Unit - VI

Integral Equation

Integral Equation.

An Integral equation is an equation in which an unknown function appear under one or more integral sign.

Ex:

for $a \le x \le b$, $a \le t \le b$

i) $u(x) = \int_a^x k(x,t)\, u(t)\, dt$

ii) $u(x) = f(x) + \int_a^b k(x,t)\, u(t)\, dt$

iii) $u(x) = \lambda \int_a^b k(x,t)\, [u(t)]^2\, dt$.

where $u(x)$ is unknown function $f(x)$, $k(x,t)$ are known function λ, a, b. are constants.

Linear and non linear integral equation.

An Integral equation is called Linear if only linear operations are performed in it upon the unknown function.

An I. E is not linear, is known as non linear Integral Equation.

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Ex: Linear I.E

i) $u(x) = \int_a^x k(x,t)u(t)\,dt$

ii) $u(x) = f(x) + \int_a^b k(x,t)u(t)\,dt$

non-linear I.E

$u(x) = \lambda \int_a^b k(x,t)[u(t)]^2\,dt.$

2.m. General form of I.E

The most general type of Linear I.E is of the form.

2m ($v(x).u(x) = f(x) + \lambda \int_a k(x,t)u(t)\,dt.$) 2m

where the upper limit may be either variable x are or fixed points b.

The Function $f(x)$, $v(x)$, $k(x,t)$ are known function

$u(x)$ is unknown function

λ is a non zero real (or) complex parameter.

The Function 2m ($k(x,t)$ is known as kernal 2m of the integral Equation) 2m

Types of Linear I.E :

Consider the Linear I.E

$$v(x).u(x) = f(x) + \lambda \int_a k(x,t)\,u(t)\,dt \quad \to (1)$$

i) If $v(x) = 0$

$$(1) \Rightarrow \quad 0 = f(x) + \lambda \int_a k(x,t)\,u(t)\,dt$$

is known as Linear I.E of first kind.

ii) If $v(x) = 1$

$$(1) \Rightarrow \quad u(x) = f(x) + \lambda \int_a k(x,t)\,u(t)\,dt$$

is known as Linear I.E of second kind.

iii) If $v(x) = 1$, $f(x) = 0$.

$$(1) \Rightarrow \quad u(x) = \lambda \int_a k(x,t)\,u(t)\,dt$$

is known as homogeneous integral Equation

iv) If $v(x) \neq 0$

$$(1) \to \quad v(x).u(x) = f(x) + \lambda \int_a k(x,t)\,u(t)\,dt$$

is known as Linear I.E of third kind.

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volterra I.E

A Linear I.E is of the form :

$$v(x).u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt$$

where a is constant

$v(x)$, $f(x)$, $K(x,t)$ are known function

$u(x)$. is unknown function

λ is a non-zero real (or) complex parameter

This equation is called volterra I.E

Special Cases:

i) If $v(x)=0$

$$0 = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt$$ is

known as volterra I.E of first kind

ii) If $v(x)=1$

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt$$ is

known as volterra I.E of second kind

iii) If $v(x)=1$ $f(x)=0$

$$u(x) = \lambda \int_a^x K(x,t)\, u(t)\, dt$$ is

known as homogenous volterra I.E

Fredholm I.E

A linear I.E is of the form

$$v(x).u(x) = f(x) + \lambda \int_a^b k(x,t)\,u(t)\,dt$$

where a, b are constant

$v(x), f(x), k(x,t)$ are known function.

$u(x)$ is unknown function.

λ is a non-zero real (or) complex parameter.

This equation is called Fredholm I.E

Special cases:

i) If $v(x)=0$.

$$0 = f(x) + \lambda \int_a^b k(x,t)\; u(t)\,dt.$$ is

Known as Fredholm I.E of first kind

ii) If $v(x)=1$

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\,u(t)\,dt$$ is

Known as Fredholm I.E of second kind

iii) If $v(x)=1$ $f(x)=0$

$$u(x) = \lambda \int_a^b k(x,t)\; u(t)\,dt$$ is

Known as homogenous Fredholm I.E

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Singular I.E

when one or both limit of integration became infinite (or) when the kernel became infinite at one or more point with in the range of integration, The integral equation is known as singular I.E.

Example

i) $u(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-(x-t)} u(t)\, dt$

ii) $u(x) = f(x) + \lambda \int_{0}^{\infty} \frac{1}{(x-t)^{\alpha}} u(t)\, dt, \; 0<\alpha<1$

Convolution type of I.E

If the kernel $K(x,t)$ of the integral equation is a function of the difference of x and t.

i.e) $K(x,t) = K(x-t)$

where K is the certain function of one variable

Ex:

$u(x) = f(x) + \lambda \int_{a}^{x} K(x,t)\, u(t)\, dt$

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Types of Kernal:

1. Symmetric kernal:

A kernal $k(x,t)$ is symmetric ~~or~~ or complex symmetric or hermition if $k(x,t) = \overline{k(x,t)}$
where bar denotes the complex conjugate.

A real kernal $k(x,t)$ is called symmetric, if
if $k(x,t) = k(t,x)$

ii) Seperable (or) Degenerate kernal.

A kernal which is particularly useful for solving Fredholm I.E

$$k(x,t) = \sum_{i=1}^{n} a_i(x) b_i(t)$$

where n is finite and a_i, b_i are linearly Independent set of function, such a kernal is called separable.

iii) Transposted kernal

A kernal $k^T(x,t) = k(t,x)$ is called transposted kernal.

iv) Iterated kernal.

a) Consider the fredholm I.E of second kind.

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt.$$

Then the iterated kernal $k_n(x,t)$, $n=1,2,\ldots$ are defined as follows.

1. $K_1(x,t) = K(x,t)$
2. $K_n(x,t) = \int_a^b K(x,s) K_{n-1}(s,t)\, ds$ $\quad n=2,3,\ldots$

Consider the volterra I.E of second kind

1. $u(x) = f(x) + \lambda \int_a^x K(x,t) u(t)\, dt$

then the itterate kernal $k_n(x,t)$, $n=1,2,\ldots$ are defined as follows.

1. $K_1(x,t) = K(x,t)$
2. $K_n(x,t) = \int_a^x K(x,s) \cdot K_{n-1}(s,t)\, ds.$

Resolvant (or) Resiprocal kernal

Consider the integral equations, $u(x) = f(x) + \lambda \int_a^b K(x,t) u(t)\, dt \to ①$

and

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t)\, dt \to ②.$$

solving the above equations ① & ② and the solutions are given by

$$u(x) = f(x) + \lambda \int_a^b R(x,t,\lambda) f(t)\, dt$$

and

$$u(x) = f(x) + \lambda \int_a^b \Gamma(x,t,\lambda) f(t)\, dt$$

where $R(x,t,\lambda)$ and $\Gamma(x,t,\lambda)$ are called Resolvant kernal.

2m (or) 5m.

Eigen values and Eigen functions:

Consider the homogenous Fredholm I.E

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt \rightarrow (1).$$

unit 3 (or) 4.

then the values of the parameter λ for which equation (1) has non-zero solution $(u(x) \neq 0)$ are called Eigen values and Every non-zero solution of equation (1) is called Eigen function corresponding to the Eigen values.

Differentiation under the sign of Integral (Leibnitz Rule)

Let $F(x,t)$ and $\frac{\partial F}{\partial x}$ are continuous function on both x and t.

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\, dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}\, dt + F[x, H(x)]\frac{dH}{dx} - F[x, G(x)]\frac{dG}{dx}$$

Solution of Integral Equations.

1. verify that the given function $u(x) = 1/2$ is a solution of the I.E $\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \sqrt{x}$.

5m

Solution:

Given I.E is

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \sqrt{x} \rightarrow (1)$$

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Also given that

$u(x) = \frac{1}{2}$

$u(t) = \frac{1}{2}$.

L.H.S of (1)

$$L.H.S = \int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt$$

$$= \int_0^x \frac{\frac{1}{2}}{\sqrt{x-t}}\, dt$$

$$= \frac{1}{2}\int_0^x \frac{1}{\sqrt{x-t}}\, dt \qquad \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a.(n+1)}$$

$$= \frac{1}{2}\int_0^x (x-t)^{-\frac{1}{2}}\, dt$$

$$= \frac{1}{2}\left[\frac{(x-t)^{-\frac{1}{2}+1}}{-1(-\frac{1}{2}+1)}\right]_0^x$$

$$= \frac{1}{2}\left[\frac{(x-t)^{\frac{1}{2}}}{-\frac{1}{2}}\right]_0^x$$

$$= -\left[(x-t)^{\frac{1}{2}}\right]_0^x$$

$$= -\left[(0) - (x)^{\frac{1}{2}}\right]$$

$$= x^{\frac{1}{2}}$$

$$= \sqrt{x}$$

L.H.S = R.H.S.

$\therefore$ $u(x)$ is also given I.E

2. S.T the solution $u(x)=1$ is a solution of Fredholm I.E $u(x)+\int_0^1 x(e^{xt}-1)u(t)dt = e^x - x$

proof:

Given I.E is

$$u(x)+\int_0^1 x(e^{xt}-1)u(t)\,dt = e^x - x \quad \to (1)$$

Also given that.

$$u(x)=1$$

$$u(t)=1$$

L.H.S of (1).

$$L.H.S = u(x)+\int_0^1 x(e^{xt}-1)u(t)dt$$

$$= 1+\int_0^1 x(e^{xt}-1).1\,dt$$

$$= 1+x\int_0^1 (e^{xt}-t)\,dt$$

$$= 1+x\left[\frac{e^{xt}}{x}-t\right]_0^1$$

$$= 1+x\left[\left(\frac{e^x}{x}-1\right)-\left(\frac{1}{x}-0\right)\right]$$

$$= 1+x\left[\frac{e^x}{x}-1-\frac{1}{x}\right]$$

$$= 1+e^x-x-1$$

$$= e^x-x$$

L.H.S = R.H.S

$\therefore u(x)$ is also given I.E.

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3. S.T the function $u(x) = (1+x^2)^{-3/2}$ is a solution volterra I.E

$$u(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{(1+x^2)} u(t)\,dt.$$

proof :

Given I.E is

① $$u(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{(1+x^2)} u(t)\,dt \quad \to ①.$$

Also given that.

$$u(x) = (1+x^2)^{-3/2}$$

$$u(t) = (1+t^2)^{-3/2}$$

R.H.S of ①.

$$R.H.S = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} u(t)\,dt$$

$$= \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} (1+t^2)^{-3/2}\,dt$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \int_0^x t(1+t^2)^{-3/2}\,dt$$

put $s = t^2$

$$\frac{ds}{dt} = 2t$$

when $t = 0 \Rightarrow s = 0$

$t = x \Rightarrow s = x^2$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \int_0^{x^2} t\,(1+s)^{-3/2} \frac{ds}{2t}$$

$$= \frac{1}{1+x^2} - \frac{1}{2(1+x^2)} \int_0^{x^2} (1+s)^{-3/2}\,ds$$

$$= \frac{1}{1+x^2} - \frac{1}{2(1+x^2)} \left[\frac{(1+s)^{-3/2+1}}{-3/2+1} \right]_0^{x^2}$$

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$$= \frac{1}{1+x^2} - \frac{1}{2(1+x^2)}\left[\frac{(1+s)^{-1/2}}{-1/2}\right]_0^{x^2}$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[(1+s)^{-1/2}\right]_0^{x^2}$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[(1+x^2)^{-1/2} - 1\right]$$

$$= \frac{1}{1+x^2} + \frac{(1+x^2)^{-1/2}}{(1+x^2)} - \frac{1}{(1+x^2)}$$

$$= (1+x^2)^{-3/2}$$

R.H.S = L.H.S.

$\therefore$ $u(x)$ is also given I.E

4. S.T the function $u(x) = e^x\left(2x - \frac{2}{3}\right)$ is a solution of fredholm I.E

$$u(x) + 2\int_0^1 e^{x-t}\, u(t)\, dt = 2xe^x$$

Solution:

Given I.E is

$$u(x) + 2\int_0^1 e^{x-t}\, u(t)\, dt = 2xe^x \quad \to (1)$$

Also given that

$$u(x) = e^x\left(2x - \frac{2}{3}\right)$$

$$u(t) = e^t\left(2t - \frac{2}{3}\right)$$

L.H.S of (1).

$$L.H.S = u(x) + 2\int_0^1 e^{x-t} u(t)\,dt$$

$$= e^x(2x - 2/3) + 2\int_0^1 e^{x-t} e^t (2t - 2/3)\,dt$$

$$= e^x(2x - 2/3) + 2\int_0^1 e^x (2t - 2/3)\,dt$$

$$= e^x(2x - 2/3) + 2e^x \int_0^1 (2t - 2/3)\,dt$$

$$= e^x(2x - 2/3) + 2e^x \left[\frac{2t^2}{2} - \frac{2t}{3}\right]_0^1$$

$$= e^x(2x - 2/3) + 2e^x \left[t^2 - 2t/3\right]_0^1$$

$$= e^x(2x - 2/3) + 2e^x \left[(1 - 2/3) - (0 - 0)\right]$$

$$= e^x(2x - 2/3) + 2e^x (1/3)$$

$$= e^x 2x - 2/3 e^x + 2/3 e^x$$

$$= 2xe^x$$

$= R.H.S$ of (1).

$\therefore u(x)$ is also given I.E

5. S.T the function $u(x) = 3$ is the soln of the volterra I.E

$$\int_0^x (x-t)^2 u(t)\,dt = x^3$$

solution:

Given I.E is.

$$\int_0^x (x-t)^2 u(t)\,dt = x^3 \rightarrow (1)$$

Also given that

$u(x) = 3$

$u(t) = 3$.

L.H.S of ①.

$$L.H.S = \int_0^x (x-t)^2 u(t)\, dt.$$

$$= \int_0^x (x-t)^2 \; 3 \; dt$$

$$= 3 \int_0^x (x-t)^2 \, dt.$$

$$= 3 \left[\frac{(x-t)^3}{3(-1)} \right]_0^x$$

$$= 3 \left[\frac{(x-t)^3}{-3} \right]_0^x$$

$$= -\left[(x-t)^3 \right]_0^x$$

$$= -\left[(0 - \cancel{0}) - (x^3) \right].$$

$$= x^3$$

$= R.H.S.$

$\therefore u(x)$ is also given I.E

6. Show that the function $u(x) = 1-x$ is the solution of the integral Eqn $\int_0^x e^{x-t} u(t)\, dt = x$.

5m
(7)

solution:

Given I.E is

$$\int_0^x e^{x-t} u(t)\, dt = x. \longrightarrow ①.$$

Also given that

$u(x) = 1-x$.

$u(t) = 1-t$.

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L.H.S of ①.

$$L.H.S = \int_0^x e^{x-t}\, u(t)\, dt$$

$$= \int_0^x e^{x-t}\,(1-t)\, dt$$

$$= e^x \int_0^x e^{-t}\,(1-t)\, dt$$

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$u = (1-t)$ $\qquad dv = e^{-t}\,dt$

$u' = -1$ $\qquad v = -e^{-t}$

$u'' = 0.$ $\qquad v_1 = e^{-t}$

$$\Rightarrow \int u\,dv = e^x \left[-(1-t)e^{-t} + e^{-t} - 0\right]_0^x$$

$$= e^x \left[\left(-(1-x)e^{-x} + e^{-x}\right) - \left(-(1-0)1 + 1\right)\right]$$

$$= e^x \left[\left(-e^{-x} + xe^{-x} + e^{-x} + 1 - 1\right)\right]$$

$$= e^x \left[x e^{-x}\right]$$

$L.H.S = x$

$\therefore u(x)$ is also given I.E.

7. S.T the function $y(x) = 2-x$ is the soln of the I.E $\int_0^x e^{x-t}\, y(t)\, dt = e^x + x - 1$

proof:

Given I.E

$$\int_0^x e^{x-t}\, y(t)\, dt = e^x \cancel{+x-1} \rightarrow ①$$

Also given that

$y(x) = 2 - x$

$y(t) = 2 - t.$

L.H.S of ①.

$$L.H.S = \int_0^x e^{x-t} (2-t)\, dt.$$

$$= e^x \int_0^x e^{-t} (2-t)\, dt$$

By Bernoulli's formula.

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$u = (2-t)$ $dv = e^{-t}\, dt.$

$u' = -1$ $v = -e^{-t}$

$u'' = 0$ $v_1 = e^{-t}$

$$= e^x \left[-(2-t)e^{-t} + e^{-t} + 0 \right]_0^x$$

$$= e^x \left[\left(-(2-x)e^{-x} + e^{-x} \right) - \left(-(2-0)1 + 1 \right) \right]$$

$$= e^x \left[-2e^{-x} + xe^{-x} + e^{-x} + 2 - 1 \right]$$

$$= e^x \left[-2e^{-x} + xe^{-x} + 1 + e^{-x} \right].$$

$$= e^x \left[-e^{-x} + xe^{-x} + 1 \right]$$

$L.H.S = e^x + x - 1.$

$\therefore u(x)$ is also given I.E.

8. verify the function $u(x) = x - \frac{x^3}{6}$ is a soln of I.E

$$u(x) = x - \int_0^x \sinh(x-t)\, u(t)\, dt.$$

proof:

Given I.E

$$u(x) = x - \int_0^x \sinh(x-t)\, u(t)\, dt \rightarrow ①$$

Also given that

$$u(x) = x - \frac{x^3}{6}$$

$$u(t) = t - \frac{t^3}{6}$$

R.H.S of ①

$$R.H.S = x - \int_0^x \sinh(x-t)\, u(t)\, dt$$

$$= x - \int_0^x \sinh(x-t)\left(t - \frac{t^3}{6}\right) dt$$

By using Bernoulli's ~~fa~~ formula:

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$u = t - \frac{t^3}{6}$ $\qquad$ $dv = \sinh(x-t)\,dt$

$v = -\cosh(x-t)$

$u' = 1 - \frac{t^2}{2}$ $\qquad$ $v_1 = \sinh(x-t)$

$v_2 = -\cosh(x-t)$

$u'' = -t$ $\qquad$ $v_3 = \sinh(x-t)$

$u''' = -1$

$u^{iv} = 0$

$$= x - \left[-\left(t - \frac{t^3}{6}\right)\cosh(x-t) - \left(1 - \frac{t^2}{2}\right)\sinh(x-t) + (-t)(-\cosh(x-t)) - (-1)(\sinh(x-t)\right]_0^x$$

$$= x - \left[-\left(t - \frac{t^3}{6}\right)\cosh(x-t) - \left(1 - \frac{t^2}{2}\right)\sinh(x-t) + t(\cosh(x-t)) + (\sinh(x-t)\right]_0^x$$

$$= x - \left[\left(-\left(x - \frac{x^3}{6} \right)(1) - \left(1 - \frac{x^2}{2} \right)(0) + x + 0 \right) - \left(0 - 1(\sinh x) + 0 + \sinh(x) \right) \right]$$

$$= x - \left[-x + \frac{x^3}{6} - \cancel{0} + \cancel{\frac{0^2}{2}} + x \right]$$

$$= x - \left[\frac{x^3}{6} \right]$$

$$R.HS = x - \frac{x^3}{6}$$

$\therefore u(x)$ is a soln of I.E.

9. S.T the function $u(x) = \cos 2x$ is a soln of the I.E $u(x) = \cos x + 3\int_0^{\pi} K(x,t)u(t)dt$

where $K(x,t) = \begin{cases} \sin x \cos t & 0 \leq x \leq t \\ \cos x \sin t & t \leq x \leq \pi \end{cases}$

proof.

Given I.E

$$u(x) = \cos x + 3\int_0^{\pi} K(x,t)\, u(t)\, dt \quad \rightarrow (1).$$

where

$$K(x,t) = \begin{cases} \sin x \cos t & 0 \leq x \leq t \\ \cos x \sin t & t \leq x \leq \pi \end{cases}$$

$$K(t,x) = \begin{cases} \sin t \cos x & 0 \leq t \leq x \\ \cos t \sin x & x \leq t \leq \pi \end{cases}$$

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Also given that

$u(x) = \cos 2x$

$u(t) = \cos 2t$

R.H.S of (1)

$$R.H.S = \cos x + 3\int_0^x \sin t \cos x \cos 2t \, dt$$

$$+ 3\int_x^\pi \cos t \sin x \cos 2t \, dt$$

$$= \cos x + 3\cos x \int_0^x \cos 2t \sin t \, dt$$

$$+ 3\sin x \int_x^\pi \cos 2t \cos t \, dt$$

$$= \cos x + 3\cos x \left[\frac{1}{2}\left(\sin 3t - \sin t\right)\right]_0^x$$

$$+ 3\sin x \left[\frac{1}{2}\left(\cos 3t + \cos t\right)\right]_x^\pi$$

$$= \cos x + \frac{3\cos x}{2}\left[\sin 3t - \sin t\right]_0^x + \frac{3\sin x}{2}\left[\cos 3t + \cos t\right]_x^\pi$$

$$= \cos x + \frac{3\cos x}{2}\left[\frac{-\cos 3t}{3} + \cos t\right]_0^x + \frac{3\sin x}{2}\left[\frac{\sin 3t}{3} + \sin t\right]_x^\pi$$

$$= \cos x + \frac{3\cos x}{2}\left[\left(\frac{-\cos 3x}{3} + \cos x\right) - \left(\frac{-1}{3} + 1\right)\right]$$

$$+ \frac{3\sin x}{2}\left[(0+0) - \left(\frac{\sin 3x}{3} + \sin x\right)\right]$$

$$= \cos x - \frac{\cos 3x \cos x}{2} + \frac{3}{2}\cos^2 x - \cos x - \frac{\sin 3x \sin x}{2}$$

$$- \frac{3}{2}\sin^2 x$$

$$= \frac{3}{2}\left[\cos^2 x - \sin^2 x\right] - \frac{1}{2}\left[\cos 3x \cos x + \sin 3x \sin x\right]$$

$$= \frac{3}{2}\cos 2x - \frac{1}{2}\cos 2x$$

R.H.S $= \cos 2x$

$\therefore u(x)$ is a soln of I.E

10 S.T the function $u(x) = \sqrt{x}$ is the soln of the I.E

$$u(x) - \int_0^1 K(x,t)\,u(t)\,dt = \sqrt{x} + \frac{x}{15}\left(4x^{3/2} - 7\right)$$

where $K(x,t) = \begin{cases} x/2\,(2-t) & 0 \le x \le t \\ t/2\,(2-x) & t \le x \le 1 \end{cases}$

Solution:

Given I.E

$$u(x) - \int_0^1 K(x,t)\,u(t)\,dt = \sqrt{x} + \frac{x}{15}\left(4x^{3/2} - 7\right) \rightarrow (1)$$

where

$$K(x,t) = \begin{cases} x/2\,(2-t) & 0 \le x \le t \\ t/2\,(2-x) & t \le x \le 1 \end{cases}$$

$$K(t,x) = \begin{cases} t/2\,(2-x) & 0 \le t \le x \\ x/2\,(2-t) & x \le t \le 1 \end{cases}$$

Also given that

$u(x) = \sqrt{x}$

$u(t) = \sqrt{t}$

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L.H.S of (i)

$$L.H.S = u(x) - \int_0^1 K(x,t)\,u(t)\,dt$$

$$= \sqrt{x} - \int_0^x t/_2 (2-x) (\sqrt{t})\,dt$$

$$- \int_x^1 \frac{x}{2} (2-t) (\sqrt{t})\,dt$$

$$= \sqrt{x} - (2-x) \int_0^x \frac{t}{2} (\sqrt{t})\,dt$$

$$- \frac{x}{2} \int_x^1 (2-t) (\sqrt{t})\,dt$$

$$= \sqrt{x} - \frac{(2-x)}{2} \int_0^x t^{3/2}\,dt - \frac{x}{2} \int_x^1 (2\sqrt{t} - t^{3/2})\,dt$$

$$= \sqrt{x} - \left(\frac{2-x}{2}\right) \int_0^x t^{3/2}\,dt - \frac{x}{2} \int_x^1 (2\sqrt{t} - t^{3/2})\,dt$$

$$= \sqrt{x} - \left(\frac{2-x}{2}\right) \left[\frac{t^{5/2}}{5/2}\right]_0^x - \frac{x}{2} \left[\frac{2t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2}\right]_x^1$$

$$= \sqrt{x} - \left(\frac{2-x}{2}\right)\left(\frac{2x^{5/2}}{5}\right) - \frac{x}{2}\left[\left(\frac{4}{3} - \frac{2}{5}\right)\right.$$

$$\left. - \frac{4}{3} x^{3/2} - \frac{2}{5} x^{5/2}\right]$$

$$= \sqrt{x} - \frac{2}{5} x^{5/2} + x^{7/2} - \frac{2x}{3} + \frac{x}{5} + \frac{2x^{5/2}}{3} - \frac{x^{7/2}}{5}$$

$$= \sqrt{x} - \frac{7x}{15} + \frac{4}{15} x^{5/2}$$

$$= \sqrt{x} + \frac{x}{15} (4x^{3/2} - 7)$$

$= R.H.S$

$u(x)$ is the solution of I-E

11) S.T the function $u(x) = 1/\pi\sqrt{x}$ is the soln of the I.E $\int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt = 1$

Solution:

Given I.E

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt = 1 \longrightarrow (1)$$

Also given that

$$u(x) = \frac{1}{\pi\sqrt{x}}$$

$$u(t) = \frac{1}{\pi\sqrt{t}}$$

L.H.S of (1).

$$L.H.S = \int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt$$

$$= \int_0^x \frac{1/\pi\sqrt{t}}{\sqrt{x-t}}\,dt$$

$$= \int_0^x \frac{1}{\pi\sqrt{t}} \frac{1}{\sqrt{x-t}}\,dt$$

$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{xt-t^2}}\,dt$$

$\left(t-\frac{x}{2}\right)^2 = t^2 - xt + \frac{x^2}{4}$

$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{(x/2)^2 - (t-x/2)^2}}\,dt$$

$-\left(t-\frac{x}{2}\right)^2 = -t^2 + xt - \frac{x^2}{4}$

$\frac{x^2}{4} - \left(t-\frac{x}{2}\right)^2 = xt - t^2$

$$= \frac{1}{\pi}\left[\sin^{-1}\left(\frac{t-x/2}{x/2}\right)\right]_0^x$$

$$= \frac{1}{\pi}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= \frac{1}{\pi}\left[\sin^{-1}(1) + \sin^{-1}(1)\right]$$

$$= 1/\pi\left(\pi/2 + \pi/2\right)$$

$= \frac{1}{\pi} (\pi)$

$= 1$

$= R.H.S$

$\therefore u(x)$ is a solution of I.E

12. S.T the function $u(x) = x e^{x}$ is the solution of the volterra I.E

$$u(x) = \sin x + 2\int_0^x \cos(x-t)\, u(t)\, dt.$$

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solution

Given I.E

$$u(x) = \sin x + 2\int_0^x \cos(x-t)\, u(t)\, dt \quad \to (1)$$

Also given that

$u(x) = x e^{x}$

$u(t) = t e^{t}$

R.H.S of (1).

$$R.H.S = \sin x + 2\int_0^x \cos(x-t)\,(t e^{t})\, dt$$

$\int e^{ax}\cos(bx+c)\,dx$

$= \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right].$

$\int u\,dv = uv - \int v\,du$

$u = t.$ $\quad dv = e^{t}\cos(x-t)\,dt$

$du = 1\,dt$ $\quad v = \frac{e^{t}}{2}\left[\cos(x-t) - \sin(x-t)\right]$

$$= \sin x + 2\left\{\left[\frac{t e^{t}}{2}\left(\cos(x-t) - \sin(x-t)\right)\right]_0^x - \int_0^x \frac{e^{t}}{2}\left(\cos(x-t) - \sin(x-t)\right)dt\right.$$

$$= \sin x + \left[x e^{x} - 0\right] - \int_0^x \frac{e^{t}\cos(x-t)\,dt}{\cancel{2}} + \int_0^x \frac{e^{t}\sin(x-t)\,dt}{\cancel{2}}$$

$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$

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$= \sin x + xe^x - \left[\frac{e^t}{2}\left(\cos(x-t) - \sin(x-t)\right)\right]_0^x$

$+ \left[\frac{e^t}{2}\left(\sin(x-t) + \cos(x-t)\right)\right]_0^x$

$= \sin x + xe^x - \left(\frac{e^x}{2} - \frac{1}{2}(\cos x - \sin x)\right)$

$+ \left(\frac{e^x}{2} - \frac{1}{2}\left(\sin x + \cos x\right)\right)$

$= \sin x + xe^x - \frac{e^x}{2} - \frac{e^x}{2} + \frac{\cos x}{2} - \frac{\sin x}{2} + \frac{e^x}{2}$

$- \frac{\sin x}{2} - \frac{\cos x}{2}$

$= \sin x + xe^x - \sin x$

$= xe^x$

$\therefore u(x)$ is a solution of I.E.

Conversion of differential eqn into Integral eqn.

Note :

Conversion of multiple integral into single ordinary integral is given by

$$\int_a^x f(t)\,dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)\,dt$$

proof :

Consider the Linear D.E of order n

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$$\frac{d^n y}{dx^n} + c_1(x)\frac{d^{n-1}y}{dx^{n-1}} + c_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + c_n(x)\,y = \phi(x) \longrightarrow ①$$

with continuous coefficient $c_i(x)$, $i = 1, 2, 3, \ldots n$ and with initial condition $y(a) = a_0$, $y'(a) = a_1$, $y''(a) = a_2 \ldots y^{n-1}(a) = a_{n-1} \longrightarrow ②$.

Let us introduces an unknown function $u(x)$, such that

$$\frac{d^n y}{dx^n} = u(x) \longrightarrow ③.$$

Integrating ③ on both sides with respect to 'x' from a to x, we get

$$\left[\frac{d^{n-1}y}{dx^{n-1}}\right]_a^x = \int_a^x u(x)\,dx$$

$$\frac{d^{n-1}y}{dx^{n-1}} - a_{n-1} = \int_a^x u(x)\,dx$$

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_a^x u(x)\,dx + a_{n-1} \longrightarrow ④,$$

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_a^x u(t)\,dt + a_{n-1} \longrightarrow ⑤$$

Again integrating ean ④ with respect to 'x' from a to x.

$$\left[\frac{d^{n-2}y}{dx^{n-2}}\right]_a^x = \int_a^x u(x)\,dx^2 + \int_a^x a_{n-1}\,dx$$

$$\frac{d^{n-2}y}{dx^{n-2}} - a_{n-2} = \int_a^x u(x)\, dx^2 + a_{n-1}(x-a)$$

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_a^x u(x)\, dx^2 + a_{n-1}(x-a) + a_{n-2} \quad \mapsto (6).$$

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_a^x u(t)\, dt^2 + a_{n-1}(x-a) + a_{n-2}$$

$$= \int_a^x (x-t)\, u(t)\, dt + a_{n-1}(x-a) + a_{n-2} \quad \mapsto (7).$$

Again integrating eqn (6) with respect to 'x' from a to x.

$$\left[\frac{d^{n-3}y}{dx^{n-3}}\right]_a^x = \int_a^x u(x)\, dx^3 + a_{n-1}\int_a^x (x-a)\,dx + \int_a^x a_{n-2}\, dx.$$

$$\frac{d^{n-3}y}{dx^{n-3}} - a_{n-3} = \int_a^x u(x)\, dx^3 + a_{n-1}\frac{(x-a)^2}{2} + a_{n-2}(x-a)$$

$$\frac{d^{n-3}y}{dx^{n-3}} = \int_a^x u(x)\, dx^3 + a_{n-1}\frac{(x-a)^2}{2!} + a_{n-2}(x-a) + a_{n-3} \quad \mapsto (8).$$

$$\frac{d^{n-3}y}{dx^{n-3}} = \int_a^x u(t)\, dt^3 + a_{n-1}\frac{(x-a)^2}{2!} + a_{n-2}(x-a) + a_{n-3} \quad \mapsto (9).$$

$$= \int_a^x \frac{(x-t)^2}{2!} u(t)\, dt + a_{n-1}\frac{(x-a)^2}{2!} + a_{n-2}(x-a) + a_{n-3} \quad \mapsto (9)$$

III'ly, we get

$$\frac{dy}{dx} = \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} u(t)\, dt + (a_{n-1})\frac{(x-a)^{n-2}}{(n-2)!} + (a_{n-2})\frac{(x-a)^{n-3}}{(n-3)!} + \dots + a_2(x-a) + a_1 \quad \mapsto (10).$$

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Again, Integrating

$$y = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} u(t)\,dt + (a_{n-1})\frac{(x-a)^{n-1}}{(n-1)!} + (a_{n-2})\frac{(x-a)^{n-2}}{(n-2)!}$$

$$+ \cdots + a_1 (x-a) + a_0 \rightarrow (11)$$

Now Substitue eqn (3), (5), (7), (9), (10), (11) in (1), we get.

$$\phi(x) = u(x) + a_{n-1} C_1(x) + \{a_{n-2} + (x-a) a_{n-1}\} C_2(x)$$

$$+ \cdots + \left\{ a_0 + a_1 (x-a) + \cdots + a_{n-1} \frac{(x-a)^{n-1}}{(n-1)!} \right\} C_n(x)$$

$$+ \int_a^x \left[C_1(x) + \frac{(x-t)}{1!} C_2(x) + \frac{(x-t)^2}{2!} C_3(x) + \cdots + \frac{(x-t)^{n-1}}{(n-1)!} C_n(x) \right] u(t)\,dt$$

let $\phi(x) - g(x) = f(x)$

$$\phi(x) = u(x) + g(x) - \int_a^b K(x,t) u(t)\,dt.$$

let $\phi(x) - g(x) = f(x)$

$$f(x) + \int_a^b K(x,t) u(t)\,dt = u(x)$$

which is the required I.E.

Problem

1. Form the I.E corresponding to the D.E $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ with initial

Condition $y(0) = 0$, $y'(0) = -1$

Solution:

Given D.E.

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$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \longrightarrow (1).$$

Let consider $\frac{d^2y}{dx^2} = u(x) \longrightarrow (2).$

Integrating (2) w.r.to x from 0 to x.

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx.$$

$$\frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} + 1 = \int_0^x u(x)\,dx.$$

$$\frac{dy}{dx} = \int_0^x u(x)\,dx - 1 \longrightarrow (3)$$

$$\frac{dy}{dx} = \int_0^x u(t)\,dt - 1 \longrightarrow (4).$$

Again $\int$ing (3) w.r.to x from 0 to x.

$$[y]_0^x = \int_0^x u(x)\,dx^2 - [x]_0^x$$

$$y(x) - y(0) = \int_0^x u(x)\,dx^2 - x$$

$$y(x) = \int_0^x u(x)\,dx^2 - x$$

$$= \int_0^x u(t)\,dt^2 - x$$

$$y(x) = \int_0^x (x-t)\,u(t)\,dt - x \rightarrow (5)$$

Sub eqn (3), (4), (5) in (1).

$$(1) \Rightarrow u(x) - 5\left[\int_0^x u(t)dt - 1\right] + 6\left[\int_0^x (x-t)u(t)dt - x\right] = 0$$

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$$u(x) + \int_0^x \left[-5 + 6(x-t)\right] u(t)\,dt + 5 - 6x = 0$$

$$u(x) = 6x - 5 - \int_0^x [-5 + 6(x-t)] u(t)\,dt$$

$$u(x) = 6x - 5 + \int_0^x [5 - 6(x-t)] u(t)\,dt$$

which is required I.E

2. Form the I.E corresponding to the D.E $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y = 0$ with initial Condition $y(0) = 1$, $y'(0) = 0$.

Solution :

Given D.E

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y = 0 \rightarrow (1)$$

let us consider $\frac{d^2y}{dx^2} = u(x) \rightarrow (2)$

Integrating (2) w.r.to 'x' from 0 to x

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - 0 = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} = \int_0^x u(x)\,dx \rightarrow (3)$$

$$\frac{dy}{dx} = \int_0^x u(t)\,dt \rightarrow (4)$$

Again $\int$ing ③ w.r to x from 0 to x.

$$[y]_0^x = \int_0^x u(x)\,dx^2$$

$$y(x) - y(0) = \int_0^x u(x)\,dx^2$$

$$y(x) - 1 = \int_0^x u(x)\,dx^2$$

$$y(x) = \int_0^x u(x)\,dx^2 + 1$$

$$y(x) = \int_0^x u(t)\,dt^2 + 1$$

$$y(x) = \int_0^x (x-t)u(t)\,dt + 1 \longrightarrow ⑤$$

Sub eqn ③, ④, ⑤ in ①

$$① \Rightarrow u(x) - 2x\left[\int_0^x u(t)\,dt\right] - 3\left[\int_0^x (x-t)u(t)\,dt + 1\right] = 0.$$

$$u(x) = 3 + \int_0^x \left[2x + 3(x-t)\right] u(t)\,dt$$

$$u(x) = 3 + \int_0^x \left[5x - 3t\right] u(t)\,dt.$$

which is required I.E

3 Form the I.E corresponding to the D.E $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ with initial Condition $y(0) = 1$, $y'(0) = 0$.

proof:

Given I.E

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \longrightarrow ①$$

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Let us consider $\frac{d^2y}{dx^2} = u(x) \rightarrow$ (2)

Integrating (2) w.r.to 'x' from 0 to x

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx}(x) - y'(0) = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - 0 = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} = \int_0^x u(x)\,dx \rightarrow (3)$$

$$\frac{dy}{dx} = \int_0^x u(t)\,dt \rightarrow (4)$$

Again integrating (3) w.r.to 'x' from 0 to x.

$$[y]_0^x = \int_0^x u(x)\,dx^2$$

$$y(x) - y(0) = \int_0^x u(x)\,dx^2$$

$$y(x) - 1 = \int_0^x u(x)\,dx^2$$

$$y(x) = \int_0^x u(x)\,dx^2 + 1$$

$$y(x) = \int_0^x u(t)\,dt^2 + 1 \rightarrow (5)$$

$$y(x) = \int_0^x (x-t)\,u(t)\,dt + 1 \rightarrow (5)$$

Sub eqn (3), (4), (5) in eqn (1), we get.

$$u(x) + x\left[\int_0^x u(t)\,dt\right] + \left[\int_0^x (x-t)\,u(t)\,dt + 1\right] =$$

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$$u(x) + \int_0^x \left[x + (x-t) \right] u(t)\,dt + 1$$

$$u(x) = -1 - \int_0^x \left[x+(x-t) \right] u(t)\,dt$$

which is required I.E

4. Form the I.E Corresponding to the D.E $\frac{d^2y}{dx^2} + xy = 1$ with initial condition

$y(0) = 0, \quad y'(0) = 0$

solution :

Given I.D

$$\frac{d^2y}{dx^2} + xy - 1 = 0 \rightarrow (1)$$

Let us consider $\frac{d^2y}{dx^2} = u(x) \rightarrow (2)$

Integrating (2) w.r.to 'x' from 0 to x.

$$\left[\frac{dy}{dx} \right]_0^x = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} = \int_0^x u(x)\,dx \rightarrow (3)$$

$$\frac{dy}{dx} = \int_0^x u(t)\,dt \rightarrow (4)$$

Again integrating (3) w.r.to 'x' from 0 to 'x'

$$[y]_0^x = \int_0^x u(x)\,dx^2$$

$$y(x) - y(0) = \int_0^x u(x)\,dx^2$$

$$y(x) - 0 = \int_0^x u(x)\,dx^2 \Rightarrow y(x) = \int_0^x u(x)\,dx^2$$

$$y(x) = \int_0^x u(t)\,dt^2 \longrightarrow (5)$$

Sub eqn (3), (4), (5) in (1) we get.

$$u(x) + x\left[\int_0^x (x-t)\,u(t)\,dt\right] - 1 = 0$$

$$u(x) + x\int_0^x (x-t)\,u(t)\,dt - 1 = 0$$

$$u(x) = 1 - x\int_0^x (x-t)\cdot u(t)\,dt$$

$$u(x) = 1 - \int_0^x (x^2 - xt)\,u(t)\,dt$$

$\therefore$ $u(x)$ is required I.E

5. If $y''(x) = f(x)$ and satisfies the initial condition $y(0) = y_0$ and $y'(0) = y_0'$

S.T $y = y_0 + x y_0' + \int_0^x (x-t) f(t)\,dt$

solution:

Given that

$$y''(x) = f(x) \longrightarrow (1)$$

Sing (1) w.r.to x from 0 to x

$$[y'(x)]_0^x = \int_0^x f(x)\,dx$$

$$y'(x) - y'(0) = \int_0^x f(x)\,dx$$

$$y'(x) = y_0' + \int_0^x f(x)\,dx.$$

Again $\int$ing w.r to x from 0 to x

$$[y]_0^x = \int_0^x y_0'\,dx + \int_0^x f(x)\,dx^2$$

$$y(x) - y(0) = y_0'[x]_0^x + \int_0^x f(t)\,dt^2$$

$$y(x) = y_0 + y_0' x + \int_0^x (x-t) f(t)\,dt$$

6. S.T the linear D.E of 2^{nd} order $\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x)$ with initial condition $y(0) = c_0$, $y'(0) = c_1$ are transforming to non-homogeneous volterra I.E of 2^{nd} kind.

Solution:

Given that

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x) \rightarrow (1)$$

Assume that $\frac{d^2y}{dx^2} = u(x) \rightarrow (2)$.

$\int$ing (2) w.r to x from 0 to x.

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx.$$

$$\frac{dy}{dx} - c_1 = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} = c_1 + \int_0^x u(x)\,dx \rightarrow (3)$$

Again $\int$ing (3) w.r.to 'x' from 0 to x.

$$[y]_0^x = c_1 x + \int_0^x u(x)\,dx^2$$

$$y(x) - y(0) = c_1 x + \int_0^x u(x)\,dx^2$$

$$y(x) - c_0 = c_1 x + \int_0^x u(x)\,dx^2$$

$$y(x) = c_0 + c_1 x + \int_0^x u(x)\,dx^2$$

$$y(x) = c_0 + c_1 x + \int_0^x u(t)\,dt^2 \rightarrow (4)$$

$$y(x) = c_0 + c_1 x + \int_0^x (x-t)u(t)\,dt \rightarrow (5)$$

Sub (2), (5), (4) in (1).

$$u(x) + a_1(x)\left[c_1 + \int_0^x u(t)\,dt\right] + a_2(x)\left[c_0 + c_1 x + \int_0^x (x-t)u(t)\,dt\right] = f(x)$$

$$u(x) + c_1 a_1(x) + c_0 a_2(x) + c_1 x a_2(x) + \int_0^x \left[a_1(x) + a_2(x)(x-t)\right]u(t)\,dt = f(x)$$

$$u(x) = f(x) - c_1 a_1(x) - c_0 a_2(x) - c_1 x\, a_2(x)$$

$$- \int_0^x \left[a_1(x) + a_2(x)(x-t)\right] u(t)\, dt$$

$\therefore$ which is required non-homogeneous volterra I.E.

T. Form the I.E corresponding to the D.E $\frac{d^2y}{dx^2} + y = \cos x$ with initial condition $y(0) = 0$ and $y'(0) = 1$

solution:

Given that

$$\frac{d^2y}{dx^2} + y = \cos x \longrightarrow (1).$$

Assume that $\frac{d^2y}{dx^2} = u(x) \longrightarrow (2).$

$\int$ing (2) w.r.to 'x' from 0 to 'x'; we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\, dx$$

$$\frac{dy}{dx} - y'(0) = \int_0^x u(x)\, dx$$

$$\frac{dy}{dx} - 1 = \int_0^x u(x)\, dx$$

$$\frac{dy}{dx} = \int_0^x u(x)\, dx + 1 \longrightarrow (3)$$

Again (3) w.r.to 'x' from 0 to x we get

$$[y]_0^x = \int_0^x u(x)\, dx^2 + x$$

$$y(x) - y(0) = \int_0^x u(x)\, dx^2 + x$$

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$$y(x) = x + \int_0^x u(x)\,dx^2$$

$$y(x) = x + \int_0^x u(t)\,dt^2 \rightarrow (4)$$

Sub (2), (3), (4) in eqn (1)

$$u(x) + x + \int_0^x (x-t)u(t)\,dt = \cos x$$

$$u(x) = \cos x - x - \int_0^x (x-t)\,u(t)\,dt.$$

$$u(x) = \cos x - x + \int_0^x (t-x)\,u(t)\,dt.$$

$\therefore u(x)$ is required I.E

8 Form the I.E corresponding to the D.E $y'' + y = 0$ with initial condition $y(0) = 0$, $y'(0) = 1$

solution

Given that

$$\frac{d^2y}{dx^2} + y = 0 \rightarrow (1)$$

Assume that $\frac{d^2y}{dx^2} = u(x) \rightarrow (2)$

$\int$ing (2) w.r.to 'x' from 0 to x

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - 1 = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} = \int_0^x u(x)\,dx + 1 \rightarrow (3)$$

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Again $\int$ing ③ w.r.to x from 0 to x.

$$[y]_0^x = x + \int_0^x u(x)\,dx^2$$

$$y(x) - y(0) = x + \int_0^x u(x)\,dx^2$$

$$y(x) = x + \int_0^x u(x)\,dx^2$$

$$y(x) = x + \int_0^x u(t)\,dt^2 \rightarrow ④.$$

Sub ②, ③, ④ in ①.

$$u(x) + x + \int_0^x (x-t)u(t)\,dt = 0.$$

$$u(x) = -x - \int_0^x (x-t)\,u(t)\,dt$$

$$u(x) = -x + \int_0^x (t-x)\,u(t)\,dt$$

$\therefore u(x)$ is required I.E

Q Form the I.E corresponding to the D.E. $y'' + (1+x^2)y = \cos x$ with initial condition $y(0) = 0$ and $y'(0) = 2$.

solution:

Given that

$$\frac{d^2y}{dx^2} + (1+x^2)y = \cos x \rightarrow ①$$

Assume that $\frac{d^2y}{dx^2} = u(x) \rightarrow ②$

$\int$ing ② w.r.to x from 0 to x.

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

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$$\frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} - 2 = \int_0^x u(x)\,dx$$

$$\frac{dy}{dx} = 2 + \int_0^x u(x)\,dx \rightarrow (5)$$

Againting (3) w.r to 'x' from 0 to x

$$[y]_0^x = 2x + \int_0^x u(x)\,dx^2$$

$$y(x) - y(0) = 2x + \int_0^x u(x)\,dx^2$$

$$y(x) = 2x + \int_0^x u(t)\,dt^2 \rightarrow (4)$$

Sub eqn (2), (5), (4) in (1).

$$u(x) + (1+x^2)\left[2x + \int_0^x (x-t)\,u(t)\,dt\right] = \cos x$$

$$u(x) + 2x(1+x^2) + (1+x^2)\int_0^x (x-t)\,u(t)\,dt = \cos x$$

$$u(x) + 2x + 2x^3 + (1+x^2)\int_0^x (x-t)\,u(t)\,dt = \cos x$$

$$u(x) = \cos x - 2x - 2x^3 \mp \int_0^x (1+x^2)(x-t)\,u(t)\,dt$$

$$u(x) = \cos x - 2x(1+x^2) - \int_0^x \left[x^3 - x - x^2 t \mp t\right] u(t)\,d$$

$$u(x) = \cos x - 2x(1+x^2) + \int_0^x (x^3 - x - x^2 t - t)\,u(t)$$

$\therefore$ $u(x)$ is required I.E

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10. Form the I.E Corresponding to the D.E $\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} + e^x y = x$ with initial condition $y(0)=1$, $y'(0)=-1$

solution :

Given that

$$\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} + e^x y = x \longrightarrow (1)$$

Assume that $\frac{d^2y}{dx^2} = u(x) \longrightarrow (2)$

Sing (2) w.r to x from 0 to x we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$y'(x) - y'(0) = \int_0^x u(x)\,dx$$

$$y'(x) + 1 = \int_0^x u(x)\,dx$$

$$y'(x) = \int_0^x u(x)\,dx - 1 \longrightarrow (3)$$

$$y'(x) = \int_0^x u(t)\,dt$$

Again ding (3) w.r to x from 0 to x

$$[y(x)]_0^x = \int_0^x u(x)\,dx^2 - x$$

$$y(x) - y(0) = x + \int_0^x u(x)\,dx^2 - x$$

$$y(x) = x + \int_0^x u(x)\,dx^2 + 1 - x$$

$$y(x) = x + \int_0^x (x-t)u(t)\,dt + 1 - x \mapsto (4)$$

Sub ean (2), (3), (4) in (1) we get

$$u(x) - \sin x\left[\int_0^x u(t)\,dt - 1\right] + e^x\left[-x + \int_0^x (x-t)u(t)\,dt\right] = x$$

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$$u(x) - \sin x \left[\int_0^x u(t)\, dt - 1 \right] + e^x \left[x + \int_0^x (x-t) u(t)\, dt - x + 1 \right] = x$$

$$u(x) - \sin x + xe^x + \left\{ \int_0^x \left[e^x (x-t) \right] u(t)\, dt = x \right.$$

$$u(x) = \sin x - x - xe^x - \int_0^x \left[e^x (x-t) \right] u(t)\, dt$$

$$u(x) = \sin x - x(1+e^x) - \int_0^x \left[e^x (x-t) \right] u(t)\, dt$$

$\therefore u(x)$ is required I.E

$$u(x) - \sin x \int_0^x u(t)\, dt + \sin x + e^x \int_0^x (x-t) u(t)\, dt - xe^x + e^x = x$$

$$u(x) + \sin x - e^x(x-1) - \int_0^x \left[\sin x - e^x (x-t) \right] u(t)\, dt = x$$

$$u(x) = x - \sin x + xe^x - e^x + \int_0^x \left[\sin x - e^x (x-t) \right] u(t)\, dt$$

$$u(x) = x - \sin x + e^x(x-1) + \int_0^x \left[\sin x - e^x (x-t) \right] u(t)\, dt$$

$\therefore u(x)$ is required I.E

1. Convert $y''(x) - \sin x\, y'(x) + e^x y(x) = x$ with initial condition $y(0) = 1$, $y'(0) = -1$ to a volterra I.E of second kind. Conversely the original D.E with initial condition from the I.E of obtain

proof:

Given that

$$y''(x) - \sin x\, y'(x) + e^x y(x) = x \rightarrow (1)$$

$$y''(x) = \sin x \, y'(x) - e^x y(x) + x \rightarrow (2)$$

$\int$ing (2) w.r to x from 0 to x.

$$[y'(x)]_0^x = \int_0^x \sin x \, y'(x) dx - \int_0^x e^x y(x) dx + \frac{x^2}{2}$$

$$y'(x) - y'(0) = \int_0^x \sin x \, y'(x) dx - \int_0^x e^x y(x) dx + \frac{x^2}{2}$$

$u = \sin x \qquad dv = y'(x) dx$

$du = \cos x \, dx \qquad v = y(x)$

$$y'(x) + 1 = \frac{x^2}{2} + [\sin x \, y(x)]_0^x - \int_0^x \cos x \, y(x) dx - \int_0^x e^x y(x) dx$$

$$y'(x) = \frac{x^2}{2} - 1 + \sin x \, y(x) - \int_0^x [\cos x + e^x] y(x) dx$$

Again $\int$ing w.r to x from 0 to x.

$$[y(x)]_0^x = \frac{x^3}{6} - x + \int_0^x \sin x \, y(x) dx - \int_0^x [\cos x + e^x] y(x) dx^2$$

$$y(x) - 1 = \frac{x^3}{6} - x + \int_0^x \sin x \, y(x) dx - \int_0^x [\cos x + e^x] y(x) dx^2$$

$$y(x) = \frac{x^3}{6} - x + 1 + \int_0^x \sin t \, y(t) dt - \int_0^x [\cos t + e^t] y(t) dt^2$$

$$y(x) = \frac{x^3}{6} - x + 1 + \int_0^x \sin t \, y(t) dt - \int_0^x (x-t)(\cos t + e^t) y(t) \, dt$$

$$y(x)=\frac{x^3}{6}-x+1+\int_0^x\left[\sin t-(x-t)(\cos t+e^t)\right]y(t)\,dt \to (5)$$

which is the required I.E.

Conversely,

Let

$$y(x)=\frac{x^3}{6}-x+1+\int_0^x\left[\sin t-(x-t)(\cos t+e^t)\right]y(t)\,dt \to (4)$$

Diff (4) w.r.to x

$$y'(x)=\frac{x^2}{2}-1+0+\frac{d}{dx}\int_0^x\left[\sin t-(x-t)(\cos t+e^t)\right]y(t)\,dt$$

$$y'(x)=\frac{x^2}{2}-1+\int_0^x\frac{\partial}{\partial x}\left[\sin t-(x-t)(\cos t+e^t)\right]y(t)\,dt+\left[\sin x-(x-x)(\cos x+e^x)\right]y(x)\frac{dx}{dx}$$

$$=\frac{x^2}{2}-1+\int_0^x\left[0-(\cos t+e^t)(1-0)\right]y(t)\,dt+\sin x\,y(x)$$

$$y'(x)=\frac{x^2}{2}-1-\int_0^x(\cos t+e^t)y(t)\,dt+\sin x\,y(x)$$

$$y'(x)=\frac{x^2}{2}-1+\sin x\,y(x)-\int_0^x(\cos t+e^t)y(t)\,dt \to (5)$$

Again Diff w.r.to x

$$y''(x)=x-0+\sin x\,y'(x)+y(x)\cos x-\frac{d}{dx}\int_0^x(\cos t+e^t)y(t)\,dt$$

$$y''(x)=x+\sin x\,y'(x)+\cos x\,y(x)-\int_0^x\frac{\partial}{\partial x}(\cos t+e^t)y(t)\,dt-(\cos x+e^x)y(x)\frac{dx}{dx}$$

$$y''(x)=x+\sin x\,y'(x)-e^x y(x)$$

$$y''(x)-\sin x\,y'(x)+e^x y(x)=x.$$

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put $x=0$ in (4)

$y(0) = 0 - 0 + 1 + 0$

$y(0) = 1$

put $x=0$ in (5)

$y'(0) = 0 - 1 + 0 - 0$

$y'(0) = -1$

2. Convert $y''(x) - 3y'(x) + 2y(x) = 4\sin x$ with initial condition $y(0) = 1$, $y'(0) = -2$ to a volterra I.E of second kind, conversely the original D.E with initial condition from the I.E obtain.

Solution :

Given that

$$y''(x) - 3y'(x) + 2y(x) = 4\sin x \rightarrow (1)$$

$$y''(x) = 3y'(x) - 2y(x) + 4\sin x \rightarrow (2)$$

$\int$ing (2) w.r.to 'x' from '0' to 'x' we get

$$[y'(x)]_0^x = 3\int_0^x y'(x)\,dx - 2\int_0^x y(x)\,dx + 4\int_0^x \sin x\,dx$$

$$y'(x) - y'(0) = 3[y(x)]_0^x - 2\int_0^x y(x)\,dx + 4\int_0^x \sin x\,dx$$

$$y'(x) + 2 = 3[y(x) - y(0)] - 2\int_0^x y(x)\,dx + 4[-\cos x]_0^x$$

$$y'(x) = 3y(x) - 3 - 2\int_0^x y(x)\,dx - 4\cos x + 4 - 2$$

$$y'(x) = 3y(x) - 4\cos x - 1 - 2\int_0^x y(x)\,dx$$

Again $\int$ing w.r to x from 0 to x.

$$[y(x)]_0^x = 3\int_0^x y(x)\,dx - x - 4\int_0^x \cos x\,dx - 2\int_0^x y(x)\,dx^2$$

$$y(x) - y(0) = 3\int_0^x y(x)\,dx - x - 4\sin x - 2\int_0^x y(x)\,dx^2$$

$$y(x) = 1 - x - 4\sin x + 3\int_0^x y(x)\,dx - 2\int_0^x y(x)\,dx^2.$$

$$y(x) = 1 - x - 4\sin x + 3\int_0^x y(t)\,dt - 2\int_0^x (x-t)\,y(t)\,dt$$

$$y(x) = 1 - x - 4\sin x + \int_0^x [3 - 2(x-t)]\,y(t)\,dt.$$

$$y(x) = 1 - x - 4\sin x + \int_0^x [3 - 2x + 2t]\,y(t)\,dt. \quad \to (5)$$

$\therefore$ which is Required I.E.

Conversely.

Let

$$y(x) = 1 - x - 4\sin x + \int_0^x [3 - 2x + 2t]\,y(t)\,dt \quad \to (4)$$

Diff (4) w.r to x, we get.

$$y'(x) = 0 - 1 - 4\cos x + \frac{d}{dx}\int_0^x [3 - 2x + 2t]\,y(t)\,dt$$
$$+ [3 - 2x + 2x]\,y(x).1 - 0.$$

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$$y'(x) = -1 - 4\cos x + \int_0^x -2y(t)\,dt + 3y(x)$$

$$y'(x) = -1 - 4\cos x + 3y(x) - 2\int_0^x y(t)\,dt \rightarrow (5)$$

$$y''(x) = 0 + 4\sin x + 3y'(x) - 2\frac{d}{dx}\int_0^x y(t)\,dt$$

$$= 4\sin x + 3y'(x) - 2\int_0^x \frac{\partial}{\partial x} y(t)\,dt - 2[y(x)\cdot 1]$$

$\to 0$.

$$y''(x) = 4\sin x + 3y'(x) - 2y(x)$$

$$y''(x) - 3y'(x) + 2y(x) = 4\sin x.$$

put $x = 0$ in eqn (3).

$y(x) = 1 - 0 - 0$

$y(0) = 1$

$x = 0$ in eqn (5).

$y'(0) = -1 - 4 + 3(y(0))$

$= -1 - 4 + 3(1)$

$y'(0) = -2$

3. convert $y'' - 2xy' - 3y = 0$ with initial condition $y(0) = 1$ $y'(0) = 0$ to a volterra I.E conversely the original D.E with initial condition from the I.E obtain.

solution

Given that

$$y'' - 2xy' - 3y = 0 \longrightarrow (1).$$

$$y''(x) = 2xy' + 3y \longrightarrow (2).$$

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$\int$ing ③ w.r to x from 0 to x.

$$[y'(x)]_0^x = 2 \int_0^x x y'(x)\, dx + \int_0^x 3y\, dx.$$

$$y'(x) - y'(0) = 2 \int_0^x x y'(x)\, dx + 3 \int_0^x y\, dx.$$

$$y'(x) = 2\left[[x\, y(x)]_0^x - \int_0^x y(x)\, dx \right] + 3\int_0^x y(x)\, dx$$

$u = x$, $du = 1$, $dv = y'(x)dx$, $v = y(x)dx$

$$y'(x) = 2x\, y(x) - 2\int_0^x y(x)\, dx + 3\int_0^x y(x)\, dx.$$

$$y'(x) = 2x y(x) + \int_0^x y(x)\, dx.$$

Again $\int$ing w.r to x from 0 to x.

$$[y(x)]_0^x = 2\int_0^x x\, y(x)\, dx + \int_0^x y(x)\, dx^2$$

$$y(x) - y(0) = 2\int_0^x x\, y(x)\, dx + \int_0^x y(x)\, dx^2$$

$$y(x) - 1 = 2\int_0^x t\, y(t)\, dt + \int_0^x y(t)\, dt^2$$

$$y(x) - 1 = 2\int_0^x t\, y(t)\, dt + \int_0^x (x - t)\, y(t)\, dt$$

$$y(x) = 1 + \int_0^x [2t + x - t]\, y(t)\, dt$$

$$y(x) = 1 + \int_0^x [x + t]\, y(t)\, dt. \longrightarrow ⑤$$

which is Required I.E

conversely,

let

$$y(x) = \int_0^x [x+t]\, y(t)\, dt \to (4).$$

Diff (4) w.r.to x we get.

$$y'(x) = \frac{d}{dx}\int_0^x [x+t]\, y(t)\, dt.$$

$$y'(x) = \int_0^x \frac{\partial}{\partial x}[x+t]\, y(t)\, dt + [x+x]\, y(x).1 - 0.$$

$$y'(x) = \int_0^x y(t)\, dt + 2xy(x)$$

$$y'(x) = 2xy(x) + \int_0^x y(t)\, dt \to (5)$$

$uv = uv' + vu'$

$u = 2x, \quad v = y(x)$

Again Diff (5)

$$y''(x) = [2x\, y'(x) + y(x)\, 2] + \frac{d}{dx}\int_0^x y(t)\, dt.$$

$$y''(x) = [2xy'(x) + 2y(x) + \int_0^x \frac{\partial}{\partial x}\, y(t)\, dt.$$

$$y''(x) = 2xy'(x) + 2y(x) + y(x)$$

$$y''(x) = 2xy'(x) + 3y(x)$$

$$y''(x) - 2xy'(x) - 3y(x) = 0.$$

put $x = 0$ in eqn (3).

$y(0) = 1 + 0$

$y(0) = 1$

put $x = 0$ in eqn (5)

$y'(0) = 0.$

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Boundary value problem.

A problem is said to BVP in which the ODE is to be solved under condition involving dependent variable in the derivative at two different values of independent variable.

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2. Initial value problem.

A problem is said to IVP in which the ODE is to be solved ~~is~~ under condition involving dependent variable in the derivative at two same values of independent variable.

1 Reduce the following boundary value problem into an Integral Equation $u''(x)+\lambda u=0$ with Boundary condition $u(0)=0$ $u(l)=0$

Solution:

Given that $u''(x)+\lambda u=0 \rightarrow (1)$

$u''(x)=-\lambda u \rightarrow (2)$

on $\int$ing w.r to 'x' from 0 to x

$$\left[u'(x)\right]_0^x = -\lambda \int_0^x u(x)\,dx$$

$$u'(x)-u'(0) = -\lambda \int_0^x u(x)\,dx \quad \text{let } u'(0)=c$$

$$u'(x)-c = -\lambda \int_0^x u(x)\,dx$$

$$u'(x) = c - \lambda \int_0^x u(x)\,dx$$

Again $\int$ing

$$[u(x)]_0^x = cx - \lambda \int_0^x u(x)\,dx^2$$

$$u(x) - u(0) = cx - \lambda \int_0^x u(x)\,dx^2$$

$$u(x) = cx - \lambda \int_0^x u(t)\,dt^2$$

$$u(x) = cx - \lambda \int_0^x (x-t)\,u(t)\,dt \rightarrow (3)$$

put $x = l$ in (3)

$$u(l) = cl - \lambda \int_0^l (l-t)\,u(t)\,dt.$$

$$0 = cl \mp \lambda \int_0^l (l-t)\,u(t)\,dt$$

$$cl = \lambda \int_0^l (l-t)\,u(t)\,dt$$

$$c = \frac{\lambda}{l} \int_0^l (l-t)\,u(t)\,dt \rightarrow (4)$$

Sub (4) in (3).

$$u(x) = \frac{\lambda x}{l} \int_0^l (l-t)\,u(t)\,dt - \lambda \int_0^x (x-t)\,u(t)\,dt$$

$$u(x) = \frac{\lambda x}{l} \int_0^l (l-t)\,u(t)\,dt - \lambda \int_0^x (x-t)\,u(t)\,dt$$

$$= \lambda \left[\int_0^l \frac{x(l-t)}{l} - \int_0^x (x-t) \right] u(t)\,dt.$$

$$= \lambda \left[\int_0^x \frac{x(l-t)}{l} + \int_x^l \frac{x(l-t)}{l} - \int_0^x (x-t) \right] u(t)\,dt$$

$$= \lambda \left[\int_0^x \frac{x(l-t)}{l} - (x-t) + \int_x^l \frac{x(l-t)}{l} \right] u(t)\,dt$$

$$= \lambda \left[\int_0^x \frac{xl - xt - xl + lt}{l} + \int_x^l \frac{x(l-t)}{l} \right] u(t)\,dt$$

$$u(x) = \lambda \left[\int_0^x \frac{t(l-x)}{l} + \int_x^l \frac{x(l-t)}{l} \right] u(t)\,dt$$

$$u(x) = \lambda \int_0^l k(x,t)\, u(t)\,dt$$

where,

$$k(x,t) = \begin{cases} \frac{t(l-x)}{l} & 0 \le t \le x \\ \frac{x(l-t)}{l} & x \le t \le l \end{cases}$$

2. Find the I.E corresponding to the boundary value problem $\frac{d^2y}{dx^2} + xy = 1$, $y(0) = 0$, $y(1) = 1$

Solution :

Given that

$$y''(x) + x\,y(x) = 1 \longrightarrow ①$$

$$y''(x) = 1 - x\,y(x) \longrightarrow ②$$

on $\int$ ing w.r.to 'x' from 0 to x

$$[y'(x)]_0^x = x - \int_0^x x\,y(x)\,dx$$

$$y'(x) - y'(0) = x - \int_0^x x\,y(x)\,dx \quad \therefore y'(0) = C$$

$$y'(x) = c + x - \int_0^x x\,y(x)\,dx$$

Again Jing

$$[y(x)]_0^x = cx + \frac{x^2}{2} - \int_0^x x\,y(x)\,dx^2$$

$$y(x) - y(0) = cx + \frac{x^2}{2} - \int_0^x x\,y(x)\,dx^2$$

$$y(x) = cx + \frac{x^2}{2} - \int_0^x x\,y(x)\,dx^2$$

$$y(x) = cx + \frac{x^2}{2} - \int_0^x (x-t)\,t\,y(t)\,dt \to (3).$$

put $x=1$ in (3).

$$y(1) = c + \frac{1}{2} - \int_0^1 (1-t)\,t\,y(t)\,dt$$

$$1 = c + \frac{1}{2} - \int_0^1 (1-t)\,t\,y(t)\,dt$$

$$c = 1 - \frac{1}{2} + \int_0^1 (t-t^2)\,y(t)\,dt \to (4).$$

$$c = \frac{1}{2} + \int_0^1 (t-t^2)\,y(t)\,dt \to (4)$$

Sub (4) in (3).

$$y(x) = \frac{x}{2} + \int_0^1 x(t-t^2)\,y(t)\,dt + \frac{x^2}{2} - \int_0^x (x-t)t\,y(t)\,dt.$$

$$= \frac{x}{2} + \frac{x^2}{2} + \int_0^1 x(t-t^2)\,y(t)\,dt - \int_0^x (x-t)t\,y(t)\,dt$$

$$= x/2 + x^2/2 + \left[\int_0^1 x(t-t^2) - \int_0^x (x-t)t\right] y(t)\,dt.$$

$$= \frac{x}{2} + \frac{x^2}{2} + \left[\int_0^x x(t-t^2) + \int_x^1 x(t-t^2) - \int_0^x (x-t)t\right] y(t)\,dt$$

$$= \frac{x}{2} + \frac{x^2}{2} + \left[\int_0^x xt - xt^2 - xt + t^2 + \int_x^1 x(t-t^2) \right] y(t)\,dt$$

$$y(x) = \frac{x}{2} + \frac{x^2}{2} + \left[\int_0^x t^2(1-x) + \int_x^1 x(t-t^2) \right] y(t)\,dt$$

$$y(x) = f(x) + \int_0^1 K(x,t)\, y(t)\,dt$$

where,

$$f(x) = \frac{x}{2} + \frac{x^2}{2}$$

$$K(x,t) = \begin{cases} t^2(1-x) & 0 \le t \le x \\ x(t-t^2) & x \le t \le 1 \end{cases}$$

3 obtain Fredholm I.E of 2^{nd} kind corresponding to boundary value problem $\frac{d^2u}{dx^2} + \lambda u(x) = x$

with boundary condition $u(0)=0$, $u'(1)=0$

Solution:

Given that

$$u''(x) + \lambda u(x) = x \longrightarrow (1)$$

$$u''(x) = x - \lambda u(x) \longrightarrow (2)$$

on $\int$ing (2) w.r.to x from 0 to x

$$[u'(x)]_0^x = \frac{x^2}{2} - \lambda \int_0^x u(x)\,dx$$

$$u'(x) - u'(0) = \frac{x^2}{2} - \lambda \int_0^x u(x)\,dx$$

$$u'(x) - c = \frac{x^2}{2} - \lambda \int_0^x u(x)\,dx$$

$$u'(x) = c + \frac{x^2}{2} - \lambda \int_0^x u(x)\,dx \rightarrow (3)$$

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$$u'(x) = c + \frac{x^2}{2} - \lambda \int_0^x u(t)\,dt \longrightarrow (4)$$

Again $\int$ing (3) w.r.to 'x' from 0 to x

$$[u(x)]_0^x = cx + \frac{x^3}{6} - \lambda \int_0^x u(x)\,dx^2$$

$$u(x) - u(0) = cx + \frac{x^3}{6} - \lambda \int_0^x u(x)\,dx^2$$

$$u(x) = cx + \frac{x^3}{6} - \lambda \int_0^x (x-t)\,u(t)\,dt \rightarrow (5)$$

put $x=1$ in eqn (4),

$$u'(1) = c + \frac{1}{2} - \lambda \int_0^1 u(t)\,dt$$

$$0 = c + \frac{1}{2} - \lambda \int_0^1 u(t)\,dt$$

$$c = -\frac{1}{2} + \lambda \int_0^1 u(t)\,dt \longrightarrow (6)$$

Sub (6) in (5),

$$u(x) = -\frac{x}{2} + \lambda \int_0^1 x\,u(t)\,dt + \frac{x^3}{6} - \lambda \int_0^x (x-t)u(t)\,dt$$

$$= -\frac{x}{2} + \frac{x^3}{6} - \lambda \left[\int_0^1 x - \int_0^x (x-t) \right] u(t)\,dt$$

$$= -\frac{x}{2} + \frac{x^3}{6} - \lambda \left[\int_0^x x + \int_x^1 x - \int_0^x (x-t) \right] u(t)\,dt$$

$$= \frac{x^3}{6} - \frac{x}{2} - \lambda \left[\int_0^x x - x + t + \int_x^1 x \right] u(t)\,dt$$

$$= \frac{x^3}{6} - \frac{x}{2} - \lambda \left[\int_0^x t + \int_x^1 x \right] u(t)\,dt$$

$$u(x) = \frac{x^3}{6} - \frac{x}{2} - \lambda \left[\int_0^x t + \int_x^1 x \right] u(t)\,dt$$

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$$u(x) = f(x) + \lambda \int_0^1 K(x,t)\, u(t)\, dt.$$

where

$$f(x) = \frac{x^3}{6} - \frac{x}{2}$$

$$K(x,t) = \begin{cases} t \;;\; 0 \le t \le x \\ x \;;\; x \le t \le 1. \end{cases}$$

4. a) If $y''(x) + \lambda y(x) = 0$ and y satisfies the boundary condition $y(0) = 0$, $y(l) = 0$ S.T

$$y(x) = \frac{\lambda x}{l} \int_0^l (l-t)\, y(t)\, dt - \lambda \int_0^x (x-t)\, y(t)\, dt$$

b) S.T for result of part (a) can be written as $y(x) = \lambda \int_0^l K(x,t)\, y(t)\, dt$ where

$$K(x,t) = \begin{cases} \frac{t(l-x)}{l} & , t < x \\ \frac{x(l-t)}{l} & , t > x \end{cases}$$

c) verify directly that the expression obtain, satisfies the D.E and B.C

solution:

(a) Given that

$$y''(x) + \lambda y(x) = 0 \longrightarrow ①$$

$$y''(x) = -\lambda y(x) \longrightarrow ②$$

on $\int$ing w.r.to 'x' from 0 to x

$$\left[y'(x) \right]_0^x = -\lambda \int_0^x y(x)\, dx$$

$$y'(x) - y'(0) = -\lambda \int_0^x y(x)\, dx \qquad \therefore \text{let } y'(0) = c$$

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$$y'(x) - c = -\lambda \int_0^x y(x)\,dx$$

$$y'(x) = c - \lambda \int_0^x y(x)\,dx$$

Again $\int$ing

$$[y(x)]_0^x = cx - \lambda \int_0^x y(x)\,dx^2$$

$$y(x) - y(0) = cx - \lambda \int_0^x y(x)\,dx^2$$

$$y(x) = cx - \lambda \int_0^x y(t)\,dt^2$$

$$y(x) = cx - \lambda \int_0^x (x-t)\,y(t)\,dt \rightarrow (3)$$

put $x = l$ in (3)

$$y(l) = cl - \lambda \int_0^l (l-t)\,y(t)\,dt$$

$$0 = cl - \lambda \int_0^l (l-t)\,y(t)\,dt$$

$$cl = \lambda \int_0^l (l-t)\,y(t)\,dt$$

$$c = \frac{\lambda}{l} \int_0^l (l-t)\,y(t)\,dt \rightarrow (4)$$

sub (4) in (3)

$$y(x) = \frac{\lambda x}{l} \int_0^l (l-t)\,y(t)\,dt - \lambda \int_0^x (x-t)\,y(t)\,dt \rightarrow (5)$$

(b)

$$y(x) = \frac{\lambda x}{l} \int_0^l (l-t)\,y(t)\,dt - \lambda \int_0^x (x-t)\,y(t)\,dt$$

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$$= \lambda \left[\int_0^l \frac{x(l-t)}{l} - \int_0^x (x-t) \right] y(t)\,dt$$

$$= \lambda \left[\int_0^x \frac{x(l-t)}{l} + \int_x^l \frac{x(l-t)}{l} - \int_0^x (x-t) \right] y(t)\,dt$$

$$= \lambda \left[\int_0^x \frac{x(l-t)}{l} - (x-t) + \int_x^l \frac{x(l-t)}{l} \right] y(t)\,dt$$

$$y(x) = \lambda \left[\int_0^x \frac{t(l-\cancel{x})}{l} + \int_x^l \frac{x(l-t)}{l} \right] y(t)\,dt$$

$$y(x) = \lambda \int_0^l K(x,t)\, y(t)\,dt$$

where

$$K(x,t) = \begin{cases} \dfrac{t(l-x)}{l} & 0 \le t \le x \\ \dfrac{x(l-t)}{l} & x \le t \le l \end{cases}$$

(c)

Conversely,

Let $y(x) = \dfrac{\lambda x}{l} \displaystyle\int_0^l (l-t)y(t)\,dt - \lambda \int_0^x (x-t)y(t)\,dt$ $\rightarrow$ (5)

Diff eqn (5) w.r.to 'x'

$$y'(x) = \frac{\lambda}{l} \frac{d}{dx} \int_0^l x(l-t)y(t)\,dt$$

$$- \lambda \frac{d}{dx} \int_0^x (x-t)y(t)\,dt.$$

$$y'(x) = \frac{\lambda}{l} \left[\int_0^l \frac{\partial}{\partial x} x(l-t)y(t)\,dt + 0 - 0 \right]$$

$$- \lambda \left[\int_0^x \frac{\partial}{\partial x} (x-t)y(t)\,dt + 0 - 0 \right]$$

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$$y'(x) = \frac{\lambda}{l}\int_0^l (l-t)y(t)\,dt - \lambda\int_0^x y(t)\,dt$$

Again Diff w.r.to x

$$y''(x) = \frac{\lambda}{l}\frac{d}{dx}\int_0^l (l-t)y(t)\,dt - \lambda\frac{d}{dx}\int_0^x y(t)\,dt$$

$$= \frac{\lambda}{l}\left[\int_0^l \frac{\partial}{\partial x}(l-t)y(t)\,dt + 0 - 0\right]$$

$$- \lambda\left[\int_0^x \frac{\partial}{\partial x}y(t)\,dt + y(x).1 - 0\right]$$

$$= \frac{\lambda}{l}(0) - \lambda(0 + y(x))$$

$$y''(x) = -\lambda y(x)$$

$$y''(x) + \lambda y(x) = 0$$

put $x=0$ in eqn ⑥.

$$y(0) = 0.$$

put $x=l$ in eqn ⑥

$$y(l) = \frac{\lambda \cancel{l}}{\cancel{l}}\int_0^l (l-t)y(t)\,dt - \lambda\int_0^l (\cancel{l}-t)y(t)\,dt$$

$$y(l) = 0.$$