

$$Pl(A \cup B) = 1 - \min[Bel(\bar{A}), Bel(\bar{B})]$$
$$= \max[1 - Bel(\bar{A}), 1 - Bel(\bar{B})]$$
$$= \max[Pl(A), Pl(B)]$$

Note:

If we denote possibility and necessity measures by π and η respectively, then the above result (theorem) is as follows:

1) $\eta(A \cap B) = \min[\eta(A), \eta(B)]$

$\pi(A \cup B) = \max[\pi(A), \pi(B)]$ $\forall A, B \in \mathcal{P}(X)$ (*)

2) $\eta(A) = 1 - \pi(\bar{A})$, $\forall A \in \mathcal{P}(X)$

THEOREM:

Every possibility measure π on $\mathcal{P}(X)$ can be uniquely determined by a possibility distribution function $r : X \to [0,1]$ via the formula $\pi(A) = \max_{x \in A} r(x)$, $\forall A \in \mathcal{P}(X)$

Proof:

We prove this theorem by induction on the cardinality of A.

If the cardinality of A is 1, then the result is obvious.

Assume that the given equation is satisfied for $|B| = n-1$

and let $B = \{x_1, x_2, \dots x_{n-1}\}$

now,

Prove that the equation is satisfied for $|A| = n$, where $A = \{x_1, x_2, \dots x_n\}$

now,

$$\pi(A) = \pi\left[\{x_1, x_2, \dots x_n\}\right]$$

$$= \pi\left[\{x_1, x_2 \dots x_{n-1}\} \cup \{x_n\}\right]$$

$$= \max\left[\pi\{x_1, x_2, \dots x_{n-1}\}, \pi\{x_n\}\right]$$

[using (*)]

$$= \max\left[\max\{\pi\{x_1\}, \pi\{x_2\} \dots \pi\{x_{n-1}\}\}, \pi\{x_n\}\right]$$

$$= \max\{\pi\{x_1\}, \pi\{x_2\} \dots \pi\{x_n\}\}$$

$$= \max_{x \in A}\{r(x)\}$$

Remark:

1. If r is a possibility distribution on $X = \{x_1, x_2, \dots x_n\}$, then we denote it by

$$r = (e_1, e_2 \dots e_n), \quad \text{where } e_i = r(x_i)$$

r is said to be the length of n.

r is ordered in the sense that

$$e_i \geq e_j \quad \text{for } i < j$$

The set of all r with length n is denoted by ${}^{n}R$ and set of all r is denoted by R.

It is clear that $R = \bigcup_{n \in N} {}^{n}R$

For ${}^{1}r = ({}^{1}\rho_1, {}^{1}\rho_2, \ldots {}^{1}\rho_n)$

${}^{2}r = ({}^{2}\rho_1, {}^{2}\rho_2, \ldots {}^{2}\rho_n) \in {}^{n}R$,

we say that ${}^{1}r \leq {}^{2}r \iff {}^{1}\rho_i \leq {}^{2}\rho_i, \forall i$

If ${}^{i}r \overset{\text{join}}{\vee} {}^{j}r = (\max({}^{i}\rho_1, {}^{j}\rho_1), \max({}^{i}\rho_2, {}^{j}\rho_2) \ldots \max({}^{i}\rho_n, {}^{j}\rho_n))$

and ${}^{i}r \overset{\text{meet}}{\wedge} {}^{j}r = (\min({}^{i}\rho_1, {}^{j}\rho_1), \min({}^{i}\rho_2, {}^{j}\rho_2) \ldots \min({}^{i}\rho_n, {}^{j}\rho_n))$

Then, ${}^{n}R$ become a lattice

2. III^rly, If $X = \{x_1, x_2, \ldots x_n\}$ and π is a possibility distribution in terms of a basic assignment m, then all the focal elements are nested.

Suppose that $A_1 \subset A_2 \subset \ldots \subset A_n (= X)$,

then $m(A) = 0, \forall A \neq A_i$ and $\sum_{i=1}^{n} m(A_i) = 1$

Now, $m = (\mu_1, \mu_2, \ldots \mu_n)$, where $\mu_i = m(A_i) \; \forall i$

is called a basic distribution of

length n

The set of all m with length n is denoted by ${}^{n}M$ and set of all m is denoted by M.

It is clear that $M = \bigcup_{n \in N} {}^{n}M$

Clearly, $\rho_i = r(x_i) = \pi(\{x_i\})$
$$= \sum_{k=1}^{n} m(A_k)$$
$$= \sum_{k=1}^{n} \mu_k \quad , \quad \forall \, i = 1, 2, \ldots n$$

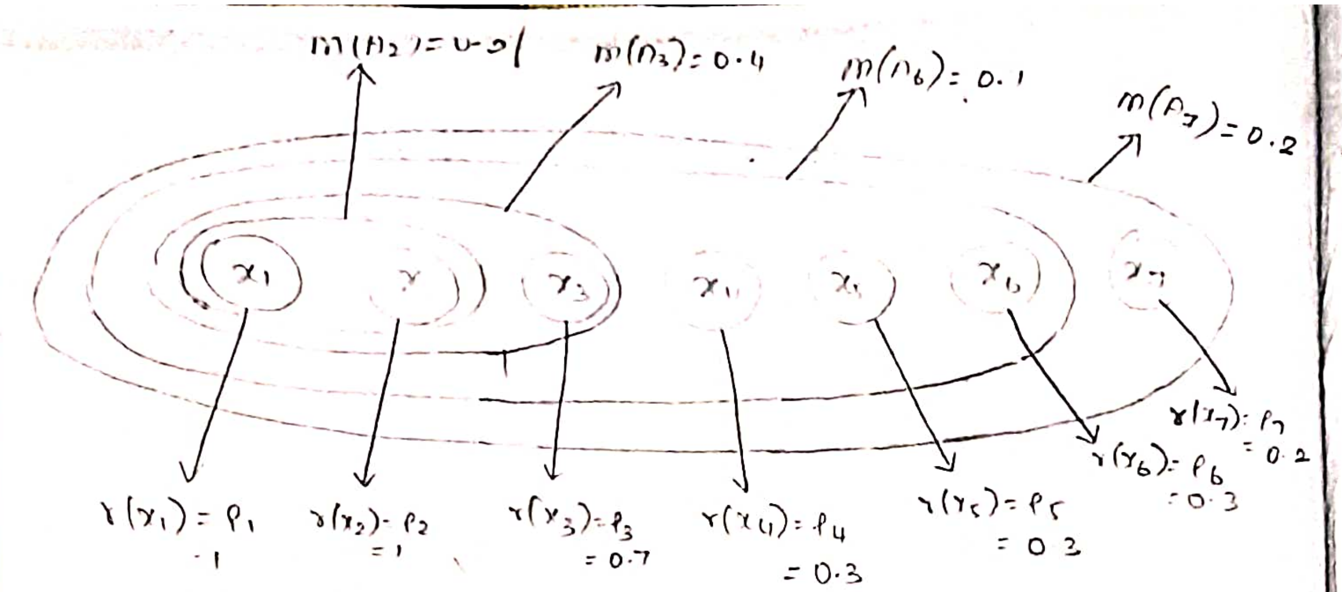
Thus,
$$\rho_1 = \mu_1 + \mu_2 + \cdots + \mu_n$$
$$\rho_2 = \mu_2 + \mu_3 + \cdots + \mu_n$$
$$\vdots$$
$$\rho_n = \mu_n$$

and
$$\rho_i - \rho_{i+1} = \mu_i \quad \text{with} \quad \rho_{n+1} = 0$$

It is clear that, there is a 1-1 correspondance between R and M and it is defined by $t : R \longrightarrow M$ which is 1-1 and onto.

Example:

Obtain the possibility distribution and the degree of possibility for the set $A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. where the basic distribution is represented in the following diagram.



Soln:

Here $m(A_1) = m(A_4) = m(A_5) = 0$

$$\therefore \quad m = (m(A_1), \dots m(A_7)) = (\mu_1, \mu_2 \dots \mu_7)$$
$$= (0, 0.3, 0.4, \cancel{0.4}, 0, 0.1, 0.2)$$

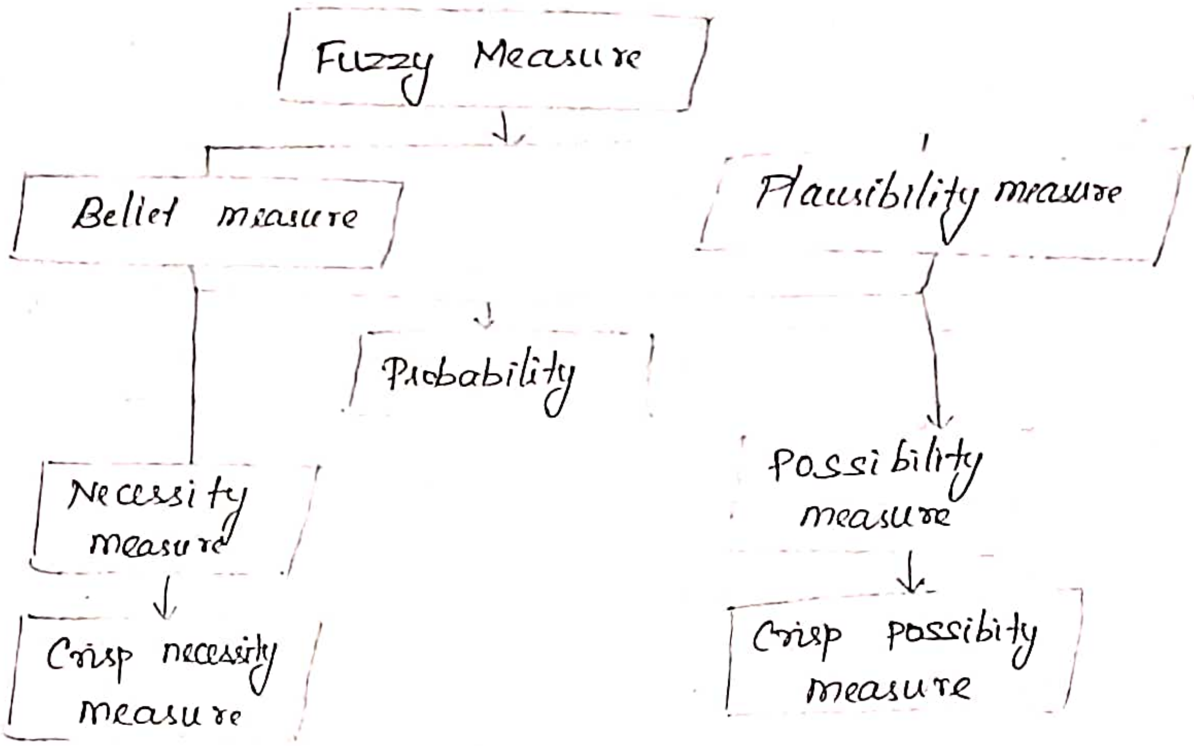
We obtain the possibility distribution

$$r = (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7)$$
$$= (1, 1, 0.7, 0.3, 0.3, 0.3, 0.2)$$

Degree of possibility for A is given by

$$\pi(A) = \max(\rho_1, \rho_2 \dots \rho_7)$$
$$= \max(1, 1, 0.7, 0.3, 0.3, 0.3, 0.2)$$
$$= 1$$

Relationship among classes of fuzzy measures:



(The concept of fuzzy measure provides us with the large frame work within which various special classes of measures can be formulated including the classical probabity measure.

The most fundamental property of fuzzy measure is their monotonicity with respect to subset relationship.)

The special classes of fuzzy measures that are introduced in this unit are currently the most prominent types of measures in the literature. (This is summarized in the above figure.)

(Two large classes of fuzzy measures are

Belief measures and ~~probability~~ Plausibility measures. they are mutually dual in the sense that one of them can be uniquely determined from the other.)

Together from the form, (a theory that usually referred to as a mathematical theory of evidence.

This theory is fully characterised by a paired dual axioms

$$Bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_i Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) + \cdots + (-1)^{n+1} Bel(A_1 \cap A_2 \cdots \cap A_n)$$

and

$$Pl(A_1 \cap A_2 \cdots \cap A_n) \leq \sum_i Pl(A_i) - \sum_{i<j} Pl(A_i \cup A_j) + \cdots + (-1)^{n+1} Pl(A_1 \cup A_2 \cdots \cup A_n)$$

Each pair consisting of a Belief measure and its dual plausibility measure is represented by the basic probability assignment which assigns degrees of evidence or Belief to certain specific success of the universal set.

(The non-zero elements of degrees of evidence is called focal elements.

When all focal elements are singleton, the corresponding Belief measure is equal to the dual Plausibility measure.

In these cases, Belief and plausibility measures are not distinguished.

(The dual properties of subadditivity merge into a single property of additivity expressed by equation $Bel(A \cup B) = Bel(A) + Bel(B)$, which is a fundamental property of the classical probability measure.

i.e., The probability measure are the merging of Belief and plausibility measure is expressed in the above figure.)

(An important property of probability measure is that each of them can be uniquely represented by a function defined on elements of the Universal set rather than on its subsets. This function is called a Probability distribution function.)

(When focal elements are nested, we obtain a special subclass of plausibility measure and its dual subclass of Belief measure. These measures are called possibility and Necessity measures respectively.)

(As in the case with probability measure, Possibity and Necessity measures can also be uniquely charactered by function defined on the Universal set.

These functions are called a possibility distribution function and a necessity distribution function.

When the values of these functions are restricted to zeros and ones, the measures are referred to as crisp possibility measure and crisp necessity measures.