UNIT-II

Variational problem with moving boundaries.

Consider the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

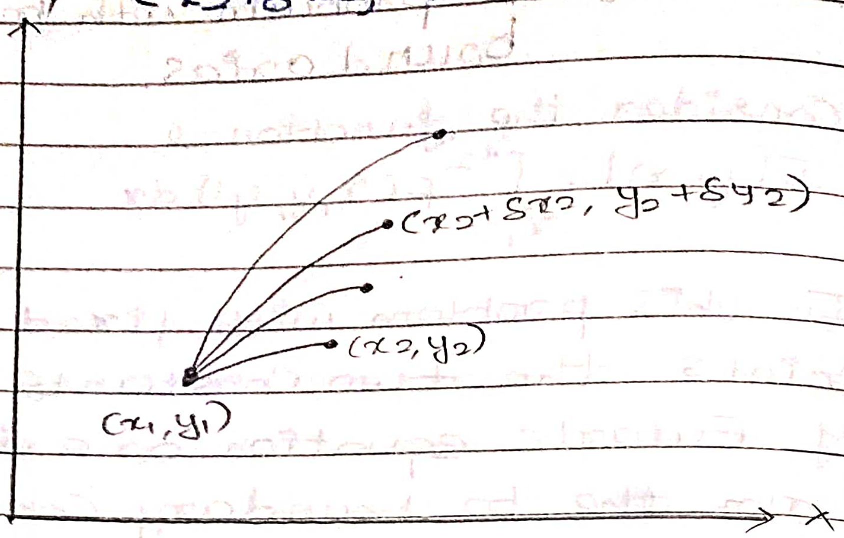
In this problem with fixed boundary points. the two constants is a solution of Euler's equation are determine from the to boundary condition at the point (x_1, y_1) and (x_2, y_2).

In the case of moving boundary problem one (or) both of these conditions in the general solution of Euler's equation have to be obtained from the vanishing of the variation δI.

Let us assume that one of the boundary problem one (or) both of these conditions in the general solution of Euler's equation have to be obtained from the vanishing of the variation δI.

Let us assume that one of the boundary point (x_1, y_1) is fixed while the other boundary point (x_2, y_2) can move and passes to the point.

$(x_2+\delta x_2,\ y_2+\delta y_2)$



A pencil of extremal through a given point.

Now, we consider the value of the functional $I[y(x,c_1c_2)]$ reduces to a function of the parameter c_1 and c_2 of the limit of Integration. We shall call the admissible curve $y=y(x)$ and $y=y(x)+\delta y(x)$ are closed if $|\delta y|$ and $|\delta y'|$ are also small. Now the point (x_1,y_1) from the pencil of extremal $y=y(x),c_1$. If the curve of the pencil $y=y(x,c_1)$ do not intersect in the neighbourhood of the extremals then $I[y(x,c_1)]$ may be considered as a single value function of (x_2,y_2). Let us determine the variation of the functional $I[y(x,c_1)]$, when the boundary points (x_2,y_2) moves the boundary point (x_2,y_2) moves to $(x_2+\delta x_2,\ y_2+\delta y_2)$. Thus the increment ΔI is given by,

$$\Delta I = I[y(x)+\delta y(x)] - I[y(x)]$$

$$= \int_{x_1}^{x_2+\delta x_2} F[x, y+\delta y, y'+\delta y']\,dx - \int_{x_1}^{x_2} F[x, y, y']\,dx$$

$$= \int_{x_1}^{x_2} F[x, y+\delta y, y'+\delta y']\,dx + \int_{x_2}^{x_2+\delta x_2} F[x, y+\delta y, y'+\delta y']\,dx$$

$$- \int_{x_1}^{x_2} F[x, y, y']\,dx$$

$$= \int_{x_1}^{x_2} \{F[x, y+\delta y, y'+\delta y'] - F[x, y, y']\}\,dx$$

$$+ \int_{x_2}^{x_2+\delta x_2} F[x, y+\delta y, y'+\delta y']\,dx$$

By using mean value theorem, we can write the second term of R.H.S on,

$$\int_{x_2}^{x_2+\delta x_2} F[x, y+\delta y, y'+\delta y']\,dx = [F]_{x_2+\theta\delta x_2}(x_2+\delta x_2 - x_2)$$

$$= [F]_{x_2+\theta\delta x_2} \cdot \delta x_2$$

where θ lies between 0 and 1 but by virtue of continuity of F.

$$F|_{x_2+\theta\delta x_2} = F|_{x_2+\epsilon}$$

Here ϵ is infinitesimal such that $\epsilon \to 0$ as $\delta x_2 \to 0$ and $\delta y_2 \to 0$. Then we write the above eqn as,

$$\int_{x_2}^{x_2+\delta x_2} F[x, y+\delta y, y'+\delta y']\,dx = F|_{x=x_2}\,\delta x_2 + \epsilon\delta x_2 \quad \text{①}$$

using Taylor's formula,
We now transform the first term of R.H.S as

$$\int_{x_1}^{x_2} \left\{ F[x, y+\delta y, y'+\delta y'] - F[x, y, y'] \right\} dx$$

$$= \int_{x_1}^{x_2} [F_y \delta y + F_{y'} \delta y'] \, dx + R$$

where R is an infinitesimal of the higher order δy (or) $\delta y'$. Further the linear part on the RHS of the above integral can be alter by integration by parts and reduce to,

$u = F_{y'}$ $\quad dv = \delta y' \, dx$

$du = \frac{d}{dx} F_{y'} \, dx$ $\quad v = \delta y$

$$\int_{x_1}^{x_2} [F_y \delta y + F_{y'} \delta y'] \, dx = \int_{x_1}^{x_2} F_y \delta y \, dx + [F_{y'} \delta y]_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y \frac{d}{dx} F_{y'} \, dx$$

$$= [F_{y'} \delta y]_{x_1}^{x_2} - \int_{x_1}^{x_2} \left[F_y - \frac{d}{dx} F_{y'} \right] \delta y \, dx$$

Since the value of the functional are taken only on the external, the integral of the second term of the above expression vanish.

$$F_y - \frac{d}{dx} F_{y'} = 0$$

and the expression becomes

$$[F_{y'} \delta y]_{x_1}^{x_2}$$

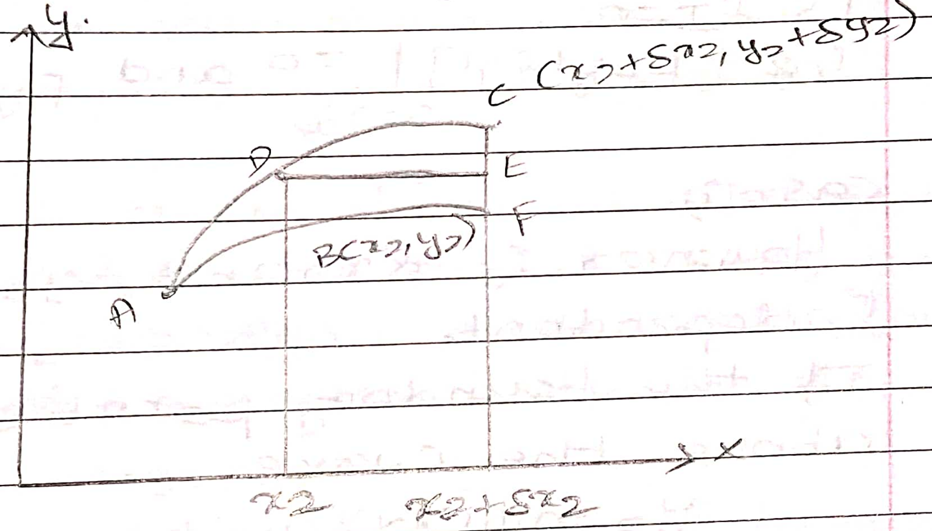
Since x_1 is a fixed point, we get $(\delta y)_{x_1} = 0$

Now are going to find $[F_{y'} . \delta y]_{x_2}$

Note that $[\delta y]_{x_2}$ is not equal to δy_2

Since δy_2 is the increment of y_2 than the boundary point is displaced to $(x_2 + \delta x_2, y_2 + \delta y_2)$ but $[\delta y_2]_{x_2}$ is the increament of the ordinate at the point x_2.

When passing form the extremum joining (x_1, y_1) and (x_2, y_2) to (x_1, y_1) and $(x_2 + \delta x_2, y_2 + \delta y_2)$.



Here, $BD = [\delta y]_{x_2}$

$FC = \delta y_2$

$EC = y'(x_2) . \delta x_2$

$BD = FC - EC$

$(\delta y_2)_{x_2} = \delta y_2 - y'(x_2) \delta x_2$

Thus,

$$\int_{x_1}^{x_2} \left\{ F[x, y + \delta y, y' + \delta y'] - F[x, y, y'] \right\} dx = F_{y'} \Big|_{x_1 = x_2} (\delta y_2 - y'(x_2) \delta x_2) \quad \text{(2)}$$

Adding ① & ②

$$\int_{x_1}^{x_2}\{F[x,y+\delta y,y'+\delta y']-F[x,y,y']\}dx+\int_{x_1}^{x_2+\delta x_2}F[x,y+\delta y,y'+\delta y']dx$$

$$=F\Big|_{x=x_2}\delta x_2+F_{y'}\Big|_{x=x_2}(\delta y_2-y'(x_2)\delta x_2)$$

$$\delta I=F-y\delta y'\Big|_{x=x_2}\delta x_2+F_{y'}\Big|_{x=x_2}\delta y_2=0 \rightarrow ③$$

By virture of the fact that δx_2 and δy_2 are Independent.

The necessary condition for extremal is $\delta I=0$

i,e $[F-y'\delta y']\Big|_{x=x_2}=0$ and $F_{y'}\Big|_{x=x_2}=0$

Case(i)

However if δx_2 and δy_2 are not independent.

If the boundary point (x_2,y_2) move above the curve.

$$y_2=\phi(x_2)$$

$\therefore$ Then $\delta y_2=\phi'(x_2)\delta x_2$

③ $\Rightarrow [F+(\phi'-y')F_{y'}]_{x=x_2}\delta x_2=0 \rightarrow ④$

Since δx_2 is arbitrary, we must have

$$[F+(\phi'-y')F_{y'})_{x=x_2}=0 \rightarrow ⑤$$

Which provide the case when f is given by,

$$A(x,y)(1+y'^2)^{1/2}$$

Where $A(x,y)$ doesnot vanish at the movable boundary point x_2,

In this case of eqn ⑤ reduces to
$A(x,y)(1+\phi'(y))(1+y'^2)^{1/2}=0$ at $x=x_2$
Since $A(x,y)\neq 0$ at $x=x_2$, we have
$$y'=-\frac{1}{\phi'} \text{ at } x=x_2$$
which is the orthogonality condition

1. Find the shortest distance b/w the parabola $y=x^2$ and the straight line $x-y=5$.

proof:
The problem is to find the extremum of the functional
$$I[y(x)]=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$$

Subject to condition the left end of the extremal moves along $y=x^2$ while the right end moves along $x-y=5$. Thus the transversality condition is given by,
$$[F+(\phi'-y')F_{y'}]=0$$

$$\left[\sqrt{1+y'^2}+(2x-y')\frac{1}{2\sqrt{1+y'^2}}2y'\right]=0 \rightarrow ①$$

and $$\left[\sqrt{1+y'^2}+(1+y')\frac{1}{2\sqrt{1+y'^2}}2y'\right]_{x=x_2}=0 \rightarrow ②$$ $(\because \phi=x^2,\ \phi'=2x)$

Since the general solution of Euler's equation for above functional is,
$$y=c_1x+c_2$$
where c_1, c_2 are constant

DATE:
PAGE:

$y' = c_1$

Further both end point lie on the extremal.

$y = c_1 x + c_2$

Hence, we have

$$\left.\begin{aligned} c_1 x_1 + c_2 &= x_1^2 \\ c_1 x_2 + c_2 &= x_2 - 5 \end{aligned}\right\} \rightarrow (3)$$

Replacing $y' = c_1$ in (1)

$$\left[\sqrt{1+c_1^2} + \frac{(2x - c_1)\, c_1}{\sqrt{1+c_1^2}}\right] = 0$$

$(1+c_1)^2 + (2x - c_1) c_1 = 0$

$1 + c_1^2 + 2x c_1 - c_1^2 = 0$

$2x c_1 = -1 \rightarrow (4)$

from (2)

$$\left[\sqrt{1+c_1^2} + \frac{(1-c_1) c_1}{\sqrt{1+c_1^2}}\right] = 0$$

$(1+c_1^2) + (1-c_2) c_1 = 0$

$1 + c_1^2 + c_1 - c_1^2 = 0$

$1 + c_1 = 0$

$c_1 = -1$

$(4) \Rightarrow 2x(-1) = -1$

$2x = 1$

$x = 1/2$

put $x = x_1$, $x_1 = 1/2$, $c_1 = -1$ in (3)

$-\frac{1}{2} + c_2 = \frac{1}{4}$

$c_2 = \frac{1}{4} + \frac{1}{2}$

$c_2 = \frac{3}{4}$

$$-\frac{x}{2} + \frac{3}{4} = x_2 - 5$$

$$2x_2 = \frac{23}{4}$$

$$x_2 = \frac{23}{8}$$

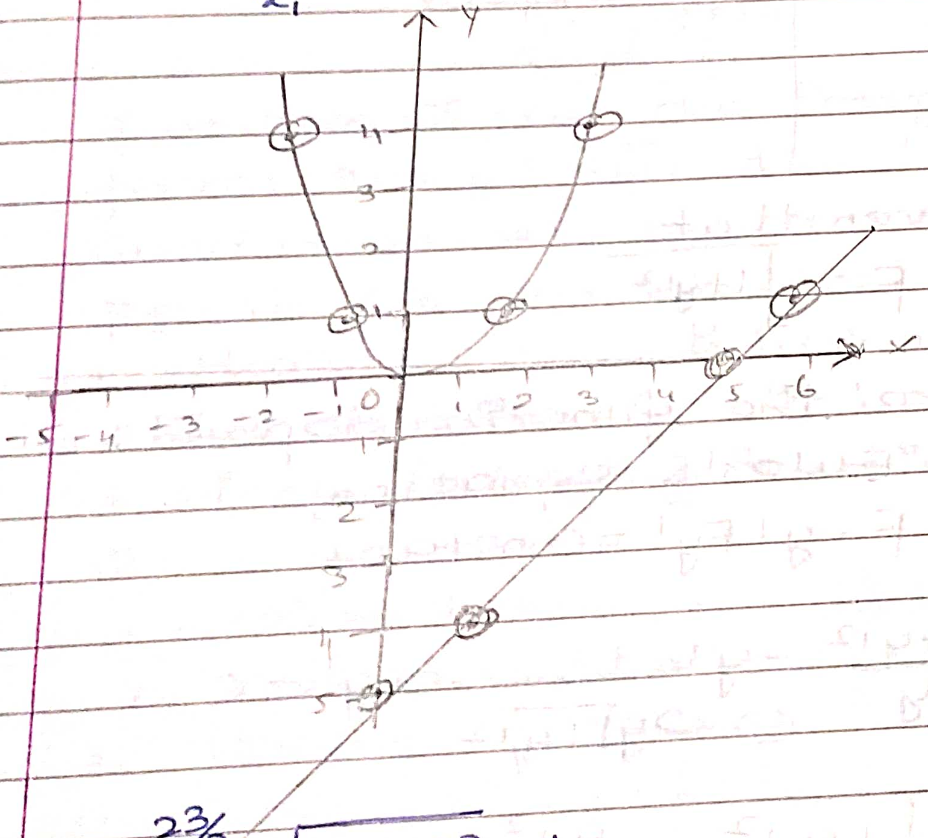
$\therefore$ The required extremal is,

$$y = c_1 x + c_2$$

$$y = -x + \frac{3}{4}$$

$\therefore$ The shortest distance between the given parabola & straightline is,

$$L = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$



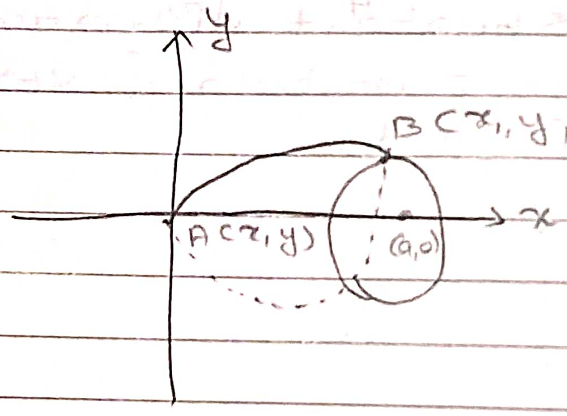
$$= \int^{23/8} \sqrt{1+(-1)^2}\, dx$$

$$= \int_{1/2}^{23/8} \sqrt{2}\, dx \Rightarrow \sqrt{2}\left[x\right]_{1/2}^{23/8} \Rightarrow \sqrt{2}\left[\frac{23}{8} - \frac{1}{2}\right]$$

$$L = \sqrt{2}\left(\frac{19}{8}\right)$$

2. Using only the basis necessary condition $\delta I = 0$ find the curve on which an extremum of the functional
$$I[y(x)] = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx, \quad y(0) = 0$$
can be achieved if the second point (x_1, y_1) can move along the circumference $(x-9)^2 + y^2 = 9$.

Solution:-



Given that
$$F = \frac{\sqrt{1+y'^2}}{y}$$

Here the function depends on y and y'
By Euler's equation,
$$F - y' F_{y'} = \text{constant}$$

$$\frac{\sqrt{1+y'^2}}{y} - y' \frac{1}{2y\sqrt{1+y'^2}} \cdot 2y' = c$$

$$\Rightarrow \frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}} = c$$

$$\frac{1 + y'^2 - y'^2}{y^2\sqrt{1+y'^2}} = c$$

$$\frac{1}{y^2\sqrt{1+y'^2}} = c$$

$$\frac{1}{y^2(1+y'^2)} = c$$

$$1+y'^2 = \frac{c}{y^2}$$

The solution is the two parameter family of circle,

$$(x-c_1)^2 + y^2 = c_2^2 \rightarrow (1)$$

The boundary condition is,

$$y(0) = 0$$

$$(1) \Rightarrow (0-c_1)^2 + 0 = c_2^2$$

$$c_1^2 = c_2^2$$

$$c_1 = c_2$$

Further, since the integral is of the form $A(x,y)\sqrt{1+y'^2}$, the transversality condition at the movable boundary point (x_1, y_1) reduces to the orthogonality condition.

Thus the external will be the arc of a circle belonging to which is orthogonal to,

$$(x_1-9)^2 + y_1^2 = 9$$

$$x_1^2 + 81 - 18x_1 + y_1^2 = 9$$

$$x_1^2 - 18x_1 + y_1^2 = -72 \rightarrow (2)$$

$$(1) \Rightarrow (x_1-c_1)^2 + y_1^2 = c_1^2$$

$$x_1^2 + c_1^2 - 2x_1c_1 + y_1^2 = c_2^2$$

$$x_1^2 - 2x_1c_1 + y_1^2 = 0 \rightarrow (3)$$

$$(2) - (3) \Rightarrow$$

$$-18x_1 + 2c_1x_1 = -72$$

$$2x_1(c_1 - 9) = -72$$

DATE:
PAGE:

$x_1(c_1-9) = -36$

$x_1(9-c_1) = 36 \rightarrow (4)$

In the view of the orthogonality of the two circles at (x_1, y_1) the tangent to $(x-c_1)^2+y^2=c_2^2$ at B passes through the centre $(9,0)$ of the given circle.

This yields,

$(9-c_1)x_1 = 9c_1 \rightarrow (5)$

from (4) & (5) we have

$c_1 = 4$

(4) $\Rightarrow (9-4)x_1 = 36$

$5x_1 = 36$

$x_1 = 36/5$

So that the required extremal are the arc of the circle,

$(x-c_1)^2+y^2=c_2^2$

$(x-4)^2+y^2=16$

$x^2-8x+16+y^2=16$

$x^2+y^2-8x=0$

$y^2=8x-x^2$

$y=\pm\sqrt{8x-x^2}$

3. Find the extremum of the functional $I=\int_{x_1}^{x_2}[y'^2+z'^2+2yz]\,dx$ with $y(0)=0$, $z(0)=0$

are the point (x_2, y_2, z_2) moves in a fixed point $x=x_2$.

Solution:-

Given that $F=y'^2+z'^2+2yz$

By Euler's eqn,

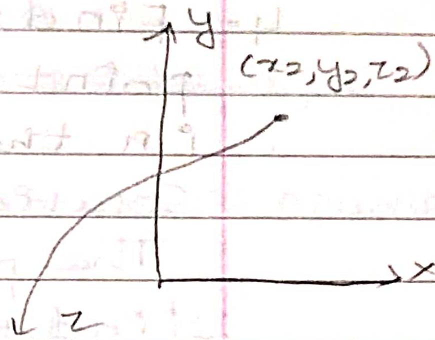
$$F_y - \frac{d}{dx} F_{y'} = 0 \qquad\qquad F_z - \frac{d}{dx} F_{z'} = 0$$

$$2z - \frac{d}{dx}(2y') = 0 \qquad\qquad 2y - \frac{d}{dx}(2z') = 0$$

$$2z - 2y'' = 0 \qquad\qquad 2y - 2z'' = 0$$

$$z - y'' = 0 \qquad\qquad y - z'' = 0$$

$$z = y'' \qquad\qquad y = z''$$



$$\Rightarrow y = y^{IV} \qquad\qquad z = z^{IV}$$

$$y^{IV} - y = 0 \qquad\qquad z^{IV} - z = 0$$

$\therefore$ The general solution of this equation is,

$$y(x) = c_1 \cosh x + c_2 \sinh x + c_3 \cos x + c_4 \sin x \rightarrow ①$$

$$z(x) = c_1 \cosh x + c_2 \sinh x - c_3 \cos x - c_4 \sin x \rightarrow ②$$

Subject to the condition,

$$y(0) = 0, \quad z(0) = 0$$

$$① \Rightarrow y(0) = c_1 + c_3 \qquad\qquad ② \Rightarrow z(0) = c_1 - c_3$$

$$c_1 + c_3 = 0 \qquad\qquad c_1 - c_3 = 0$$

$$c_3 = 0$$

$$\therefore y(x) = c_2 \sinh x + c_4 \sin x \rightarrow ③$$

$$z(x) = c_2 \sinh x - c_4 \sin x \rightarrow ④$$

Further the condition of the fixed point at x_2 given,

$$y'(x_2) = 0, \quad z'(x_2) = 0$$

$$y'(x_2) = c_2 \cosh x_2 + c_4 \cos x_2 = 0$$

$$\Rightarrow c_2 \cosh x_2 + c_4 \cos x_2 = 0$$

$$z'(x_2) = c_2 \cosh x_2 - c_4 \cos x_2$$

$$\Rightarrow c_2 \cosh x_2 - c_4 \cos x_2 = 0$$

If $\cos x_2 \neq 0$, then $c_2 = 0$ and c_4

$\therefore$ The extremum is attained at $y=0$ and $z=0$

If $\cos x_2 = 0$

Then $c_2 = 0$, c_4 remains arbitrary in this case the extremum is,

$$y = c_4 \sin x$$

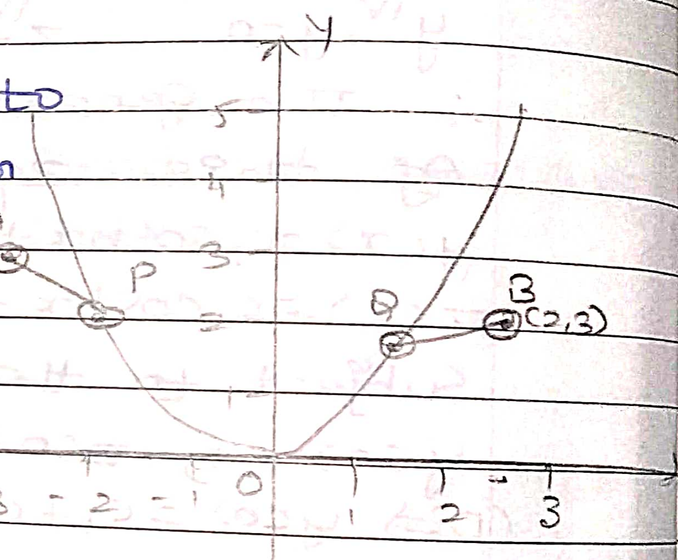
$$z = -c_4 \sin x.$$

4. Find the shortest path from the point $A(-2,3)$ to the point $B(2,3)$ in the region $y \le x^2$.

Solution:-

The problem is to find the extremum the function

$$I[y(x)] = \int_{-2}^{2} \sqrt{1+y'^2}\,dx$$



Subject to the condition $y \le x^2$.

Clearly, the extremals of $I[y(x)]$ are straight line.

i.e) $y = c_1 + c_2 x$

If F is the integral in $I[y(x)]$ then,

$F_{y'y'} \neq 0 \Rightarrow$ one side variation.

Let $F = \sqrt{1+y'^2}$

$$F_{y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$F_{y'y'} = \frac{\sqrt{1+y'^2} - y'\frac{y'}{\sqrt{1+y'^2}}}{(1+y'^2)}$$

DATE:
PAGE:

$$= \frac{(1+y'^2) - y'^2}{(1+y'^2)^{3/2}}$$

$$F_{y'y'} = \frac{1}{(1+y'^2)^{3/2}} \neq 0$$

Hence by the result of one sides variation it desired extremal with consist of the straight line AP and BQ Both tangent to the parabola $y=x^2$ and of the portion PQ

let the abcissa of P and Q be $-\bar{x}, \bar{x}$ respectively.

The condition of the tangent of AP and BQ at P and Q.

$$c_1 + c_2\bar{x} = \bar{x} \rightarrow ①$$

$$c_2 = 2\bar{x}$$

since the tangent Q passes through the point $(-2, 3)$

$$c_1 + 2c_2 = 3 \rightarrow ②$$

Solving ① & ②

$$① - ② \Rightarrow c_2(\bar{x}-2) = \bar{x}^2 - 3$$

$$2\bar{x}(\bar{x}-2) = \bar{x}^2 - 3$$

$$\Rightarrow \bar{x} = 1, 3$$

$$\therefore \bar{x}_1 = 1, \quad \bar{x}_2 = 3$$

the second value is clearly admissable.

$$\therefore \bar{x} = 1, \quad c_2 = 2\bar{x} \Rightarrow c_2 = 2$$

$$② \Rightarrow c_1 + 4 = 3$$

$$c_1 = -1$$

The required extemal is,

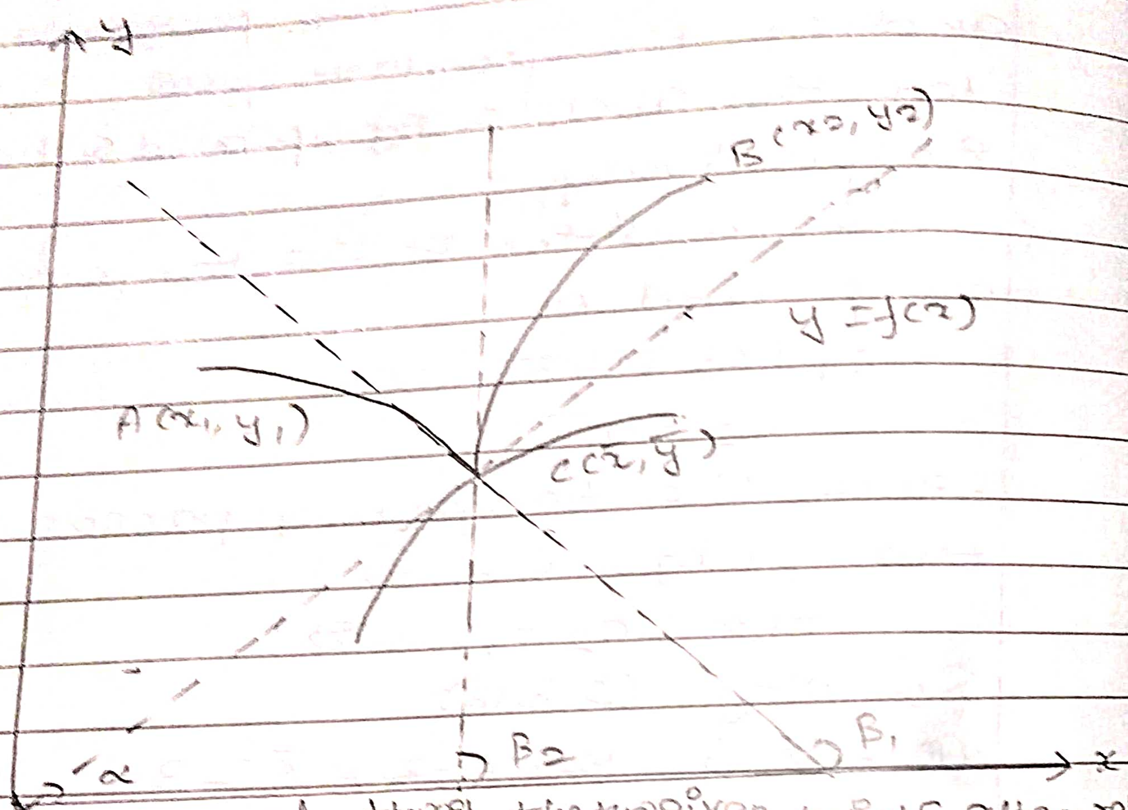
$AP \Rightarrow y = -1-2x \qquad \therefore 2 \leq x \leq -1$

$PQ \rightarrow y = x^2 \qquad -1 \leq x \leq 1$

$QB \rightarrow y = -1+2x \qquad -1 \leq x \leq 2$

Reflection of extremals:

To find the curve that extremize the functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx$ and passes through $A(x_1, y_1)$ and $B(x_2, y_2)$ the curve must at B only after reflection from a given curve $y = f(x)$.



extremal through the two given points after reflection a given curve.

It is conceivable that at the point of reflection $C(\bar{x}, \bar{y})$, the desired extremal may have a corner pointers so that $y'(\bar{x}-0)$ and $y'(\bar{x}+0)$ are en distinct.

Thus the functional can be written as,

$$I[y(x)] = \int_{x_1}^{\bar{x}} F[x, y, y']\,dx + \int_{\bar{x}}^{x_2} F[x, y, y']\,dx$$

DATE:
PAGE:

So that the necessary condition for an extremal is given by,

$\delta I = 0$

Then the above equation redued to

$$\delta \int_{x_1}^{\bar{x}} F(x, y, \bar{y}) dx + \delta \int_{\bar{x}}^{x_2} F(x, y, \bar{y}) dx = 0 \rightarrow (1)$$

Evidently the curves AC and CB are both extremal because if we assume that one of the curve is already found and us vary the other one done.

Then the problem reduces to finding the extremals of

$\int_{x_1}^{\bar{x}} F dx$ (or) $\int_{\bar{x}}^{x_1} F dx$ with fixed bounded

point.

Thus for calculating the variation of the functional (1)

We will assume that the functionals are considered only on extremal with corner point C.

Thus,

$$\delta \int_{x_1}^{\bar{x}} F(x, y, y') dx = [F + (\phi' - y') F_{y'}]_{x = \bar{x} - 0} \delta x$$

$$\delta \int_{\bar{x}}^{x_2} F(x, y, y') dx = -[F + (\phi' - y') F_{y'}]_{x = \bar{x} + 0} \delta x$$

$$(1) \Rightarrow [F + (\phi' - y') F_{y'}]_{x = \bar{x} - 0} = [F + (\phi' - y') F_{y'}]_{x = \bar{x} + 0}$$

$\rightarrow (2)$

This notation assumes a particular simple form for a functional of the form.

$$I[y(x)] = \int_{x_1}^{x_2} A(x,y)\sqrt{1+y'^2}\,dx$$

In this case eqn ② becomes,

$$A(x,y)\left[\sqrt{1+y'^2} + \frac{(f'-y')y'}{\sqrt{1+y'^2}}\right]_{x=\bar{x}-0}$$

$$= A(x,y)\left[\sqrt{1+y'^2} + \frac{(f'-y')\;y'}{\sqrt{1+y'^2}}\right]_{x=\bar{x}+0}$$

Assuming $A(x,y) \neq 0$, the above relation reduces to,

$$\left[\frac{1+f'y'}{\sqrt{1+y'^2}}\right]_{x=\bar{x}-0} = \left[\frac{1+f'y'}{\sqrt{1+y'^2}}\right]_{x=\bar{x}+0}$$

We find that,

$$y'(\bar{x}-0) = \tan\beta_1$$

$$y'(\bar{x}+0) = \tan\beta_2$$

$$f(\bar{x}) = \tan\alpha$$

The above equation reduced to,

$$-\frac{(1+\tan\alpha\tan\beta_1)}{\sqrt{1+\tan^2\beta_1}} = \frac{(1+\tan\alpha\tan\beta_2)}{\sqrt{1+\tan^2\beta_2}}$$

$$-\frac{(1+\tan\alpha.\tan\beta_1)}{\sec\beta_1} = \frac{(1+\tan\alpha.\tan\beta_2)}{\sec\beta_2}$$

$$-\frac{1+\frac{\sin\alpha}{\cos\alpha}.\frac{\sin\beta_1}{\cos\beta_1}}{\sec\beta_1} = \frac{1+\frac{\sin\alpha}{\cos\alpha}.\frac{\sin\beta_2}{\cos\beta_2}}{\sec\beta_2}$$

$$\Rightarrow -\frac{(\cos\alpha\cos\beta_1 + \sin\alpha\sin\beta_1)}{\cos\alpha\cos\beta_1 \sec\beta_1}$$

$$= \frac{\cos\alpha\cos\beta_2 + \sin\alpha\sin\beta_2}{\cos\alpha\cos\beta_2 \sec\beta_2}$$

$-[\cos(\alpha-\beta_1)] = \cos(\alpha-\beta_2)$

which express the equality of the angle of reflection.

Refraction of extremals:-

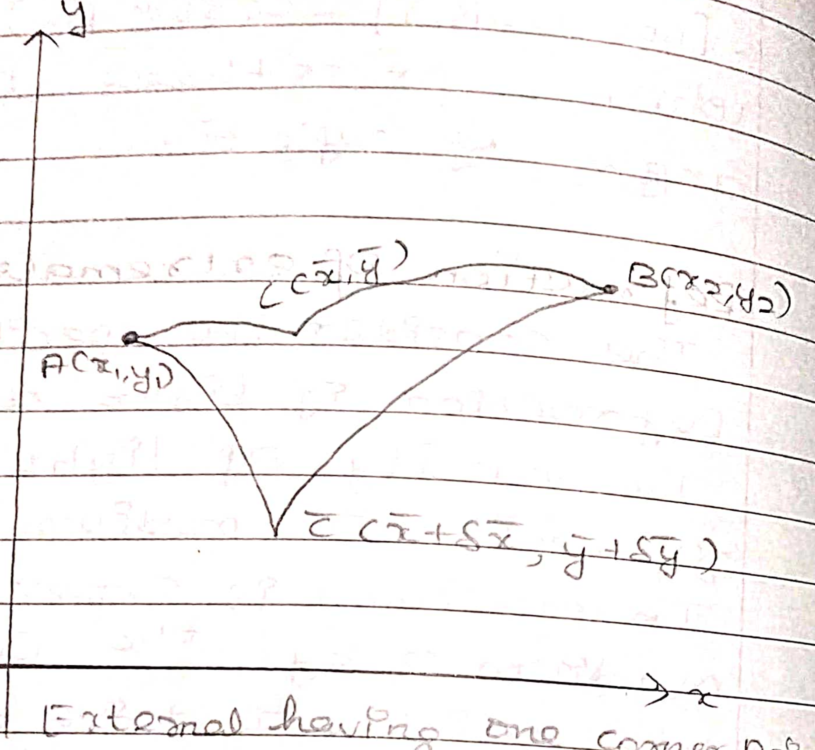
The consider the specific problem of Refraction of light as follows. The velocity of light in medium is v_1 while in medium 2 it is v_2. In medium 1 is seperated from medium 2 by the curve $y=\phi(x)$. We wish to drive the log of refraction of light ray travelling from a point A in medium 1 to a point B medium B subject to the condition that the light ray travers dis path in the shortest time interval in accordance with Fermat's principle. Clearly the problem reduces to finding the minimum of the functional

$$I[y(x)] = \int_a^c \frac{\sqrt{1+y'^2}}{v_1}\,dx + \int_c^b \frac{\sqrt{1+y'^2}}{v_2}\,dx$$

$$I[y(x)] = \int_a^b F_1(y')\,dx + \int_c^b F_2(y')\,dx \rightarrow ①$$

DATE:
PAGE:

where a and b are the abscissa of A and B and c is the abscissa of a point.
where the extremal from A to B meets the curve $y = y(x)$.



Extremal having one corner point

It is conceivable that the extremum suffers a jump in so at the point C. The necessary condition for an extremum is $\delta z = 0$

Then $[F_1 + (\phi' - y') F_{y'}]_{x=c-0} = [F_2 - (\phi' - y') F_{2y'}]_{c+0}$

since in the present problem F_1 and F_2 given by eqn ① are functions of y' only. the extremals are straight line s.t

$$y = y_1(x) = lx + m$$

is the extremal in medium 1 and $y = y_2(x) = px + 2$ is the extremal in medium 2.

Substituting for t_1 and t_2 in the above equation we get,

$$\frac{\sqrt{1+y_1'^2}}{\gamma_1} + \frac{(\phi'-y_1')}{\gamma_1\sqrt{1+y_1'^2}} = \frac{\sqrt{1+y_2'^2}}{\gamma_2} + \frac{(\phi'-y_2')y_2'}{\gamma_2\sqrt{1+y_2'^2}}$$

$$\frac{(1+y_1'^2)+\phi' y_1' - y_1'^2}{\gamma_1\sqrt{(1+y_1'^2)}} = \frac{(1+y_2')^2+\phi' y_2' - y_2'^2}{\gamma_2\sqrt{1+y_2'^2}}$$

$$\frac{1+\phi' y_1'}{\gamma_1\sqrt{1+y_1'^2}} = \frac{1+\phi' y_2'}{\gamma_2\sqrt{1+y_2'^2}} \rightarrow (3)$$

Let γ be the angle from with x axis by the tangent to the boundary of the curve $y=\phi(x)$ at the point C

Let α be the angle to the left hand ray with the x axis and β be the angle of the right hand ray with the x axis

$\therefore \phi' = \tan\gamma$

$y_1' = \tan\alpha, \; y_2' = \tan\beta$

(3) $\Rightarrow$

$$\frac{1+\tan\gamma\tan\alpha}{\gamma_1\sqrt{1+\tan^2\alpha}} = \frac{1+\tan\gamma\tan\beta}{\gamma_2\sqrt{1+\tan^2\beta}}$$

$$\frac{1+\frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\gamma}{\cos\gamma}}{\gamma_1(\sec\alpha)} = \frac{1+\frac{\sin\gamma}{\cos\gamma}\cdot\frac{\sin\beta}{\cos\beta}}{\gamma_2\sec\beta}$$

$$\frac{\cos\gamma\cos\alpha+\sin\gamma\sin\alpha}{\cos\alpha\cdot\cos\gamma\,\gamma_1\sec\alpha} = \frac{\cos\gamma\cos\beta+\sin\gamma\sin\beta}{\cos\gamma\cos\beta\,\gamma_2\sec\beta}$$

$$\cos(\gamma-\alpha) = \frac{\cos(\gamma-\beta)}{\gamma_2\cos\gamma}$$

$$\frac{\cos(\gamma-\alpha)}{\gamma_1} = \frac{\cos(\gamma-\beta)}{\gamma_2}$$

DATE:
PAGE:

where $r-\alpha$ and $r-\beta$ are the angles b/w the ray and the tangent to the boundary curve.
Introducing in the place the angles and ϕ b/w the normal to the boundary curve and the incident refracted rays.
We obtain a well known law of the refraction of light.
(Snell's law)

As $\frac{\sin\theta}{\sin\phi} = \frac{r_1}{r_2}$ = constant.

from the foregoin discussion it should not. However be thought that a corner point on an extremal occurs only in the problem involving diflection and refraction of extremal such a point can exist on an extremal of the functional.
$\int_{x_1}^{x_2} F[x, y, y']dx$ where F is three times differentiable and the admissable curve passes through the point $A[x_1, y_1]$ and $B[x_2, y_2]$.
Let us know determine the condition must be satisfies the solution of the extremal problems of the above function. We assume that extremal this problem as one corner point P at $[\bar{x}, \bar{y}]$ as shown in the figure.

Now AC and CB are the integral curves of Euler's equation for the functional.

If C moves in any function, we get

$$I[y(x)] = \int_{x_1}^{\bar{x}} F[x, y, y']\,dx + \int_{\bar{x}}^{x_2} [x, \bar{y}, \bar{y}']\,dx$$

The necessary condition for an extremum is $\delta I = 0$

$$\therefore [F - y' F_{y'}]_{x=\bar{x}-0}\, \delta\bar{x} + [F_{y'}]_{x=\bar{x}-0}\, \delta\bar{y} -$$

$$[F - \bar{y} F_{y'}]_{x=\bar{x}+0}\, \delta\bar{x} - [F_{y'}]_{x=\bar{x}+0}\, \delta\bar{y} = 0$$

Since $\delta\bar{x}$ & $\bar{F_y}$ are independent, we get

$$[F - y' F_{y'}]_{x=\bar{x}-0} = [F - \bar{y} F_{y'}]_{x=\bar{x}+0}$$

$$[F_{y'}]_{x=\bar{x}-0} = [F_{y'}]_{x=\bar{x}+0}$$

These condition are known as Weistrass Erdmann corner condition.

5. Find the extremals with the corner points of the function

$$I[y(x)] = \int_{x_1}^{x_2} y'^2 [1 - y']^2\,dx.$$

Solution:-

Given,

$$I[y(x)] = \int_{x_1}^{x_2} y'^2 [1 - y']^2\,dx$$

where,

$F = y'^2[1-y']^2$

Here the function depends upon only the extremals are straight like.

i, e) $y = c_1 x + c_2$

Now at the corner point condition By weistrass Bradamann corner point condition is,

$[F - y' F_{y'}]_{x=\bar{x}-0} = [F - y' F_{y'}]_{x=\bar{x}+0}$

$[F_{y'}]_{x=\bar{x}-0} = [F_{y'}]_{x=\bar{x}+0}$

$F_{y'} = y'^2[1-y']^2$

$= y'^2 \cdot 2(1-y')(-1) + (1-y')^2 2y'$

$= -2y'^2(1-y') + 2y'(1-y')^2$

$= 2y'(1-y')[-y' + (1-y')]$

$F_{y'} = 2y'(1-y')(1-2y')$

$F - y'F_{y'} = y'^2[1-y']^2 - 2y'^2(1-y')(1-2y')$

$= y'^2(1-y')[(1-y') - 2(1-2y')]$

$= y'^2(1-y')(1-y'-2+4y')$

$F - y'F_{y'} = y'^2(1-y')(3y'-1)$

$F - y'F_{y'} = -y'^2(1-y')(1-3y')$

$\therefore$ The condition of the corner points becomes,

$[-y'^2(1-y')(1-3y')]_{x=\bar{x}-0} = [-y'^2(1-y')(1-3y')]_{x=\bar{x}+0}$

$[y'^2(1-y')(1-3y')]_{x=\bar{x}-0} = [y'^2(1-y')(1-3y')]_{x=\bar{x}+0}$

and

$[2y'(1-y')(1-2y')]_{x=\bar{x}-0} = [2y'(1-y')(1-2y')]_{x=\bar{x}+0}$

$$[y'(1-y')(1-2y')]_{x=\bar{x}-0} = [y'(1-y')(1-2y')]_{x=\bar{x}+0}$$

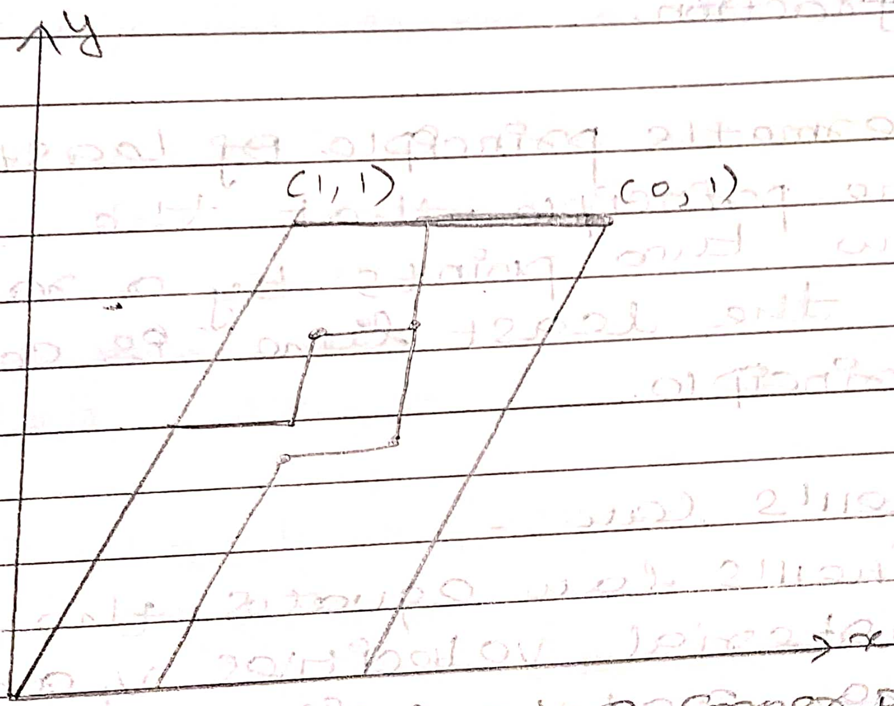
If we ignore the trivial possibility $y'(\bar{x}-0)$ and $y'(\bar{x}+0)$
we find the above equation are satisfied for either,

$y'(\bar{x}-0)=0$, $\bar{y}(\bar{x}+0)=1$

(or)

$y'(\bar{x}-0)=1$, $y'(\bar{x}+0)=0$

Thus the broken line extremal consist of segments of straight line belonging to the family $y=A_1$ and $y=x+A_2$.



extremal having two corner point

If the integral in $I[y(x)]$ is denoted by F_1 then $F_{y'y'} = 2y'(1-y')(1-2y')\neq 0$

This is a possible indication of the existance of corner points.

DATE:
PAGE:

1. Angle of incident:-
The angle b/w incident ray and normal to surface is known as Angle of incident.

2. Angle of reflected.
The angle b/w reflected ray and normal to the surface is known as angle reflected.

3. Angle of Refraction:-
The angle b/w refraction ray and the normal to the surface is Refraction.

4. Fermat's principle of least time:
The principle that the path taken b/w two points by a ray of light is the least time is called fermat principle.

5. Snell's law:-
Snell's law equates the ratio of material velocities V_1 and V_2 to the ratio of sine of incident angle θ_1 and θ_2

i.e) $\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$

6. Geodesic:-

A Geodesic on a surface is the curve along with the distance b/w any two point of the surface is minimum.

7. Abel's Integral equation:

The integral equation

$$\int_0^x \frac{y(x)}{(x-t)^\alpha} dx = \sigma(x)$$

$\sigma(x)$ is any given function and α is a constant $0 < \alpha < 1$ is called able's Integral equation.

8. Integro-differential eqn:-

An integral equation is which various derivative of the unknown function $y(x)$ are all present is called Integro-differential equation.

Ex:

$$y'(x) = y(x) - \cos x \int_0^x \sin(x-t)\, y(t)\, dt.$$