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UNIT-I

Variational problems with fixed Boundaries.

1. Define: Functional

A real-valued function F whose domain is the set of all real function $\{y(x)\}$ is known as functional. The function $y(x)$ is independent variable. Funtional are variable quantities whose values are determined by the choice on one or several functions.

EX:

The length L of a curve C whose equation is $y = y(x)$ passing through two given points $A(x_0, y_0)$ & $B(x_1, y_1)$ is given by,

$$L = \int_{x_0}^{x_1} dS$$

$$= \int_{x_0}^{x_1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

where y' denotes derivatives of y.

2. Extremum or external value:-

The calculus of variations treated with the maxima or minima of functionals $y(x)$ which are collectively called extremum of the functional.

Concept of variation and its properties.

A variable quantity $I[y(x)]$ is a functional depend on a function $y(x)$ if each function $y(x)$ belonging to a certain class of function C. There is a definite value of I.

Thus there is a correspondance b/w a given function $y(x)$ and a numbers

By the implement of variation δy of the arguement $y(x)$ of a functional I, the differences $\delta y = y(x) - y_1(x)$ b/w two functions belonging to a certain class.

A functional $I[y(x)]$ is said to be continuous if a small charges in $y(x)$ result in a small change in $I[y(x)]$

one way of specifying the closeness of $y(x)$ and $y_1(x)$ is ~~said~~ to same that the absolute value of their difference is given by $|y(x) - y_1(x)|$ is small for all x for which $y(x)$ and $y_1(x)$ are defined, when this happens we say that $y(x)$ is close to $y_1(x)$ in the sense of zero order proximity.

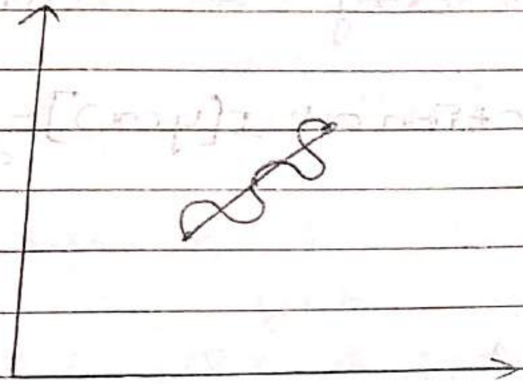
The functional $I[y(x)] = \int_a^b F[x, y(x), y'(x)]dx$

this occurs in many applications the extension of the notation of closeness of the curve $y = y(x)$ and $y = y_1(x)$ such that both

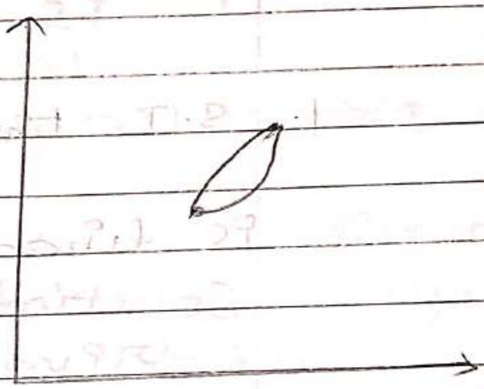
$|y(x)-y_1(x)|$ & $|y'(x)-y_1'(x)|$ are small for all value of x.

In general the curve $y=y(x)$ and $y=y_1(x)$ are said to close in the sense of nth order proximity if

$|y(x)-y_1(x)|, |y'(x)-y_1'(x)|, \ldots |y^n(x)-y_1^n(x)|$ are small for all value of x.



curve close in the sense of zero order protcimity.



curve in the sense of first order proximity

Continuous functional:-

The functional $I[y(x)]$ is said to be continuous at $y=y_0(x)$ in the sense of nth order proximity if given any +ve number ϵ then $\exists\ \delta>0$, s.t

$$\left|I[y(x)]-I[y_0(x)]\right|<\epsilon$$

whenever,

$|y(x)-y_0(x)|<\delta$, $|y'(x)-y_0'(x)|<\delta$, $|y^n(x)-y_0^n(x)|<\delta$

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Linear functional:-

Linear functional $I[y(x)]$

A linear functional defined in the normal linear space M of the function $y(x)$, this function is said to be linear

i) $I[y_1(x)+y_2(x)] = I[y_1(x)] + I[y_2(x)]$

ii) $I[c\,y(x)] = c\,I[y(x)]$

where $y(x), y_1(x), y_2(x) \in M$ & c is arbitrary constant.

1\. S.T the functional $I[y(x)] = \int_a^b [y'(x)+2y(x)]\,dx$ is linear.

Solution:-

Given that,

$$I[y(x)] = \int_a^b [y'(x)+2y(x)]\,dx$$

let

$$I[y_1(x)] = \int_a^b [y_1'(x)+2y_1(x)]\,dx$$

$$I[y_2(x)] = \int_a^b [y_2'(x)+2y_2(x)]\,dx$$

$$= \int_a^b [y_1'(x)+2y_1(x)]\,dx + \int_a^b [y_2'(x)+2y_2(x)]\,dx$$

$$= I[y_1(x)] + I[y_2(x)]$$

$$I[(c\,y(x))] = \int_a^b c[y'(x)+2y(x)]\,dx$$

$$= c\int_a^b [y'(x)+2y(x)]\,dx$$

$$I[c\,y(x)] = c\,I[y(x)]$$

$\therefore$ The given functional is linear

Euler's Equation (or) State & prove Euler's equation.

Statement:

The necessary condition for $I = \int_a^b F[x, y, y'] \, dx$ is to be extremum is that $F_y - \frac{d}{dx} F_{y'} = 0$ is called Euler's equation.

Proof:

consider the functional

$$I[y(x)] = \int_a^b F(x, y(x), y'(x)) \, dx \rightarrow (1)$$

Subject to the boundary condition $y(a) = y_1$, $y(b) = y_2$, where y_1 and y_2 are fixed point of a and b.

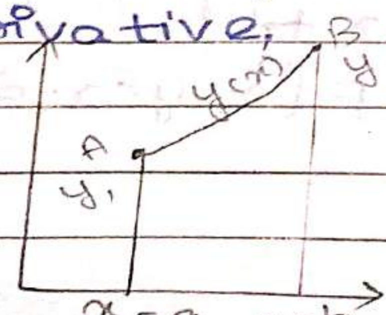
let us assume that $F[x, y(x), y'(x)]$ is three times differentiable.

W.K.T

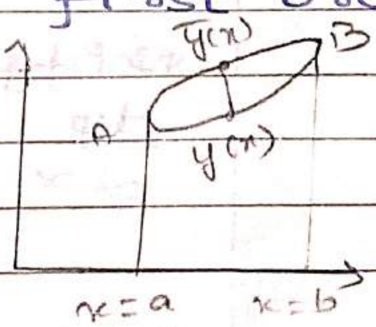
The necessary condition for the extremum of the functional is the variation must be vanish.

Now, apply this condition to the function $F[x, y(x), y'(x)]$ and assume that, the admissiable curve on ~~dy~~

which extremum is achieved admits of continuous first order derivative,



extremizing curve.



extremizing curve and admissable curve

Euler's Equation (or) State & prove Euler's equation.

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∴ dy

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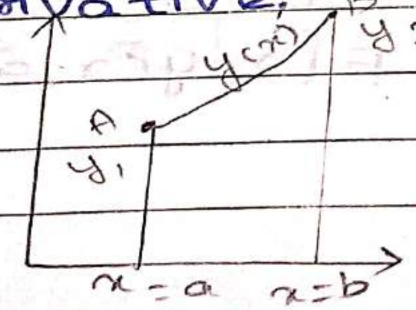
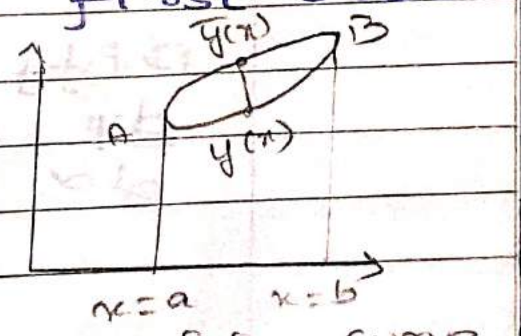




extremizing curve and admissible curve

curve.

let $y = y(x)$ be the curve extremize the functional $F[x, y, y']$ such is twice differentiable and satisfies the boundary condition.

let $y = \overline{y(x)}$ be an admissable curve closed to $y = y(x)$ s.t both $y(x)$ & $\overline{y(x)}$ can be included in one parameter family of curve

i, e) $y(x, \alpha) = y(x) + \alpha \delta y$

$= y(x) + \alpha[\overline{y}(x) - y(x)] \rightarrow (2)$

when $\alpha = 0$

$y(x, 0) = y(x)$

when $\alpha = 1$

$y(x, 1) = \overline{y(x)}$

The difference $\overline{y(x)} - y(x)$ is the variation δy of the function y on the curve of the same family.

The functional $F[x, y(x), y'(x)]$ reduces to a function α say $\psi(\alpha)$

i, e) $\psi(\alpha) = \int_a^b F[x, y(x, \alpha), y'(x, \alpha)]dx$ $\rightarrow (3)$

The necessary condition for an extremum is,

$$\frac{d\psi}{d\alpha} = 0$$

Diff (3) w.r to α

$$\frac{d\psi}{d\alpha} = \frac{d}{d\alpha}\int_a^b F[x, y(x, \alpha), y'(x, \alpha)]dx$$

$$= \int_a^b \frac{\partial}{\partial \alpha} F[x, y(x,\alpha), y'(x,\alpha)]\, dx$$

$$= \int_a^b \left[0 + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha}\right] dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial \alpha}\right] dx$$

$$= \int_a^b [F_y \delta y + F_{y'} \delta y']\, dx$$

$$= \int_a^b F_y \delta y\, dx + \int_a^b F_{y'} \cdot \delta y'\, dx \rightarrow (4)$$

Integrate the second term by integration by parts with boundary condition.

$[\delta y]_a = 0, \quad [\delta y]_b = 0$

$u = F_{y'} \quad ; \quad dv = \int y' dx$

$\frac{du}{dx} = \frac{d}{dx} F_{y'} \quad ; \quad v = \delta v$

$$\int_a^b F_{y'} \delta y'\, dx = [F_{y'} \delta y]_a^b - \int_a^b \delta y \frac{d}{dx} F_{y'} dx$$

$$= [0 - 0] - \int_a^b \frac{d}{dx} F_{y'} \delta y\, dx$$

$$\int_a^b F_{y'} \delta y'\, dx = -\int_a^b \frac{d}{dx} F_{y'} \delta y\, dx$$

$$(4) \Rightarrow \frac{d\psi}{d\alpha} = \int_a^b F_y \delta y\, dx - \int_a^b \frac{d}{dx} F_{y'} \delta y\, dx$$

$$\frac{d\psi}{d\alpha} = \int_a^b \left[F_y - \frac{d}{dx} F_{y'}\right] \delta y\, dx$$

P.T.O.

$\therefore \frac{d\psi}{d\alpha} = 0$

$$\int_a^b \left[F_y - \frac{d}{dx} F_{y'} \right] \delta y \, dx = 0$$

By fundamental theorem,

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$\therefore$ which is the Euler's equation.

problems:-

1. Test for an extremum of the functional $I[y(x)] = \int_0^1 [xy + y^2 - 2y^2y'] \, dx$, $y(0)=1$, $y(1)=2$

Solution:

Given that,

$$I[y(x)] = \int_0^1 [xy + y^2 - 2y^2y'] \, dx$$

Here,

$$F = xy + y^2 - 2y^2y'$$

$$F_y = x + 2y - 4yy'$$

$$F_{y'} = 0 + 0 - 2y^2$$

By using Euler's equation,

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$x + 2y - 4yy' - \frac{d}{dx}[-2y^2] = 0$$

$$x + 2y - 4yy' + 4yy' = 0$$

$$x + 2y = 0$$

$$2y = -x$$

$$y = -x/2$$

put $x=0$

$y(0)=0$

put $x=1$,

$y(1)=-\frac{1}{2}$

Clearly, this extremal doesnot satisfy the boundary condition.

2\. Test for an extremum of the function $I[y(x)]=\int_0^{\pi/2}[y'^2-y^2]dx$, $y(0)=0$, $y(\pi/2)=1$.

Solution:

Given that

$I[y(x)]=\int_0^{\pi/2}[y'^2-y^2]dx$, $y(0)=0$,

$y(\pi/2)=1$.

Here,

$$F=y'^2-y^2$$

$$F_y=-2y$$

$$F_{y'}=2y'$$

By using Euler's equation,

$$F_y-\frac{d}{dx}F_{y'}=0$$

$$-2y-\frac{d}{dx}2y'=0$$

$$-2y-2y''=0$$

$\div\ -2$

$$y+y''=0$$

$$y''+y=0$$

The A.E

$$m^2+1=0$$

$$m=\pm i$$

$y = c_1 \cos x + c_2 \sin x$

put $x = 0$

$y(0) = c_1$

$c_1 = 0$

put $x = \pi/2$

$y(\pi/2) = 0 + c_2$

$c_2 = 1$

$\therefore y = \sin x$

Thus the extremum can be achieved only on the curve $y = \sin x$.

Note:-

1. If F is depend on the function x and y then the Euler's equation reduced to

$$F_y = 0$$

2. If F is depend on x and y' then the Euler's equation reduced to

$$\frac{d}{dx} F_{y'} = 0$$

which has integrals $F_{y'} = c$

3. If F is depend on y and y' then the Euler's equation reduced to,

$$F - y' F_{y'} = c.$$

3. Find the extremum of the functional
$I[y(x)] = \int_{x_0}^{x} \frac{(1+y'^2)^{1/2}}{x} dx$

Solution:-

Given that $F = \frac{\sqrt{1+y'^2}}{x} \rightarrow (1)$

The function is depend on x and y'.
The Euler's equation is,

$$\frac{d}{dx} F_{y'} = 0$$

Diff (1) p.w.r to y'

$$F_{y'} = \frac{1}{x} \frac{1}{2\sqrt{1+y'^2}} 2y'$$

$$F_{y'} = \frac{y'}{x\sqrt{1+y'^2}}$$

Since,

$$\frac{d}{dx} F_{y'} = 0$$

$$\frac{d}{dx}\left[\frac{y'}{x\sqrt{1+y'^2}}\right] = 0$$

on integrating,

$$\frac{y'}{x\sqrt{1+y'^2}} = c$$

$$x = \frac{y'}{c\sqrt{1+y'^2}}$$

put $y' = \tan t$, where t be the parameter

$$x = \frac{\tan t}{c\sqrt{1+\tan^2 t}}$$

$$= \frac{\sin t/\cos t}{c\sqrt{\sec^2 t}}$$

$$= \frac{\sin t/\cos t}{c \cdot \sec t}$$

$$= \frac{\sin t/\cos t}{c/\cos t}$$

$$= \frac{\sin t}{c}$$

$x = c\sin t \rightarrow (2)$

Since $y' = \tan t$

$$\frac{dy}{dx} = \tan t$$

$$dy = \tan t \, dx$$

$$= \tan t \cdot c \cdot \cos t \, dt$$

$$dy = c\sin t \, dt$$

on $\int$ ing

$$y = -c\cos t + c_1 \rightarrow (3)$$

$(2)^2 + (3)^2 \Rightarrow$

$$x^2 + y^2 = c^2\sin^2 t + (-c\cos t + c_1)^2$$

$$= c^2\sin^2 t + c^2\cos^2 t - 2cc_1\cos t + c_1^2$$

$$= c^2 - 2cc_1\cos t + c_1^2$$

$$= c^2 + c_1^2 + 2c_1(y - c_1)$$

$$= c^2 + c_1^2 + 2c_1 y - 2c_1^2$$

$$= c^2 - c_1^2 + 2c_1 y$$

$$x^2 + y^2 - 2c_1 y + c_1^2 = c^2$$

$$x^2 + (y - c_1)^2 = c^2$$

Which is the equation of circle.

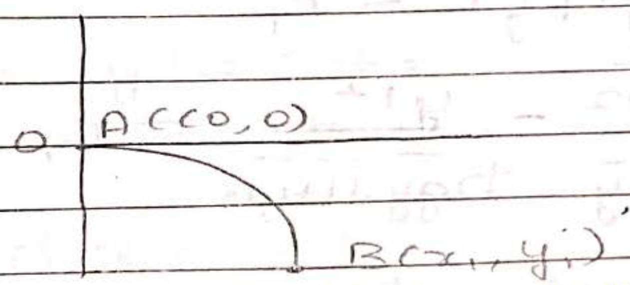
4. State and prove Branchistochrone problem.

Statement:-

The curve joining given points A and B which is tranversed by a particle moving under gravity from A to B in the shortest time, this problem is known as Brachistochrone problem.

Proof:

Fix the origin at A with x-axis horizontally and y-axis vertically downwards.



The speed of the particle is given by $\sqrt{2gy}$, where g being the acceleration due to gravity.

Thus the time taken by the particle is moving from A(0,0) to $B(x_1, y_1)$ is given by

$$T[y(x)] = \int_0^{x_1} \frac{ds}{\sqrt{2gy}}$$

$$= \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx$$

with the initial condition $y(0)=0$, $y(x_1)=y_1$.

Here, $F=\dfrac{\sqrt{1+y'^2}}{\sqrt{2gy}} \rightarrow (1)$

The function depends on y and y'

$\therefore$ The Euler's eqn is

$$F-y'F_{y'}=c_1$$

diff (1) p.w.r to y'

$$F_{y'}=\frac{1}{\sqrt{2gy}}\cdot\frac{1}{2\sqrt{1+y'^2}}\,2y'$$

$$F_{y'}=\frac{y'}{\sqrt{1+y'^2}\sqrt{2gy}}$$

$$F-y'F_{y'}=c_1$$

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}-\frac{y'^2}{\sqrt{2gy}\sqrt{1+y'^2}}=c_1$$

$$\frac{1+y'^2-y'^2}{\sqrt{2gy}\sqrt{1+y'^2}}=c_1$$

$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}}=c_1$$

$$\frac{1}{\sqrt{y}\sqrt{1+y'^2}}=c_1\sqrt{2g}$$

$$\frac{1}{\sqrt{y}\sqrt{1+y'^2}}=c$$

$$\sqrt{y}=\frac{1}{c\sqrt{1+y'^2}}$$

squaring on both sides

$$y = \frac{1}{c[1+y'^2]}$$

put $y' = \cot t$ where t is any parametere.

$$y = \frac{c}{1+\cos^2 t}$$

$$y = \frac{c}{\operatorname{cosec}^2 t}$$

$$y = c\sin^2 t$$

$$y = c\left(\frac{1-\cos 2t}{2}\right)$$

$$y = \frac{c}{2}[1-\cos 2t] \rightarrow ②$$

Since $y' = \cot t$

$$\frac{dy}{dt} = \cot t$$

$$dx = \frac{dy}{\cot t}$$

$$dx = \frac{c.\sin 2t.dt}{\cot t}$$

$$= \frac{c.2\sin t.\cos t.dt}{\cos t/\sin t}$$

$$dx = 2c\sin^2 t\ .dt$$

$$dx = 2c\left[\frac{1-\cos 2t}{2}\right]dt$$

$$dx = c[1-\cos 2t]dt$$

on ∫ing

$$x = c\left[t - \frac{\sin 2t}{2}\right]$$

$$x = \frac{c}{2}[2t - \sin 2t] \rightarrow ③$$

put $2t = t_1$ in ② & ③

$$y = \frac{c}{2}[1 - \cos t_1]$$

$$x = \frac{c}{2}[t_1 - \sin t_1]$$

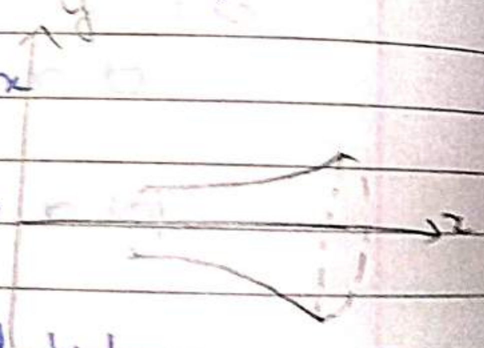
which is the eqn of circular

5. Find the curve with fixed boundary points such that its rotation about the axis of abscissa gives the rise to a surface of revolution of minimum surface area.

proof:

The area of the surface of revolution is given by,

$$s[y(x)] = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$



Here $F = y\sqrt{1+y'^2} \rightarrow ①$

The function is depends on y and y'.

The Euler's equation is given by,

$$F - y' F_{y'} = c_1$$

diff ① p.w.r to y'

$$F_{y'} = y \cdot \frac{1}{2\sqrt{1+y'^2}} 2y'$$

$$F_{y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

$F - y' F_{y'} = c_1$

$y\sqrt{1+y'^2} - \frac{y y'}{\sqrt{1+y'^2}} = c_1$

$\frac{y(1+y'^2) - y y'^2}{\sqrt{1+y'^2}} = c_1$

$\frac{y}{\sqrt{1+y'^2}} = c_1$

$y = c_1\sqrt{1+y'^2}$

put $y' = \sinh t$, t is any parametre

$y = c_1\sqrt{1+\sinh^2 t}$

$y = c_1 \cosh t \rightarrow (2)$

Since $\frac{dy}{dx} = \sinh t$

$dx = \frac{dy}{\sinh t}$

$dx = \frac{c_1 \sinh t\, dt}{\sinh t}$

$dx = c_1\, dt$

on $\int$ ing

$x = c_1 t + c_2 \rightarrow (3)$

$c_1 t = x - c_2$

$t = \frac{x - c_2}{c_1}$

$(2) \Rightarrow y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right)$

which is the equation of catenary.

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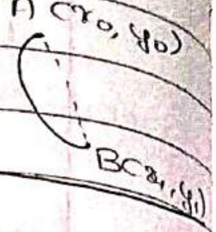
6. P.T the shortest distance b/w two fixed points in a plane is a straight line.



proof:
Assume that the two distinct point B in the xy plane.
If $y = y(x)$ is the equation of any plane curve C in the xy plane and passing through the A and B. the length of C is given by,

$$L[y(x)] = \int_{x_0}^{x_1} ds$$

$$L[y(x)] = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx$$

Here $F = \sqrt{1+y'^2} \rightarrow (1)$

F depends on y'.

$\therefore$ The Euler's equation,

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$F_y = 0$$

$$F_{y'} = \frac{1}{2\sqrt{1+y'^2}} \cdot 2 \cdot y'$$

$$F_{y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$0 - \frac{d}{dx}\left[\frac{y'}{1+y'^2}\right] = 0$$

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right]=0$$

On $\int$ing

$$\frac{y'}{\sqrt{1+y'^2}}=c$$

Squaring on both sides, we get

$$\frac{y'^2}{1+y'^2}=c$$

$$y'^2=c\left[1+y'^2\right]$$

$$y'^2=c+cy'^2$$

$$y'^2\left[1-c\right]=c$$

$$y'^2=\frac{c}{1-c}$$

Squaring on both sides, we get

$$y'=\sqrt{\frac{c}{1-c}}$$

$$y'=a$$

$$\frac{dy}{dx}=a$$

$$dy=a\,dx$$

on $\int$ing

$$y=ax+c$$

which is the eqn of straight line

7. Find the curve on which the functional $I\left[y(x)\right]=\int_0^1\left[y'^2+12xy\right]dx$ with $y(0)=0$, $y(1)=1$ can be extremized.

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Solution:

Given that

$$I[y(x)] = \int_0^1 [y'^2 + 12xy]\,dx$$

Here $F = [y'^2 + 12xy] \rightarrow (1)$

The function depends on x, y, y'

The Euler's equation is,

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$\therefore$ $F_y = 12x$

$F_{y'} = 2y'$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$12x - \frac{d}{dx}[2y'] = 0$$

$$12x - 2y'' = 0$$

$$2y'' = 12x$$

$$y'' = 6x$$

on Ing

$$y' = \frac{6x^2}{2} + c_1$$

$$y' = 3x^2 + c_1$$

$$y = \frac{3x^3}{3} + c_1 x + c_2$$

$$y = x^3 + c_1 x + c_2 \rightarrow (2)$$

Put $x = 0$

$$y(0) = c_2$$

$$c_2 = 0$$

Put $x = 1$

$$y(1) = 1 + c_1 + c_2$$

$$= 1 + c_1$$

$$c_1 = 0$$

$(2) \Rightarrow \therefore\ y = x^3$

8. Find the curve on which the functional $I[y(x)] = \int_0^1 [y^2 + x^2 y'] \, dx$, $y(0) = 0$, $y(1) = a$.

Solution:

Given

$$I[y(x)] = \int_0^1 [y^2 + x^2 y'] \, dx$$

Here $F = y^2 + x^2 y' \rightarrow (1)$

The function depends on x, y, y'.

$\therefore$ The Euler's equation is,

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$F_y = 2y$$

$$F_{y'} = x^2$$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$2y - \frac{d}{dx}[x^2] = 0$$

$$2y - 2x = 0$$

$$y = x \rightarrow (2)$$

put $x = 0$

$$y(0) = 0$$

$$0 = 0$$

put $x = 1$

$$y(1) = 1$$

$$a = 1$$

The second boundary condition doesnot satisfies extremal.

Write the Euler's eqn of n-function variational problem for the functional of n-function.

Consider the functional

$$I[y(x)] = \int_a^b F[x, y_1(x), y_2(x), \ldots, y_n(x), y_1'(x), y_2'(x), \ldots, y_n'(x)]\,dx$$

where the functional F is differentiable 3 times with respect to all of its arguements.

To find the necessary functional consider the following boundary condition of $y_1(x), y_2(x), \ldots y_n(x)$ such that

$y_1(a) = y_1, y_2(a) = y_2, \ldots, y_n(a) = y_n \rightarrow (1)$

$y_1(b) = z_1, y_2(b) = z_2, \ldots, y_n(b) = z_n \rightarrow (2)$

where $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n$ are constant and they varing only of the function $y_i(x)$.

Then the above functional reduces to a functional depend on only one of the function $y_i(x)$.

Thus the function $y_i(x)$ having continuous derivatives must satisfies the Euler's equation

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0, \quad i = 1, 2, \ldots n$$

where the boundary condition of $y_i(x)$ at $x = a$ and $x = b$ are utilized from this equation.

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problems:

1. Derive the differential equation of the lines of proparation of light in an optically nonhomogeneous medium in which the speed of light is $c(x,y,z)$

Solution:

By Fermats Law,
Light is propagates from one point to another point along a curve for which the time T process of light will be minimum.
If the equation of desired part of the light ray B.
$y = y(x)$, $z = z(x)$ then,

$$T = \int_{x_0}^{x_1} \frac{ds}{c(x,y,z)}$$

$$T = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2+z'^2}}{c(x,y,z)}\, dx$$

Here,

$$F = \frac{\sqrt{1+y'^2+z'^2}}{c(x,y,z)} \rightarrow (1)$$

The function F depends x, y, z, y', z'.
∴ The Euler's eqn

$$F_y - \frac{d}{dx} F_{y'} = 0, \quad F_z - \frac{d}{dx} F_{z'} = 0$$

Diff eqn (1) p.w.r to y

$$F_y = - \frac{\sqrt{1+y'^2+z'^2}}{c^2(x,y,z)} \frac{\partial c}{\partial y}$$

Diff ① p.w.r to 'y''

$$F_{y'} = \frac{1}{c(x,y,z)} \frac{1}{2\sqrt{1+y'^2+z'^2}} 2y'$$

$$F_{y'} = \frac{y'}{c(x,y,z)\sqrt{1+y'^2+z'^2}}$$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$-\frac{\sqrt{1+y'^2+z'^2}}{c^2(x,y,z)} \frac{\partial c}{\partial y} - \frac{d}{dx}\left[\frac{y'}{c(x,y,z)\sqrt{1+y'^2+z'^2}}\right] = 0$$

|||'y,

$$\frac{\sqrt{1+y'^2+z'^2}}{c^2(x,y,z)} \frac{\partial c}{\partial z} + \frac{d}{dx}\left[\frac{z'}{c(x,y,z)\sqrt{1+y'^2+z'^2}}\right] = 0$$

|||'y,

$$\frac{\sqrt{1+y'^2+z'^2}}{c^2(x,y,z)} \frac{\partial c}{\partial z} + \frac{d}{dx}\left[\frac{z'}{c(x,y,z)\sqrt{1+y'^2+z'^2}}\right] = 0$$

$\therefore$ which is the required differential eqn.

Functional depends on higher order derivative.

(or)

Derive the Euler's poisson equation
let us consider the extremum of functional of the form,

$$I[y(x)] = \int_a^b F[x, y(x), y'(x), y''(x), \ldots, y^n(x)]dx \quad \to (1)$$

Consider the functional F to be differentiable (n+2) times with respect to all its arguement.
The boundary conditions are taken in the form,

$$\left.\begin{aligned} y(a) = y_0,\ y'(a) = y_0',\ \ldots,\ y^{n-1}(a) = y_1^{n-1} \\ y(b) = y_1,\ y'(b) = y_1',\ \ldots,\ y^{n-1}(b) = y_1^{n-1} \end{aligned}\right\} \to (2)$$

The boundary points of the value y together with all their derivative up to the order n-1. We further assume that the extremum of the functional I is attained on a curve $y = f(x)$ which is 2n times differentiable any admissible comparison curve $y = \overline{y(x)}$ is also 2n times differentiable.
It is clear that $y = y(x)$ and $y = \overline{y(x)}$ can be included in a one parametre family of curve,

i.e) $y(x, \alpha) = y(x) + \alpha\, \delta y$
$= y(x) + \alpha\,[\overline{y(x)} - y(x)] \to (3)$

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When $\alpha = 0$

$y(x, 0) = y(x)$

When $\alpha = 1$

$y(x, 1) = \overline{y(x)}$

Now on the curve of the above family the functional ① reduces to a functional α, say $\psi(\alpha)$

i.e) $\psi(\alpha) = \int_a^b F[x, y(x,\alpha), y'(x,\alpha), \ldots, y^n(x,\alpha)]dx$ → ④

Since the extremizing curve corresponding to $\alpha = 0$, we must have

$$\frac{d\psi}{d\alpha} = 0$$

Now diff ④ w.r to α, we get

$$\frac{d\psi}{d\alpha} = \frac{d}{d\alpha}\int_a^b F[x, y(x,\alpha), y'(x,\alpha), \ldots, y^n(x,\alpha)]dx$$

$$= \int_a^b \frac{\partial}{\partial\alpha} F[x, y(x,\alpha) \cdot y'(x,\alpha), \ldots, y^n(x,\alpha)]dx$$

$$= \int_a^b \left[0 + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial\alpha} + \frac{\partial F}{\partial y'}\cdot\frac{\partial y'}{\partial\alpha} \ldots \frac{\partial F}{\partial y^n}\cdot\frac{\partial y^n}{\partial\alpha}\right]dx$$

$$\frac{d\psi}{d\alpha} = \int_a^b [F_y \delta y + F_{y'} \delta y' + \ldots + F_{y^n} \delta y^n]dx$$ → ⑤

Integrate the above equation with boundary condition

$$\int_a^b F_{y'} \delta y' dx = [F_{y'} \delta y]_a^b - \int_a^b \delta y \, \frac{d}{dx} F_{y'} dx$$

$$\int_a^b F_{y'} \delta y' dx = -\int_a^b \frac{d}{dx} F_{y'} \delta y \, dx$$ → ⑥

$$\int_a^b F_{y''}\,\delta y''\,dx = \left[F_{y''}\,\delta y'\right]_a^b - \int_a^b \delta y' \frac{d}{dx} F_{y''}\,dx$$

$$= 0 - \int_a^b \frac{d}{dx} F_{y''}\,\delta y'\,dx$$

$$= -\left[\frac{d}{dx} F_{y'}\,\delta y\right]_a^b + \int_a^b \delta y \frac{d^2}{dx^2} F_{y''}\,dx$$

$$= 0 + \int_a^b \frac{d^2}{dx^2} F_{y''}\,\delta y\,dx \rightarrow (7)$$

$$\int_a^b F_{y^n}\,\delta y^n\,dx = (-1)^n \int_a^b \frac{d^n}{dx^n} F_{y^n}\,\delta y\,dx \rightarrow (8)$$

Sub (6), (7), (8) in (3)

$$\frac{d\psi}{d\alpha} = \int_a^b F_y\,\delta y\,dx - \int_a^b \frac{d}{dx} F_{y'}\,\delta y\,dx + \int_a^b \frac{d^2}{dx^2} F_{y''}\,\delta y\,dx$$

$$- \dots (-1)^n \int_a^b \frac{d^n}{dx^n} F_{y^n}\,\delta y\,dx$$

$$\frac{d\psi}{d\alpha} = \int_a^b \left[F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \dots (-1)^n \frac{d^n}{dx^n} F_{y^n}\right] \delta y\,dx$$

$$\therefore \frac{d\psi}{d\alpha} = 0$$

$$0 = \int_a^b \left[F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \dots (-1)^n \frac{d^n}{dx^n} F_{y^n}\right] \delta y\,dx$$

By fundamental theorem,

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \dots (-1)^n \frac{d^n}{dx^n} F_{y^n} = 0$$

: which is known as Euler's Poisson equation.

1. Find the extremum of the functional $I[y(x)] = \int_0^1 (1+y'')^2 dx$ with $y(0)=0$, $y'(0)=1$, $y(1)=1$, $y'(1)=1$

Solution:

Given, $I[y(x)] = \int_0^1 (1+y'')^2 dx$

Here, $F = (1+y'')^2 \rightarrow (1)$

The function F depends on y''

The Euler's equation is given by,

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0 \rightarrow (2)$$

Diff (1) p.w.r to y

$$F_y = 0$$

$$F_{y'} = 0$$

$$F_{y''} = 0$$

$$(2) \Rightarrow 0 - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') = 0$$

$$2y^{iv} = 0$$

$$y^{iv} = 0$$

On $\int$ing

$$y''' = c_1$$

$$y'' = c_1 x + c_2$$

$$y' = c_1 \frac{x^2}{2} + c_2 x + c_3 \rightarrow (2)$$

$$y = \frac{c_1 x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4 \rightarrow (3)$$

Put $x = 0$

$$y(0) = c_4$$

$$c_4 = 0 \rightarrow (A)$$

put $x=1$

$$y(1) = \frac{c_1}{6} + \frac{c_2}{2} + c_3 + 0$$

$$1 = \frac{c_1}{6} + \frac{c_2}{2} + c_3$$

$$6 = c_1 + 3c_2 + 6c_3 \rightarrow (B)$$

put $x=0$ in (*)

$$y'(0) = c_3$$

$$c_3 = 1 \rightarrow (C)$$

put $x=1$ in (*)

$$y'(1) = \frac{c_1}{2} + c_2 + c_3$$

$$1 = \frac{c_1}{2} + c_2 + 1$$

$$c_1 + 2c_2 = 0 \rightarrow (D)$$

(D) $\Rightarrow c_1 = -2c_2$

(B) $\Rightarrow 6 = -2c_2 + 3c_2 + 6$

$$c_2 = 0$$

(D) $\Rightarrow 0 = c_1 + 0$

$$c_1 = 0$$

(3) $\Rightarrow y = 1 \times x$

$$y = x$$

2. Determine the optimum functional that minimize the integral in this following equation,

$$I[y(x)] = \int_{x_0}^{x_1} (16y^2 - y''^2)\, dx$$

Solution:

Given,

$$I[y(x)] = \int_{x_0}^{x_1} (16y^2 - y''^2)\, dx$$

Here,

$$F = [16y^2 - y''^2] \rightarrow (1)$$

The function F depends on y and y''

The Euler's equation given by,

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0 \rightarrow (2)$$

Diff (1) p.w.r to x,

$$F_y = 32y$$

$$F_{y'} = 0$$

$$F_{y''} = -2y''$$

(2) $\Rightarrow$

$$32y - 0 + \frac{d^2}{dx^2}(-2y^2) = 0$$

$$32y - \frac{d^2}{dx^2}(-2y^2) = 0$$

$$32y - 2y^{iv} = 0$$

The A.E is

$$m^4 - 16 = 0$$

$$m^4 = 16$$

$$m^2 = \pm 4$$

$$m^2 = 4$$

$$m = \pm 2$$

$$m^2 = -4$$

$$m = \pm 2i$$

$$\therefore y = Ae^{2x} + Be^{-2x} + C\cos 2x + D\sin 2x$$

3. Determine the external of the functional $I[y(x)] = \int_{-l}^{l} \left[\frac{1}{2}\mu y''^2 + ey\right] dx$

which satisfies the boundary condition $y(-l) = y'(l) = y'(l) = 0$

Solution:-

Given,

$$I[y(x)] = \int_{-l}^{l} \left[\frac{1}{2}\mu y''^2 + ey\right] dx$$

Here,

$$F = \left[\frac{1}{2}\mu y''^2 + ey\right] \rightarrow (1)$$

The function F depends on y and y''

$\therefore$ The Euler's equation is,

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0 \rightarrow (2)$$

Diff eqn (1) p.w.r to x

$$F_y = e$$

$$F_{y'} = 0$$

$$F_{y''} = \frac{1}{2}[\mu 2y'']$$

$$F_{y''} = \mu y''$$

$$(2) \Rightarrow e - 0 + \frac{d^2}{dx^2}(\mu y'') = 0$$

$$e + \mu y^{iv} = 0$$

$$\mu y^{iv} = -e$$

$$y^{iv} = -\frac{e}{\mu}$$

on $\int$ing

$$y''' = -\frac{e}{\mu} x + c_1$$

$$y''' = -\frac{\rho}{\mu}\frac{x^2}{2} + c_1 x + c_2$$

$$y' = -\frac{\rho}{\mu}\frac{x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3 \rightarrow (3)$$

$$y = -\frac{\rho}{\mu}\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4 \rightarrow (4)$$

$$y'(l) = -\frac{\rho l^3}{6\mu} + \frac{c_1 l^2}{2} + c_2 l + c_3$$

$$0 = -\frac{\rho l^3}{6\mu} + \frac{c_1 l^2}{2} + c_2 l + c_3 \rightarrow (A)$$

$$y'(-l) = \frac{\rho l^3}{6\mu} + \frac{c_1 l^2}{2} - c_2 l + c_3$$

$$0 = -\frac{\rho l^3}{6\mu} + \frac{c_1 l^2}{2} - c_2 l + c_3 \rightarrow (B)$$

$$y(l) = -\frac{\rho l^4}{24\mu} + c_1\frac{l^3}{6} + \frac{c_2 l^2}{2} + c_3 l + c_4$$

$$0 = -\frac{\rho l^4}{24\mu} + \frac{c_1 l^3}{6} + \frac{c_2 l^2}{2} + c_3 l + c_4 \rightarrow (C)$$

$$y(-l) = -\frac{\rho l^4}{24\mu} - \frac{c_1 l^3}{6} + \frac{c_2 l^2}{2} - c_3 l + c_4$$

$$0 = -\frac{\rho l^4}{24\mu} - \frac{c_1 l^3}{6} + \frac{c_2 l^2}{2} - c_3 l + c_4 \rightarrow (D)$$

$(A) + (B) \Rightarrow c_1 l^2 + 2c_3 = 0 \rightarrow (E)$

$(A) - (B) \Rightarrow -\frac{\rho l^3}{3\mu} + 2c_2 l = 0 \rightarrow (F)$

$$2c_2 l = \frac{\rho l^3}{3\mu}$$

$$c_2 = \frac{\rho l^2}{6\mu}$$

$$C + D \Rightarrow$$

$$\frac{-\rho l^4}{12\mu} + c_2 l^2 + 2c_4 = 0 \rightarrow (E)$$

$$-\frac{\rho l^4}{12\mu} + \frac{\rho l^4}{6\mu} + 2c_4 = 0$$

$$\frac{\rho l^4}{12\mu} + 2c_4 = 0$$

$$2c_4 = -\frac{\rho l^4}{12\mu}$$

$$c_4 = -\frac{\rho l^4}{24\mu}$$

$$C - D \Rightarrow$$

$$\frac{c_1 l^3}{3} + 2c_3 l = 0 \rightarrow (F)$$

$$F \times l \Rightarrow c_1 l^3 + 2c_3 l = 0$$

$$-\frac{2}{3} c_1 l^3 = 0$$

$$c_1 = 0$$

$$(4) \Rightarrow y = -\frac{\rho l^4}{24\mu} + 0 + \frac{\rho l^2 x^2}{12\mu} + 0 - \frac{\rho l^4}{24\mu}$$

$$= -\frac{\rho x^4}{24\mu} + \frac{\rho l^2 x^2}{12\mu} - \frac{\rho l^4}{24\mu}$$

$$= \frac{-\rho}{24\mu}\left[x^4 - 2l^2 x^2 + l^4\right]$$

$$y = -\frac{\rho}{24\mu}\left[x^2 - l^2\right]^2$$

4. Find the extremal of the functional $I[y(x)] = \int_{x_0}^{x_1} (2xy + y'''^2)\,dx$

Solution:-

Given,

$$I[y(x)] = \int_{x_0}^{x_1} (2xy + y'''^2)\,dx$$

Here $F = 2xy + y'''^2 \rightarrow (1)$

The function depends on x, y and y'''

The Euler's equation is,

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \frac{d^3}{dx^3} F_{y'''} = 0 \rightarrow (2)$$

Diff eqn (1) p.w.r to 'x' we get

$$F_y = 2x$$
$$F_{y'} = 0$$
$$F_{y''} = 0$$
$$F_{y'''} = 2y'''$$

$$(2) \Rightarrow 2x - 0 + 0 + \frac{d^3}{dx^3}(2y''') = 0$$

$$2x - \frac{d^3}{dx^3}(2y''') = 0$$

$$2x - 2y^{vi} = 0$$

$$2y^{vi} = +2x$$

$$y^{vi} = x$$

on ʃing,

$$y^{v} = \frac{x^2}{2} + c_1$$

$$y^{iv} = \frac{x^3}{6} + c_1 x + c_2$$

$$y''' = \frac{x^4}{24} + \frac{c_1 x^2}{2} + c_2 x + c_3$$

$$y'' = \frac{x^5}{120} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4$$

$$y' = \frac{x^6}{720} + \frac{c_1 x^4}{24} + \frac{c_2 x^3}{6} + \frac{c_3 x^2}{2} + c_4 x + c_5$$

$$y = \frac{x^7}{5040} + \frac{c_1 x^5}{120} + \frac{c_2 x^4}{24} + \frac{c_3 x^3}{6} + \frac{c_4 x^2}{2} + c_5 6 + c_6$$

Functional depends on functions of several independent variables:-

In the variational problem we consider so far Euler's equation determining the external functional involving multiple integrals leading to one or more partial differential equation.

For example: The problem for finding an extremum of the functional

$$J[u(x,y)] = \iint_{\sigma} F[x, y, u, u_x, u_y]\, dx\,dy \to (1)$$

over a region of integration σ by determining $u(x,y)$ which is continuous and as continuous derivatives upto the second order and takes prescribed value on the boundary of σ.

We further assume that F is three times differentiable. Let the extremizing surface be $u = u(x,y)$, so that the admissable curve on one parameter surface is taken as,

$u(x,y,\alpha) = u(x,y) + \alpha\eta(x,y)$

Where $\eta(x,y) = 0$ on the boundary of σ.

Then the necessary condition for an extremum is vanishing of the first variation.

i,e) $\frac{\partial}{\partial\alpha}[J(u+\alpha\eta)]_{\alpha=0} = 0$

$$(1) \Rightarrow \iint_{\sigma} [F_u \eta + F_{u_x}\eta_x + F_{u_y}\eta_y]\, dx\,dy = 0 \to (2)$$

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which may be again transformed by integration by parts.

We assume that the boundary Γ of G admints of a tangent which terms piecewise continuous.

By using Green's theorem.

$$\int_{\Gamma} u\,dx + v\,dy = \iint_{G} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \rightarrow ③$$

$$\iint_{G} \left[F_{u_x}\eta_x + F_{u_y}\eta_y\right] dx\,dy = \iint_{G} \left[\frac{\partial F}{\partial u_x}\frac{\partial \eta}{\partial x} + \frac{\partial F}{\partial u_y}\frac{\partial \eta}{\partial y}\right.$$

$$= \iint_{G} \left[\frac{\partial \eta}{\partial x}\frac{\partial F}{\partial u_x} + \frac{\partial \eta}{\partial y}\frac{\partial F}{\partial u_y}\right] dx\,dy \rightarrow ④$$

$$= \iint_{G} \left[\frac{\partial}{\partial x}\left(\eta\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial F}{\partial u_y}\right)\right] dx\,dy \rightarrow ⑤$$

from ③ & ⑤, we get

$$N = \eta\frac{\partial F}{\partial u_x}, \quad M = \eta\frac{\partial F}{\partial u_y}$$

Now Consider

$$\frac{\partial}{\partial x}\left(\eta\frac{\partial F}{\partial u_x}\right) = \frac{\partial \eta}{\partial x}\frac{\partial F}{\partial u_x} + \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)$$

$$\frac{\partial \eta}{\partial x}\frac{\partial F}{\partial u_x} = \frac{\partial}{\partial x}\left(\eta\frac{\partial F}{\partial u_x}\right) - \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)$$

|||ly

$$\frac{\partial \eta}{\partial y}\cdot\frac{\partial F}{\partial u_y} = \frac{\partial}{\partial y}\left(\eta\cdot\frac{\partial F}{\partial u_y}\right) - \eta\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)$$

Sub above values in ④

$$\iint_{G} \left[F_{u_x}\eta_x + F_{u_y}\eta_y\right] dx\,dy =$$

$$\iint_{G} \left[\frac{\partial}{\partial x}\left(\eta\frac{\partial F}{\partial u_x}\right) - \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \right.$$

$$\left.\frac{\partial}{\partial y}\left[\eta\frac{\partial F}{\partial u_y} - \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_y}\right)\right] dx\,dy$$

$$= \iint_{\Omega} \left[\frac{\partial}{\partial x}\left(\eta \frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\eta \frac{\partial F}{\partial u_y}\right)\right] dx\,dy - \iint_{\Omega} \left[\eta \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \eta \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] dx\,dy$$

$$\iint_{\Omega} \left[F_{u_x}\eta_x + F_{u_y}\eta_y\right] dx\,dy = \int_{\Gamma} M\,dx + N\,dy - \iint_{\Omega} \left[\eta \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \eta \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] dx\,dy$$

$$(2) \Rightarrow \iint_{\Omega} \left[F_u\eta + F_{u_x}\eta_x + F_{u_y}\eta_y\right] dx\,dy = \iint_{\Omega} F_u\eta\,dx\,dy + \int_{\Gamma} M\,dx + N\,dy - \iint_{\Omega} \left[\eta \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \eta \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] dx\,dy = 0$$

$$\Rightarrow \iint_{\Omega} \left[F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y}\right] \eta\,dx\,dy + \int M\,dx + N\,dy = 0$$

Since $\eta = 0$ on Γ and the above relation holds for any arbitrary continuous differential eqn.

$$\Rightarrow \iint_{\Omega} \left[F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y}\right] \eta\,dx\,dy = 0$$

By fundamental theorem,

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0$$

This eqn is called Euler's Ostrogradsky equation.

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Note:-
Consider the functional equation
$I[u(x,y)] = \iint\limits_{\Omega} F[x, y, u_x, u_y, u_{xx}, u_{xy}, u_{yy}]\,dx\,dy$
Then the extremal of this functional is given by,
$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} + \frac{\partial^2}{\partial x^2} F_{u_{xx}} + \frac{\partial^2}{\partial x \partial y} F_{u_{xy}} + \frac{\partial^2}{\partial y^2} F_{u_{yy}} = 0$$

1. Find the extremal of the functional
$I[u(x,y)] = \iint\limits_{\Omega} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dx\,dy$

Solution:-
Given that
$$I[u(x,y)] = \iint\limits_{\Omega} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dx\,dy$$
$$= \iint\limits_{\Omega} [u_x^2 + u_y^2]\,dx\,dy$$
$F = u_x^2 + u_y^2 \rightarrow (1)$
The Function depends on u_x, u_y.
$\therefore$ The Euler's Ostrogradsky equation is given by,
$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0 \rightarrow (2)$$
$F_u = 0$
$F_{u_x} = 2u_x$
$F_{u_y} = 2u_y$
$(2) \Rightarrow 0 - \frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) = 0$
$-2u_{xx} - 2u_{yy} = 0$
$\div 2 \quad u_{xx} + u_{yy} = 0$
which is a Harmonic equation.

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2. Find the extremal of the functional,

$$J[z(x,y)] = \iint_D \left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - 2z f(x,y)\right] dx$$

Solution:-

$$= \iint_D [z_{xx}^2 + z_{yy}^2 + 2 z_{xy}^2 - 2z f(x,y)]\, dx dy$$

Here,

$$F = [z_{xx}^2 + z_{yy}^2 + 2z_{xy}^2 - 2z f(x,y)] \rightarrow ①$$

The function depends on $z, z_{xx}, z_{yy}, z_{xy}$

$\therefore$ The Euler's ostrogradsky eqn is given by,

$$F_z - \frac{\partial}{\partial x} F_{zx} - \frac{\partial}{\partial y} F_{zy} + \frac{\partial^2}{\partial x^2} + F_{z\cancel{xx}} + \frac{\partial^2 F_{zxy}}{\partial x \partial y}$$

$$+ \frac{\partial^2}{\partial y^2} F_{zyy} = 0 \rightarrow ②$$

$F_z = -2f(x,y)$

$F_{zx} = 0$

$F_{zy} = 0$

$F_{zxx} = 2z_{xx}$

$F_{zxy} = 4z_{xy}$

$F_{zyy} = 2z_{yy}$

$$-2f(x,y) - 0 - 0 + \frac{\partial^2}{\partial x^2} 2z_{xx} + \frac{\partial^2}{\partial x \partial y} 4 z_{xy}$$

$$+ \frac{\partial^2}{\partial y^2} 2z_{yy} = 0$$

$$-2f(x,y) + 2z_{xxxx} + 4z_{xxyy} + 2z_{yyyy} = 0$$

$\div 2$

$$-f(x,y) + \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = 0$$

$$\frac{\partial^2 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = 0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)z=0$$

$$\nabla^2(\nabla^2 z)=0$$

$$\nabla^4 z=0$$

3. Derive the Hamilton principle or state and prove principle.

Statement:

The motion is that the integral of the difference between the kinetic and potential energy.

i, e) $\int_{t_1}^{t_2}(T-V)\,dt=0$

proof:

We consider a particle of m moving in a force field. If their position vector of the particle w.r to field origin is denoted by r.

Then by Newton's law of motion the path of the particle is generated by the equation.

$$m.\frac{d^2r}{dt^2}=f \rightarrow (1) \qquad (\because f=ma)$$

where f is the force acting on the particle.

Now consider any other path $r+\delta r$ in that the truth path and varied path coincide at two distinct instance $t=t_1$ and $t=t_2$

This demand $\delta r|_{t_1}=0$ and $\delta r|_{t_2}=0 \to (2)$

Taking the dot product of variation δr in eqn (1) and integrating the result with respecting time over the integrate (t_1, t_2) we get

$$\int_{t_1}^{t_2}\left(m\frac{d^2r}{dt^2}\delta r - f\delta r\right)dt = 0 \to (3)$$

Integrating the first term in eqn (3) by parts and using eqn (2), we get

$$m\int_{t_1}^{t_2}\frac{d^2r}{dt^2}\delta r\,dt = m\left\{\left[\delta r\frac{dr}{dt}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{dr}{dt}\delta\frac{dr}{dt}dt\right.$$

$$= -\int_{t_1}^{t_2}\delta\frac{m}{2}\left(\frac{dr}{dt}\right)^2 dt$$

$$= -\int_{t_1}^{t_2}\delta T\,dt$$

where T is a K.E $=\frac{1}{2}m\left(\frac{dr}{dt}\right)^2$

$$(3) \Rightarrow \int_{t_1}^{t_2}(-\delta T - f\delta r)\,dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}(\delta T + f\delta r)\,dt = 0 \to (4)$$

This is hamilton principle for single path.

Now consider the case, when the force field having components (x, y, z) is conversable.

which implies that

$f\,dr = X\,dx + Y\,dy + Z\,dz$

This function is called force potential and its negative say V is the potential energy of the particle.

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i, e) $f\delta r = -\delta v$

(10) $\Rightarrow \int_{t_1}^{t_2} (\delta T - \delta v) dt = 0$

$$\int_{t_1}^{t_2} (T - V) dt = 0$$

which is a hamilton principle,
lagrangian function,
i, e) $L = T - V$

4. Derive the equation of variation of a rectilinear bar:-

A displacement from the equilibrium position $u(x,t)$ into a function x and time t.

The kinetic energy of bar of length l is given by,

$$T = \frac{1}{2} \int_0^l \rho \left(\frac{\partial u}{\partial t}\right)^2 dx$$

We assume that the bar is slightly extensiable, the potential energy of a elastic bar with a constant curvature is proportional to the square of curvature.

Thus the differential dV of the potential energy of the bar is

$$dv = \frac{1}{2} k \left[\frac{\partial^2 u}{\partial x^2} \Big/ \left[1 + \left(\frac{\partial u}{\partial x}\right)^2 \right]^{3/2} \right]^2$$

where k is constant.

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Thus the potential energy of the whole bar is given by,

$$V = \frac{1}{2} k \int_0^l \left[\frac{\partial^2 u}{\partial x^2} \Big/ \left[1 + \left(\frac{\partial u}{\partial x}\right)^2 \right]^{3/2} \right]^2 dx$$

According to our assumption of slight extension the derivative of the bar from the equilibrium position are small and the term $\frac{\partial u}{\partial x}$ is ignored.

$$\therefore V = \frac{1}{2} k \int_0^l \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx$$

by Hamilton principle,

$$\int_{t_1}^{t_2} (T - V)\, dt = \int_{t_1}^{t_2} \int_0^l \left[\frac{1}{2} \rho \left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2} k \left(\frac{\partial^2 u}{\partial x^2}\right)^2 \right] dx\, dt$$