6

APPLICATIONS

6.1 GENERAL DISCUSSION

We see our problems today differently than we did in the past. We are no longer merely facing the world trying to win its secrets or battling to shape it to meet our desires. We are now facing ourselves and our own knowledge and trying to cope with the complexity we have created. Therefore, we no longer formulate the problems in our scientific inquiries or in our engineering attempts simply in terms of missing facts or inadequate measuring devices or materials. Problems are seen to be of a different order; that is, they concern questions about knowledge and information itself and can therefore be formulated independently of any particular area of knowledge. Indeed, questions in fields as diverse as psychology and engineering, economics and medicine, sociology and genetics, artificial intelligence and meteorology, or decision making and architecture begin to take on a similar form when seen in light of their common problems of handling information, uncertainty, and complexity.

New problems call for new solutions. We want to know how to simplify complexity, how to utilize information, and how to deal with uncertainty, not in any particular context but in the form of general principles and representations that hold true in any given context. For this reason, fuzzy set theory and the other mathematical frameworks introduced in this book that deal with information and uncertainty are applicable in any field in which issues of complexity arise. It is, in fact, difficult to name a field in which complexity does not arise, either as the actual focus of the study or as a by-product or attribute of the content of the inquiry. This is not to say, however, that the mathematics of information and uncertainty will be all things to all people. This field is still relatively young, and more research is needed before a clear determination can be made of exactly what these mathematical frameworks can and cannot offer. Most early attempts to discover the usefulness of these frameworks have consisted in applying them to already existing solutions, for instance, by "fuzzifying" algorithms in order to achieve better performance. In many cases, this better performance has, in fact, been achieved. The aim of the mathematics of information and uncertainty, however, is not merely to do the same thing in a different way. It is, instead, to approach problems from a higher level of question about the possibilities and limits of knowledge itself. The success of this approach in application to real-world problems, while quite promising thus far, will be fully realized only in time.

The examples presented in this chapter are intended to give a flavor of the types of application research currently taking place in the field of fuzzy sets, uncertainty, and information. The successful applications of these mathematical models and tools is extremely diverse and widespread, and the sample included here is by no means exhaustive. Each section includes a brief overview of the application area followed by some specific illustrative examples drawn from the literature. Key references in each area are incorporated into the text and additional, more specific references are given in the notes at the end of the chapter. The number of books and papers presenting these applications is quite large and the references that are included are not intended to stand as full surveys of these fields; their purpose is to aid the reader in pursuing a more extensive study in selected areas.

6.2 NATURAL, LIFE, AND SOCIAL SCIENCES

The various natural, life, and social sciences have been active areas for application of the mathematics of uncertainty and information, both for the purpose of understanding human systems and for making natural systems understandable to human thought. The applications of fuzzy set theory include explorations within psychology and cognitive science of concept formation and manipulation, memory and learning, as well as studies in the fields of sociology, economics, ecology, meteorology, biology, and others.

The applications of information theory are even more extensive; they cover virtually the whole spectrum of science. They are, however, almost exclusively based on the classical Shannon or Hartley measures of information. The principles of maximum and minimum entropy have proven to be the most important for applications; their use is best exemplified by the work of Ronald Christensen [1981b, 1986]. The alternative principles of uncertainty, such as the maximum nonspecificity principle suggested in Sec. 5.9, are too new to have been demonstrated in specific applications thus far. There is little doubt, however, that their impact on a broad spectrum of applications in science will be profound.

In this section, we illustrate the vastly diverse applications in the sciences by two simple applications of fuzzy set theory: one in the field of meteorology and the other in the area of interpersonal communication.

Meteorology

The science of meteorology deals with a vastly complex system which, in truth, encompasses the entire planet. For this reason, meteorological descriptions as well as forecasts have always relied on the kind of robust summary offered by vague linguistic terms such as *hot weather*, *drought*, or *low pressure*. Applications of fuzzy set theory to meteorology, therefore, constitute attempts to deal with the complexity of the study by taking advantage of the representation of vagueness offered in this mathematical formalism. Three different applications of fuzzy set theory within the field of meteorology are presented by Cao and Chen [1983]. We briefly examine the first of these, which makes use of a fuzzy clustering technique. (The area of fuzzy clustering is covered briefly in Sec. 6.8.)

This particular application focuses on an understanding of climatological changes based on the examination of statistical records of weather patterns collected over a period of time. For simplicity, only the two climatological stages of drought periods and wet periods are considered in this study. The data used were taken at a single monitoring station during the years 1886 through 1979 in Shanghai, China, and consisted of records of annual precipitation levels. Each of the n original measurements of annual precipitation p_i ($i \in \mathbb{N}_n$) is preprocessed by a 10-year running mean calculated by

$$p_i = \frac{p_{i-4} + \cdots + p_{i-1} + p_i + p_{i+1} + \cdots + p_{i+5}}{10}$$

After this preprocessing, a total of 85 records were obtained from the original data.

The interest of the study is to examine the points and features of alternations between wet and drought periods. Obviously, each period is characterized by annual precipitation levels that are similar to one another, and turning points or boundaries between periods are characterized by the appearance of dissimilarity between the values of annual precipitations. The exact point of change between any two of these periods may be difficult to determine crisply. In order to accommodate the inherent fuzziness between boundaries of drought and wet periods, a fuzzy similarity relation is constructed from the annual precipitation values in order to cluster together into periods those years whose precipitation levels are "similar" to one another. This similarity relation is created by first calculating a similarity index s'_{ij} between precipitations p_i and p_j ($i, j \in \mathbb{N}_{85}$). These indices are given by

$$s'_{ij} = 1 - |p_i - p_j| .$$

The indices are then normalized to the interval [0, 1] by the formula

$$s_{ij} = \frac{s'_{ij} - s'_{\min}}{s'_{\max} - s'_{\min}} ,$$

where $s'_{\min}$ and $s'_{\max}$ are the maximum and minimum similarity indices, respectively. The resulting normalized similarity indices can be viewed as constituting the membership grades of a fuzzy relation R such that

$$\mu_R(p_i, p_j) = s_{ij}.$$

The relation R is reflexive and symmetric but is generally not transitive. Therefore, the transitive closure of R must be calculated; this can be done with the algorithm given in Sec. 3.3. The resulting relation R_T is the desired similarity relation. By taking α-cuts of this relation, where $\alpha \in [0, 1]$, we arrive at crisp clusters of precipitation levels, which are similar to each other to the degree α. Figure 6.1 illustrates the matrix representation of this type of clustering that was performed on the data collected by Cao and Chen for the similarity level $\alpha = .95$. An ijth entry of 1 in this matrix indicates the presence of similarity between record i and record j of a degree at least equal to α. As can be seen from the figure, five different climatological stages or clusters are apparent. The periods of drought are found in records 1–17, 31–52, and 68–85, whereas records 18–30 and 53–67 constitute the wet periods. The changes between periods are clearly displayed in the matrix representation of the α-cut. Such an application forms a useful tool for the investigation of climatological change and forecasting by mathematically representing the fuzziness or imprecision of boundaries between the stages of interest.

Interpersonal Communication

The process of interpersonal communication consists of a vast array of different types of simultaneously communicated signals (words, voice tone, body posture, clothing, etc.), many of which conflict with each other. It is therefore difficult to determine the precise intention and meaning of the communication, both because of distortion from environmental noise and because of ambivalence on the part of the sender. Nevertheless, the receiver must respond appropriately in the face of this fuzzy or vague information. We outline here an approach suggested by Yager [1980h], which models this process and the vagueness associated with it through the use of fuzzy set theory.

Suppose that X constitutes the universal set of all possible signals x that may be communicated by the sender. Because of the distorting factors mentioned above, a clear, unique signal may not be available. Instead, the message received is a fuzzy subset M of X, in which $\mu_M(x)$ denotes the degree of certainty of the receipt of the specific signal x. In order to determine whether an appropriate response can be chosen based on the message received or whether some error was involved in the communication, an assessment of the quality of the transmission must be made. Let the maximum value of membership which any $x \in X$ attains in the set M correspond to the *strength* of the transmission. If the set M has no unique maximum, then the message is called *ambiguous*. If the support of M is large, then M is considered to be *general*. The clarity of the message can be measured by the distance between the maximum membership grade attained in M and the next largest grade of any signal x in M. When the message received is strong, unambiguous, and clear, then the signal attaining the maximum mem-

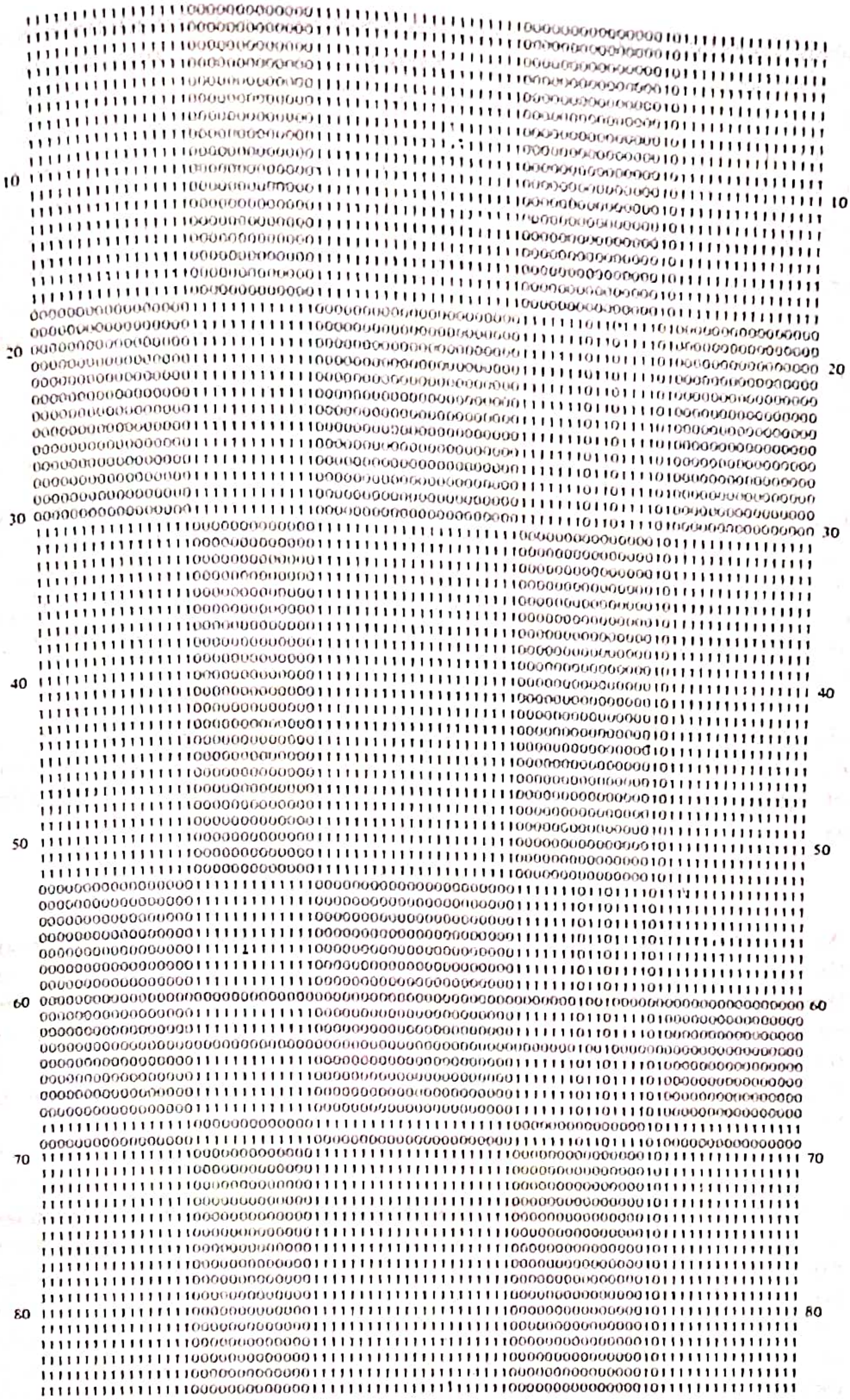

Figure 6.1. Fuzzy clustering of climatological periods: α-cut of S for $\alpha = .95$.

bership grade in M can easily be selected as the most obvious intended communication. Difficulty occurs, however, when the message is weak, ambiguous, or unclear. In this case, the receiver must determine whether the problem in the communication lies in some environmental distortion (in which case a repetition of the signal may be requested) or in the sender of the message (in which case a response must be made, which is, as far as possible, appropriate).

Usually, the receiver of the communication possesses some background information in the form of probabilities or possibilities of the signals which can be expected. If $p(x_1), p(x_2), \ldots, p(x_n)$ represent the probabilities associated with each of the signals $x_1, x_2, \ldots, x_n \in X$, then the probability of the fuzzy event of the receipt of message M is given by

$$p(M) = \sum_{x \in X} \mu_M(x) p(x).$$

The receiver can use this information to assess the consistency of the received message with his or her expectations. If the probability of the received message is high, then it can be assumed that little distortion was introduced by the environment. On the other hand, if the message is very clear and unambiguous, then an appropriate response can be made even if the probability of the signal was low.

Instead of the expectation or background information being given in probabilistic form, this information may be given in the form of a possibility distribution **r** on X. In this case, $r(x) \in [0, 1]$ indicates the receiver's belief in the possibility of signal x being sent. The total possibility of the fuzzy message M is calculated as

$$r(M) = \max_{x \in X}[\min(\mu_M(x), r(x))].$$

As in the case of probabilistic expectation, if the received message conflicts with the expected possibility of communication, then the receiver may attempt clarification by requesting a repetition of the transmission. Before this new transmission is sent, the receiver will probably have already modified his or her expectations based on the previous message. If $\mathbf{r}_0$ indicates the initial possibilistic expectations of the receiver and $\mathbf{r}_1$ is the modified expectations subsequent to the receipt of message M, then

$$r_1(x) = \min[r_0^\alpha(x), \mu_M(x)]$$

for each $x \in X$, where α indicates the degree to which past messages are considered relevant in the modification of expectations. Our procedure for signal detection now consists of the following: a test of the consistency of M against the expectations and a test of the message M for strength and clarity. If both of these values are high, the signal attaining the maximum value in M can be comfortably assumed to be the intended signal. If both tests yield low values, the expectations are modified and a repetition is requested. If only one of these tests yields a satisfactory value, then either a new signal is requested or a response is made despite the presence of doubt.

An additional complication is introduced when we consider that the receiver may also introduce distortion in the message because of inconsistency with the expectations. Let

$$s(M, r) = \max_{x \in X}[\min(\mu_M(x), r(x))] \tag{6.1}$$

correspond to the consistency of the received message with the possibilistic expectations. Then, let M' denote the message which the receiver actually hears, where

$$\mu_{M'}(x) = \mu_M^s(x) \tag{6.2}$$

for each $x \in X$. The less consistent M is with the expectations, the less M' resembles M. Since the receiver will be modifying his or her expectations based on the message thought to have been received, the new possibilistic expectation structure is given by

$$r_1(x) = \min[r_0^{1-s}(x), \mu_{M'}(x)] \tag{6.3}$$

for each $x \in X$.

Finally, once a determination has been made of the signal $x \in X$ that was sent, an appropriate response must be chosen. Let Y be the universal set of all responses and let $R \in Y \times X$ be a fuzzy binary relation in which $\mu_R(y, x)$ indicates the degree of appropriateness of response y given signal x. A fuzzy response set $A \in Y$ can be generated by composing the appropriateness relation R with the fuzzy message M,

$$A = R \circ M$$

or

$$\mu_A(y) = \max_{x \in X}[\min(\mu_R(y, x), \mu_M(x))] \tag{6.4}$$

for each $y \in Y$. The membership grade of each possible message y in fuzzy set A thus corresponds to the degree to which it is an appropriate response to the message M. A more interesting case occurs when the elements $y \in Y$ are not actual messages but instead indicate characteristics or attributes that the appropriate message should possess. This allows for creativity in formulating the actual response. The following example illustrates the use of this model of interpersonal communication.

Suppose that a young man has just proposed marriage to a young woman and is now eagerly awaiting her response. Let us assume that her answer will be chosen from the set X of the following responses:

x_1 = simple yes

x_2 = simple no

x_3 = request for time to think it over

x_4 = request for young man to ask permission of the young woman's parents

x_5 = derisive laughter

x_6 = joyful tears

Assume also that the young man has expectations of her response represented by the possibility distribution

$$\mathbf{r}_0 = (.9, .1, .7, .3, .1, .6).$$

We can see from this distribution that the young man expects a positive answer. Suppose, however, that the following message M_1 is received:

$$M_1 = .1/x_1 + .8/x_2 + .4/x_3 + .1/x_5.$$

This message, although relatively strong, unambiguous, and clear, is rather inconsistent with the young man's expectations. As measured by Eq. (6.1), the consistency is

$$s(M_1, \mathbf{r}_0) = \max[.1, .1, .4, .1] = .4.$$

Because the message is contrary to the young man's expectations, let us assume that he introduces some distortion, as specified by Eq. (6.2), such that the message he hears is

$$M_1' = .4/x_1 + .9/x_2 + .7/x_3 + .4/x_5.$$

Based on this message, he modifies his expectations according to Eq. (6.3) such that

$$r_1(x) = \min[r_0^{.6}(x), \mu_{M_1'}(x)]$$

for each $x \in X$, or

$$\mathbf{r}_1 = .4/x_1 + .25/x_2 + .7/x_3 + .25/x_5.$$

The young man has thus greatly diminished his expectation of a simple yes, somewhat increased his expectation of a simple no and of derisive laughter, and has given up all hope of the possibility of joyful tears. Suppose now that, in disbelief, he asks the young woman to repeat her answer and receives the following message:

$$M_2 = .9/x_2 + .4/x_5.$$

This message is stronger, clearer, and less general than the first answer. Its consistency with the young man's new expectations is

$$s(M_2, \mathbf{r}_1) = .25.$$

Thus, the message is highly contrary even to the revised expectations of the young man, so let us suppose that he distorts the message such that he hears

$$M_2' = .97/x_2 + .8/x_5.$$

His surprise has thus diminished the clarity of the message heard and has led him to exaggerate the degree to which he believes that the young woman has responded with derisive laughter. Let us now suppose that the response which the young man makes will have characteristics chosen from the following set Y:

y_1 = happiness y_2 = pain y_3 = surprise

y_4 = anger y_5 = patience y_6 = impatience

y_7 = affection

Let the fuzzy relation $R \in Y \times X$ represent the degree to which the young man plans to respond to a given signal x with a response having the attribute y. This relation is given by the following matrix:

$$
\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{array}
\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \\
\left[\begin{array}{cccccc}
.9 & 0 & .2 & 0 & 0 & 1 \\
0 & .9 & .1 & .2 & 1 & 0 \\
.1 & .9 & .2 & .9 & 1 & .3 \\
0 & .5 & 0 & .6 & .7 & 0 \\
.1 & 0 & .9 & 0 & 0 & .5 \\
0 & .3 & .2 & .3 & .4 & 0 \\
.9 & 0 & .9 & .3 & 0 & 1
\end{array}\right]
\end{array}
$$

Using Eq. (6.4), we can now calculate the response which the young man will make to the message M_2':

$$A = R \circ M_2' = .9/y_2 + .9/y_3 + .7/y_4 + .4/y_6.$$

The young man's response, therefore, will have the characteristics of a great deal of pain and surprise, a large degree of anger, and some impatience.

6.5 MANAGEMENT AND DECISION MAKING

The subject of the field of decision making is, as the name suggests, the study both of how decisions are actually made and how they can be made better or more successfully. Much of the focus in this field has been in the area of management, in which the decision-making process is of key importance for functions such as inventory control, investments, personnel actions, new-product development, allocation of resources, and many others. Decision making itself, however, is broadly defined to include any choice or selection of alternatives and is therefore of importance in many fields in both the "soft" social sciences and the "hard" natural and engineering sciences.

Applications of fuzzy sets within the field of decision making have, for the most part, consisted of extensions or "fuzzifications" of the classical theories of decision making. While decision making under conditions of risk and uncertainty have been modeled by probabilistic decision theories and by game theories, fuzzy decision theories attempt to deal with the vagueness or fuzziness inherent in subjective or imprecise determinations of preferences, constraints, and goals. In this section, we briefly introduce some of the simple applications of fuzzy sets in some selected areas of decision making.

Classical decision making generally deals with a set of alternatives comprising the decision space, a set of states of nature comprising the state space, a relation indicating the state or outcome to be expected from each alternative action, and, finally, a utility or objective function, which orders these outcomes according to their desirability. A decision is said to be made under conditions of *certainty* when the outcome for each action can be determined and ordered precisely. In this case, the alternative that leads to the outcome yielding the highest utility is chosen. A decision is made under conditions of *risk*, on the other hand, when the only available knowledge concerning the outcome states is their probability distributions. Again, this information can be used to optimize the utility function. When knowledge of the probabilities of the outcome states is unknown, decisions must be made under conditions of *uncertainty*. In this case, fuzzy decision theories may be used to accommodate this vagueness.

There exist several major approaches within the theories of classical crisp decision making. Decisions may, for instance, be considered to occur in a single stage or in multiple stages. The decision maker may be a single person, or a collection of multiple decision makers may be involved. The decision may involve the simple optimization of a utility function, an optimization under constraints, or an optimization given multiple criteria. The first of these three types of decision problems corresponds to statistical decision theory, the second to mathematical (linear or nonlinear) programming, and the last to theories of multicriteria decision making.

Fuzziness can be introduced at several points in the existing models of decision making. For instance, under a simple optimization of utility, any uncertainty concerning the states of the system can be handled by modeling the states as well as the utility assigned to each state with fuzzy sets. Bellman and Zadeh [1970] suggest a fuzzy model of decisions that must accommodate certain constraints C and goals G. Here, both constraints and goals are treated as fuzzy sets characterized by membership functions

$$\mu_C : X \rightarrow [0, 1]$$

and

$$\mu_G : X \rightarrow [0, 1],$$

where X is the universal set of alternative actions. The expected outcomes that result from these actions can, in many cases, be assumed to remain deterministic or probabilistic, thus restricting the introduction of fuzziness only to the goals and constraints themselves. This fuzziness allows the decision maker to frame the goals and constraints in vague, linguistic terms, which may more accurately reflect the actual state of knowledge or preference concerning these. The membership function of the fuzzy goal in this case serves much the same purpose as a utility or objective function that orders the outcomes according to preferability. Unlike the classical theory of decision making under constraints, however, in which constraints are defined on the set X of alternatives and goals are modeled as performance functions from X to another space, the symmetry between the goals and constraints under this fuzzy model allows them to be treated in exactly the same manner. This model can be extended to allow goals and constraints to be defined on different universal sets, for instance, the set X of possible actions and the set Y of possible effects or outcomes. In this case, the fuzzy constraints may be defined on the set X and the fuzzy goals on the set Y such that

$$\mu_C : X \rightarrow [0, 1]$$

and

$$\mu_G : Y \rightarrow [0, 1].$$

A function f can then be defined as a mapping from the set of actions X to the set of outcomes Y,

$$f : X \rightarrow Y,$$

such that a fuzzy goal G defined on set Y induces a corresponding fuzzy goal G' on set X. Thus,

$$\mu_{G'}(x) = \mu_G(f(x)).$$

A fuzzy decision D may then be defined as the choice that satisfies both the goals G and the constraints C. If we interpret this as a logical "and," we can model it with the intersection of the fuzzy sets G and C,

$$D = G \cap C,$$

which can easily be extended for any number of goals and constraints. If the classical fuzzy set intersection is used, the fuzzy decision D is then specified by the membership function

$$\mu_D(x) = \min[\mu_G(x), \mu_C(x)],$$

where $x \in X$. This definition of the intersection does not allow, however, for any interdependence, interaction, or trade-off between the goals and constraints under consideration. For many decision applications, this lack of compensation may not be appropriate; the full compensation or trade-off offered by the union operation that corresponds to the logical "or" (the max operator) may be inappropriate as well. Therefore, an alternative fuzzy set intersection or aggregation operation may be used to reflect a situation in which some degree of positive compensation exists among the goals and constraints.

This fuzzy model can be further extended to accommodate the relative importance of the various goals and constraints by the use of weighting coefficients. In this case, the fuzzy decision D can be arrived at by a convex combination of the n weighted goals and m weighted constraints such that

$$\mu_D(x) = \sum_{i=1}^{n} u_i \mu_{G_i}(x) + \sum_{j=1}^{m} v_j \mu_{C_j}(x)$$

where u_i and v_j are weights attached to each fuzzy goal G_i ($i \in \mathbb{N}_n$) and each fuzzy constraint C_j ($j \in \mathbb{N}_m$), respectively, such that

$$\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j = 1.$$

Once a fuzzy decision has been arrived at, it may be necessary to choose the "best" single crisp alternative from this fuzzy set. This may be accomplished in a straightforward manner by choosing the alternative $x \in X$ that attains the maximum membership grade in D. Since this method ignores information concerning any of the other alternatives, it may not be desirable in all situations. Methods that calculate the mean or center of gravity of the fuzzy set D may therefore be used instead. These concepts have been effectively utilized to extend conventional crisp mathematical programming into methods of fuzzy mathematical programming.

The fuzzy model of decision making proposed by Bellman and Zadeh [1970] can be illustrated by a simple example. Suppose we must choose one of four different possible jobs a, b, c, and d, the salaries of which are given by the function f such that

$$f(a) = 30{,}000$$
$$f(b) = 25{,}000$$
$$f(c) = 20{,}000$$
$$f(d) = 15{,}000$$

Our goal is to choose the job that will give us a high salary given the constraints that the job is interesting and within close driving distance. This first constraint of interest value is represented by the fuzzy set C_1 defined on our universal set of alternative jobs as follows:

$$C_1 = .4/a + .6/b + .8/c + .6/d.$$

Our second constraint concerning the driving distance to each job is defined by the fuzzy set C_2 such that

$$C_2 = .1/a + .9/b + .7/c + 1/d.$$

The fuzzy goal G of a high salary is defined on the universal set X of salaries by the membership function

$$\mu_G(x) = \begin{cases} 1 & \text{for } x > 40{,}000 \\ -.00125\left(\dfrac{x}{1000} - 40\right)^2 + 1 & \text{for } 13{,}000 \le x \le 40{,}000 \\ 0 & \text{for } x < 13{,}000, \end{cases}$$

where $x \in X$, and the corresponding goal G' induced on the set of alternative jobs by the function f is given by

$$G' = .875/a + .7/b + .5/c + .2/d.$$

Taking the standard fuzzy set intersection of these three fuzzy sets, we obtain the fuzzy decision D, where

$$D = G' \cap C_1 \cap C_2 = .1/a + .6/b + .5/c + .2/d.$$

Finally, we take the maximum of this set to obtain alternative b as the choice that seems best to satisfy our goal and constraints. Note that no real distinction exists here between a goal and a constraint; that is, the two concepts are symmetric.

When decisions which are made by more than one person are modeled, two differences from the case of a single decision maker can be considered: first, the goals of the individual decision makers may differ such that each places a different ordering on the alternatives; second, the individual decision makers may have access to different information upon which to base their decision. Theories known as n-person game theories deal with both of these considerations, team theories of decision making deal only with the second, and group-decision theories deal only with the first.

A fuzzy model of group decision was proposed by Blin [1974] and Blin and Whinston [1973]. Here, each member of a group of n individual decision makers is assumed to have a reflexive, antisymmetric, and transitive preference ordering P_k, $k \in \mathbb{N}_n$, which totally or partially orders a set X of alternatives. A "social choice" function must then be found which, given the individual preference orderings, produces the most acceptable overall group preference ordering. Basically, this model allows for the individual decision makers to possess different aims and values while still assuming that the overall purpose is to reach a common, acceptable decision. In order to deal with the multiplicity of opinion evidenced in the group, the social preference S may be defined as a fuzzy binary relation with membership grade function

$$\mu_S : X \times X \rightarrow [0, 1],$$

which assigns the membership grade $\mu_S(x_i, x_j)$ indicating the degree of group preference of alternative x_i over alternative x_j. The expression of this group preference requires some appropriate means of aggregating the individual preferences. One simple method computes the relative popularity of alternative x_i over x_j by dividing the number of persons prefering x_i to x_j, denoted by $N(x_i, x_j)$, by the total number of decision makers, n. This scheme corresponds to the simple majority vote. Thus,

$$\mu_S(x_i, x_j) = \frac{N(x_i, x_j)}{n}. \qquad (6.8)$$

Other methods of aggregating the individual preferences may be used to accommodate different degrees of influence exercised by the individuals in the group. For instance, a dictatorial situation can be modeled by the group preference relation S for which

$$\mu_S(x_i, x_j) = \begin{cases} 1 & \text{if } x_i \overset{k}{>} x_j \text{ for some individual } k \\ 0 & \text{otherwise,} \end{cases}$$

where $\overset{k}{>}$ represents the preference ordering of the one individual k who exercises complete control over the group decision.

Once the fuzzy relationship S has been defined, the final nonfuzzy group preference can be determined by converting S into its resolution form

$$S = \bigcup_{\alpha} \alpha S_\alpha,$$

which is the union of the crisp relations S_α comprising the α-cuts of the fuzzy relation S, $\alpha \in \Lambda_S$ (the level set of S), each scaled by α. Each value α essentially represents the level of agreement between the individuals concerning the particular crisp ordering S_α. One procedure that maximizes the final agreement level consists of intersecting the classes of crisp total orderings that are compatible with the pairs in the α-cuts S_α for increasingly smaller values of α until a single crisp total ordering is achieved. In this process, any pairs (x_i, x_j) that lead to an intransitivity are removed. The largest value α for which the unique compatible ordering on $X \times X$ is found represents the maximized agreement level of the group and the crisp ordering itself represents the group decision. This procedure is illustrated in the following example.

Assume that each individual of a group of eight decision makers has a total preference ordering $\mathbf{P}_i$ ($i \in \mathbb{N}_8$) on a set of alternatives $X = \{w, x, y, z\}$ as follows:

$$\mathbf{P}_1 = (w, x, y, z)$$

$$\mathbf{P}_2 = \mathbf{P}_5 = (z, y, x, w)$$

$$\mathbf{P}_3 = \mathbf{P}_7 = (x, w, y, z)$$

$$\mathbf{P}_4 = \mathbf{P}_8 = (w, z, x, y)$$

$$\mathbf{P}_6 = (z, w, x, y)$$

Using the membership function given in Eq. (6.8) for the fuzzy group preference ordering relation S (where $n = 8$), we arrive at the following fuzzy social preference relation:

$$\mathbf{S} = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & .5 & .75 & .625 \\ .5 & 0 & .75 & .375 \\ .25 & .25 & 0 & .375 \\ .375 & .625 & .625 & 0 \end{bmatrix} \end{array}$$

The α-cuts of this fuzzy relation S are:

$$S_1 = \varnothing$$

$$S_{.75} = \{(w, y), (x, y)\}$$

$$S_{.625} = \{(w, z), (z, x), (z, y), (w, y), (x, y)\}$$

$$S_{.5} = \{(x, w), (w, x), (w, z), (z, x), (z, y), (w, y), (x, y)\}$$

$$S_{.375} = \{(z, w), (x, z), (y, z), (x, w), (w, x), (w, z), (z, x), (z, y), (w, y), (x, y)\}$$

$$S_{.25} = \{(y, w), (y, x), (z, w), (x, z), (y, z), (x, w), (w, x), (w, z), (z, x), (z, y), (w, y), (x, y)\}$$

We can now apply the procedure to arrive at the unique crisp ordering which constitutes the group choice. All total orderings on $X \times X$ are, of course, compatible with the empty set of S_1. The total orderings $O_{.75}$ that are compatible with the pairs in the crisp relation $S_{.75}$ are

$$O_{.75} = \{(z, w, x, y), (w, x, y, z), (w, z, x, y), (w, x, z, y), (z, x, w, y), (x, w, y, z), (x, z, w, y), (x, w, z, y)\}.$$

Thus,

$$O_1 \cap O_{.75} = O_{.75}.$$

The orderings compatible with $S_{.625}$ are

$$O_{.625} = \{(w, z, x, y), (w, z, y, x)\}$$

and

$$O_1 \cap O_{.75} \cap O_{.625} = \{(w, z, x, y)\}.$$

Thus, the value .625 represents the group level of agreement concerning the social choice denoted by the total ordering (w, z, x, y).

6.6 COMPUTER SCIENCE

Applications of the mathematics of uncertainty and information within the field of computer science have been quite extensive, particularly in those endeavors concerned with the storage and manipulation of knowledge in a manner compatible with human thinking. This includes the construction of database and information storage and retrieval systems as well as the design of computerized expert systems. In this section, we give a brief overview and example of the application of fuzzy set theory in these two areas.

Database

The motivation for the application of fuzzy set theory to the design of databases and information storage and retrieval systems lies in the need to handle information that is less than ideal in the sense of being incomplete, indeterministic, contradictory, vague, imprecise, and so on. The database that can accommodate imprecise information can store and manipulate not only precise facts but also subjective expert opinions, judgments, and values that can be specified in linguistic terms. This type of information can be quite useful when the database is to be used as a decision aid in areas such as medical diagnosis, employment, investment, and geological exploration, where "soft," subjective, and imprecise data are not only common but quite valuable. In addition, it is also desirable to relieve the user of the constraint of having to formulate queries to the database in precise terms. Vague queries such as, "Which employment candidates are highly educated and moderately experienced?" or "What industries are forecasted to experience significant growth by a substantial number of experts?" often capture the relevant concerns of database users more accurately and easily than precise queries. It is important, however, that the database system incorporating imprecision be able to propagate appropriately the level of uncertainty associated with the data to the level of uncertainty associated with answers or conclusions based on the data. Precise answers should not be generated from imprecise data. An overview of some of the problems and possibilities in database technology and a discussion of the uses of imprecision in database design can be found in a paper by Gaines [1981].

Buckles and Petry [1982a, 1982b, 1983, 1984] present a model for a fuzzy relational database that contains as a special case the classical crisp model of a relational database. The model of a classical relational database consists of a series of n-dimensional relations conceptualized as tables. The columns of these tables correspond to fields or attributes and are usually called *domains*. Each domain is defined on an appropriate domain base (or universal) set. The rows are elements of the relation; they correspond to records or entries and are called *tuples*. Access to the database is accomplished through a relational algebra. This algebra consists of the procedural application of operations containing four basic elements: an operation name, the names of relations and the names of domains to be operated on, and an optional conditional expression. For instance, if our database contains a ternary relation STUDENT with domains NAME, ADDRESS, and MAJOR, we can obtain the names and addresses of all students whose major is computer science by constructing a new relation with domains NAME and ADDRESS as a projection of the original relation. The algebraic operation performing this task would be

Project(STUDENT:NAME,ADDRESS) where
MAJOR = "computer science"

The algebra also contains other relational operations such as **complement**, **union**, **intersection**, or **join**, which perform the corresponding task on the relation and domains specified in order to produce the desired information.

The fuzzy relational database proposed by Buckles and Petry differs from this crisp model in two ways: first, elements of the tuples contained in the relations may be subsets of the domain universal set and, second, a similarity relation is defined on each domain universal set. The first qualification allows the elements of tuples to consist either of singletons of the domain universal sets (as in the conventional relational database model) or of crisp subsets of the domain universal sets as in the relation MARKETS with the domains AREA, SIZE, and POTENTIAL represented by the following table:

RELATION: MARKETS		
AREA	SIZE	POTENTIAL
East	*Large*	*Good*
Midwest	*{Large, Medium}*	*{Moderate, Good}*
South	*Small*	*{Good, Excellent}*

Domain values that are not singletons may indicate, for instance, the merging of the opinions or judgments of several experts.

The second qualification is based on the assumption that in the classical database model, a crisp equivalence relation is defined on each domain universal set which groups together elements which are strictly equivalent. This identity is utilized, for example, when redundant tuples are to be eliminated or ignored. Most often, the equivalence classes generated by this relation are simply the singletons of the universal set. In the fuzzy database model, this equivalence relation is generalized to a fuzzy similarity relation. This introduction of fuzziness provides an interesting element of flexibility, since the value or meaning structures of different individual database users may be reflected by modifying the domain similarity relations appropriately. The fuzzy relational algebra used to access this fuzzy database consists of the same four components as the conventional relational algebra and, in addition, allows for the specification of a threshold level defining the minimum acceptable degree of similarity between elements in some specified domain. In the special case of the conventional database, all threshold levels are implicitly assumed to be equal to 1, thus requiring strict equivalence for the merging or elimination of tuples. In the fuzzy database, tuples may be merged if they are considered sufficiently similar.

As an example of the use of this fuzzy database model and its associated fuzzy relational algebra, suppose our database contains the opinions of a group of experts on three policy options X, Y, and Z. Two relations are contained within the database: EXPERT, which has domains NAME and FIELD and which associates the name and field of each expert, and ASSESSMENT, which has domains OPTION, NAME, and OPINION and associates the name of each expert with their expressed opinions on the policy options. These two relations are specified in Table 6.2. In addition, the following similarity relation is defined for the domain OPINION on the domain universal set {*highly favorable* (HF), *favorable* (F), *slightly favorable* (SF), *slightly negative* (SN), *negative* (N), and *highly negative* (HN)}:

$$
\begin{array}{c} \\ \text{HF} \\ \text{F} \\ \text{SF} \\ \text{SN} \\ \text{N} \\ \text{HN} \end{array}
\begin{array}{c} \begin{array}{cccccc} \text{HF} & \text{F} & \text{SF} & \text{SN} & \text{N} & \text{HN} \end{array} \\
\left[\begin{array}{cccccc}
1 & .8 & .6 & .2 & 0 & 0 \\
.8 & 1 & .8 & .6 & .2 & 0 \\
.6 & .8 & 1 & .8 & .6 & .2 \\
.2 & .6 & .8 & 1 & .8 & .6 \\
0 & .2 & .6 & .8 & 1 & .8 \\
0 & 0 & .2 & .6 & .8 & 1
\end{array}\right] \end{array}
$$

Crisp equivalence relations in which equivalence classes are singletons are assumed to be defined on domains NAME, FIELD, and OPTION.

Suppose now that our query to this fuzzy database consists of the following question:

> Which sociologists are in *considerable agreement* with Kass concerning policy option Y?

The first step is to retrieve the opinion of Kass concerning option Y. This is accomplished with the relational algebraic operation

(Project(Select ASSESSMENT where NAME = Kass and

OPTION = Y) over OPINION) giving R_1.

TABLE 6.2. EXAMPLES OF RELATIONS IN RELATIONAL DATABASE

Relation: EXPERT

NAME	FIELD
Cohen	*Sociologist*
Fadem	*Economist*
Fee	*Attorney*
Feldman	*Economist*
Kass	*Physician*
Osborn	*Sociologist*
Schreiber	*Sociologist*
Specterman	*Sociologist*

Relation: ASSESSMENT

OPTION	NAME	OPINION
X	Osborn	*Favorable*
X	Fee	*Negative*
X	Fadem	*Slightly favorable*
X	Feldman	*Highly favorable*
Y	Cohen	*Slightly negative*
Y	Osborn	*Slightly favorable*
Y	Fee	*Highly favorable*
Y	Schreiber	*Favorable*
Y	Kass	*Favorable*
Y	Fadem	*Negative*
Y	Specterman	*Highly favorable*
Y	Feldman	*Slightly negative*
Z	Osborn	*Negative*
Z	Kass	*Slightly negative*
Z	Fee	*Slightly favorable*

The resulting temporary relation R_1 on domain OPINION is given by

RELATION: R_1 OPINION
Favorable

The next step involves the selection of all sociologists from the table of experts. This is accomplished by the operation

(Project(Select EXPERT where FIELD = *Sociologist*)

over NAME) giving R_2.

Here R_2 is a temporary relation on domain NAME listing only sociologists. It is equal to

RELATION: R_2 NAME
Osborn
Schreiber
Cohen
Specterman

Next, temporary relation R_3 must be constructed on domains NAME and OPIN-

ION, which lists the opinions of the sociologists in R_2 about option Y. The algebraic expression accomplishing this is

$$\text{(Project(Select(Join } R_2 \text{ and ASSESSMENT over NAME)}$$
$$\text{where OPTION} = Y\text{) over NAME, OPINION) giving } R_3.$$

The relation R_3 is given by this table:

RELATION: R_3	
NAME	OPINION
Osborn	*Slightly favorable*
Schreiber	*Favorable*
Cohen	*Slightly negative*
Specterman	*Highly favorable*

Finally, we perform a join of relations R_1 (giving the opinion of Kass) and R_3 (giving the opinion of the sociologists) that specifies a threshold similarity level of .75 on the domain OPINION, which is chosen for this example to represent the condition of *considerable agreement*. The algebraic expression for this task is

$$\text{(Join } R_3 \text{ and } R_1 \text{ over OPINION) with}$$
$$\text{THRES(OPINION)} \geq .75, \text{ and THRES(NAME)} \geq 0.$$

The specification of a zero similarity threshold level for NAME is necessary to allow the merging of names into sets as shown in the results given by

NAME	OPINION
{Osborn, Schreiber, Specterman}	{*Slightly favorable, favorable, highly favorable*}

Note that the resulting response is less precise and contains less information than a response of

NAME	OPINION
Osborn	*Slightly favorable*
Schreiber	*Favorable*
Specterman	*Highly favorable*

In this way, the uncertainty contained in the specification of *considerable agreement* and in the similarity defined over the possible opinions is propagated to the response given. Buckles and Petry [1983] present an interesting investigation into the use of probabilistic and fuzzy information measures to determine the precision of data representation and of response generation of their fuzzy database.

Expert Systems

A computerized expert system, as the name suggests, models the reasoning process of a human expert within a specific domain of knowledge in order to make the experience, understanding, and problem-solving capabilities of the expert available to the nonexpert for purposes of consultation, diagnosis, learning, decision support, or research. Usually an expert system is distinguished from a sophisticated lookup table (which merely maps questions to answers) by the attempt to include in the expert system some sense of an understanding of the meaning and relevance of questions and information and an ability to draw nontrivial inferences from data. Computerized rule-driven control systems (which we discuss in Sec. 6.3) are sometimes considered to constitute a subset of expert systems, which emulate the reasoning of an expert human operator.

Many expert systems are designed to interact directly with the user in a dialogue format; the user generally provides the parameters of the problem of concern and the expert system provides relevant advice, judgments, or information. Often the system can also provide the user with an explanation of the reasoning process employed to arrive at the conclusions. Some specific domains for which expert systems have been designed or proposed include the areas of medical diagnosis and treatment, chemistry, technical troubleshooting, geological exploration, quality control, damage assessment, management forecast, and investment advice. Most expert systems consist of a domain-specific *knowledge base* and problem-solving or reasoning algorithms known as an *inference engine*. Some systems consist of an inference engine *shell* to which various knowledge bases can be added, thus allowing the creation of different "experts." In addition, some means of *knowledge acquisition* must be incorporated into the system. The methods of eliciting human expert knowledge (which may be implicit even to the human expert) and translating it into a form suitable for use by a computerized expert system is known as *knowledge engineering*; it constitutes one of the major challenges in the field of expert systems. Knowledge acquisition may also take place through a learning capability; this highly desirable feature allows the expert system to collect knowledge or modify rules based on feedback during operation.

The facts, relations, judgments, opinions, and "rules of thumb" contained within the expert knowledge base usually manifest varying degrees of imprecision and uncertainty. It is nevertheless desirable for an expert system to be able, like the human expert, to draw nontrivial inferences from imprecise data and vague heuristics. Thus, the management of uncertainty in the design of expert systems is of key importance for the successful modeling of the reasoning process, and the utility of fuzzy set theory and the theory of evidence for this purpose has been and continues to be extensively studied.

The inference engine of an expert system operates on a series of statements known as *production rules*, which connect antecedents with consequences, premises with conclusions, or conditions with actions. They most commonly have the form *IF A*, *THEN B*. These rules offer the advantage of a high degree of modularity; they are easily added, modified, or deleted from the rule base. There exist two major approaches to evaluating these production rules. The first is data driven and corresponds to the logical inference method of modus ponens. In this case, the available data is supplied to the expert system, which then uses it to evaluate the production rules and to draw all possible conclusions. An alternative method of evaluation is goal driven; it corresponds to the modus tollens form of logical inference. Here, the expert system searches for the data specified in the *IF* clauses of rules that will lead to the objective; these data are found either in the knowledge base, in the *THEN* clauses of other production rules, or by querying the user. Since the data driven method proceeds from *IF* clauses to *THEN* clauses in a chain through the production rules, it is commonly called *forward chaining*; because the goal driven method proceeds from the *THEN* clauses (objectives) to the *IF* clauses in its search for the required data, it is commonly called *backward chaining*. Backward chaining has the advantage of speed, since only those rules leading to the objective will be evaluated. Furthermore, if certain data are difficult to provide and only potentially necessary, then the backward chaining method is clearly superior.

The rules employed by human experts are often of an imprecise or heuristic nature; the use of bivalent truth values or the requirement of exact satisfaction of *IF* clause conditions for rule evaluation often seems unnatural in the context of human reasoning. Therefore, a great deal of research has taken place on the applications of approximate reasoning and fuzzy logic in the inference process. Whalen and Schott [1985b] describe one use of fuzzy production rules in an interactive expert system designed to aid in the selection of appropriate techniques of forecasting sales of commercial products. This system, called fINDex, operates in a goal-driven manner by asking the user an increasingly specific series of questions concerning constraints on the forecast technique to be used. These constraints consist of the type and quality of outputs required from the forecast technique and the resources such as data, time, and money that are available for implementation of the forecast. The system does not perform the forecast itself but employs the constraints input by the user to produce a small list of possible forecast techniques. This list is a fuzzy set, where membership grades of the techniques indicate their possibilities under the constraints as evaluated by the fuzzy production rules. These fuzzy production rules have the form illustrated by the following examples:

> IF the long-term historical data available are at least fair, THEN regression analysis is possible.
>
> IF medium-term accuracy is required at a level of quality at most good to very good, THEN regression analysis is possible.

The final possibility of any one technique is determined to be the minimum possibility specified by any of the production rules.

An interactive session with fINDex begins with the system querying the user about some very basic constraints such as the time and money available for the forecast. The user may answer the questions as precisely or as vaguely as desired or may decline to answer at all; this allows an accurate reflection of the precision of the user's current knowledge concerning the constraints. At any point, the user may request that the system take the answers provided thus far and evaluate the possibilities of the various techniques. In response to this request, fINDex performs a forward chaining on the production rules based on the available data in order to determine the possibility of each technique. The user may either accept this list and end the session or request a further winnowing of the techniques. In order to accomplish this latter task, fINDex must first compile a list of questions ordered by importance, which seek from the user the data necessary for further determination of the possibilities of the techniques. The most important data concern highly relevant constraints on those techniques that are currently highly possible. The relevance of a constraint such as time or money for a particular technique is based on the degree to which the technique is possible only for a narrow range of constraint values. Thus, time is a more relevant constraint for a lengthy technique than for a short one. The importance of each constraint for each technique is determined by a version of backward chaining on the production rules. The ordered list of *askable* questions about these constraints is then generated, where *askability* is high if the corresponding constraint is important (*strict* for some *highly possible* techniques), if the system possesses only vague knowledge about it, and if the question has not been asked recently. The session then continues in the question-answer mode until the user requests processing of the data.

The fINDex system manipulates vague linguistic terms such as *highly possible*, *strict*, *at least fair*, and *very good* with the use of fuzzy sets. In addition, fuzzy sets defining linguistic terms *days*, *weeks*, *months*, and *one (figure)* through *six (figures)* (for money) are defined. The use of these linguistic terms allows the compilation of fuzzy production rules to be made directly from published documentation concerning forecasting techniques. They also allow natural language responses to be made by the user at various levels of precision. The following illustrates a sample session with fINDex:

fINDex: Please give information, if possible, on the following condition. If you cannot supply information on the condition, just press return. Time available?

USER: ⟨return⟩ (no response; interpreted as unknown)

fINDex: Do you want to respond to more questions (type q) or do you want to get a reaction to the information provided already (type r)?

USER: q

fINDex: Please give information, if possible, on the following condition. If you cannot supply information on the condition, just press return. Knowledge of structural relations available?

USER: Very good
fINDex: Long term accuracy required?
USER: Poor
fINDex: More questions or a reaction?
USER: q
fINDex: Recent historical data available?
USER: Fair
fINDex: More questions or a reaction?
USER: r
fINDex:

Possibility	Technique
1	Delphi method
1	Input-output analysis
1	Panel consensus
1	Visionary forecast
1	Historical analogy
1	Trend projection
1	Regression analysis
1	Econometric analysis
.35	Life-cycle analysis
.26	Moving averages method
.26	Exponential smoothing method
.26	Box-Jenkins method
.26	X-11
.26	Intention-to-buy and surveys
.26	Economic input-output model
.26	Diffusion indexes
.26	Leading indicators
0	Market research

Do you want to stop now (type s) or continue (type c)?
USER: c
fINDEX: Delphi-coordinator available?
USER: ⟨return⟩ (no response)
fINDex: More questions or a reaction?
USER: q
fINDex: Panel of experts available?
USER: ⟨return⟩ (no response)
fINDex: More questions or a reaction?
USER: q
fINDex: Short-term accuracy required?
USER: Good to very good
fINDex: More questions or a reaction?
USER: r

fINDex: Possibility Technique*

Possibility	Technique
1	Delphi method
1	Trend projection
1	Regression analysis
1	Econometric analysis
1	Input-output analysis
.26	Exponential smoothing method
.26	Box-Jenkins method
.26	X-11
.26	Economic input-output model
.01	Moving averages method
.01	Intention-to-buy and surveys
.01	Diffusion indexes
.01	Leading indicators

Do you want to stop now (type s) or continue (type c)?
USER: c
fINDex: Turning point identification required?
USER: Very good
fINDex: More questions or a reaction?
USER: r
fINDex: Possibility Technique

Possibility	Technique
1	Regression analysis
1	Econometric analysis
.26	X-11

Do you want to stop now (type s) or continue (type c)?
USER: c
fINDex: Funds available?
USER: Lower three (three indicates a three-figure amount)
fINDex: More questions or a reaction?
USER: r
fINDex: Possibility Technique

Possibility	Technique
1	Regression analysis
.26	X-11
.18	Econometric analysis

Do you want to stop (type s) or continue (type c)?
USER: s

One of the major concerns in the design of both information storage and retrieval systems and expert systems is efficiency; these systems must be able to perform quickly enough to make interaction with human users or with a controlled process feasible, despite large amounts of stored data and large numbers of in-

* Only techniques with nonzero possibility are listed from this point on.

ference rules. The implementation of fuzzy databases and expert systems utilizing fuzzy logics may soon be made far more efficient with the introduction of integrated circuits designed specifically to implement fuzzy logic. Scattered reports indicate that prototypes for these chips have already been produced and tested. A particular VLSI implementation of fuzzy logic is described in a paper by Togai and Watanabe [1986].