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Unit - IV

Solution of Homogenous Fredholm Integral Equation of second kind with seperable (or) Degenerate Kernal

Consider a homogenous Fredholm I. E of second kind

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt \rightarrow (1)$$

Let us assume that $K(x,t)$ is seperable, we have

$$K(x,t) = \sum_{i=1}^{n} f_i(x)\, g_i(t)$$

Substitute the above in eqn (1), we get

$$u(x) = \lambda \int_a^b \sum_{i=1}^{n} f_i(x)\, g_i(t)\, u(t)\, dt \rightarrow (2)$$

$$u(x) = \lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t)\, u(t)\, dt \rightarrow (3)$$

We assume that

$$\int_a^b g_i(t)\, u(t)\, dt = c_i \quad , i = 1, 2, \ldots n \rightarrow (4)$$

From (3)

$$u(x) = \lambda \sum_{i=1}^{n} c_i f_i(x) \rightarrow (5)$$

Now we want evaluate the constant c_i

x'ly on both sides of eqn (5) successively by $g_1(x), g_2(x), \ldots g_n(x)$ and integrating over the range $[a,b]$, we get

$$\int_a^b g_1(x)u(x)dx = \lambda \sum_{i=1}^{n} c_i f_i(x) \int_a^b g_1(x)dx$$

$$= \lambda \sum_{i=1}^{n} c_i \int_a^b f_i(x) g_1(x)\, dx \rightarrow (6)$$

$$\int_a^b g_2(x)u(x)dx = \lambda \sum_{i=1}^{n} c_i \int_a^b f_i(x) g_2(x)dx \rightarrow (7)$$

$$\vdots$$

$$\int_a^b g_n(x)u(x)dx = \lambda \sum_{i=1}^{n} c_i \int_a^b f_i(x) g_n(x)dx \rightarrow (8)$$

Let us define

$$\alpha_{ji} = \int_a^b g_j(x) f_i(x)\, dx \rightarrow (9)$$

using the value of α_{ji} in (6) and using (4), we get

$$c_1 = \lambda \sum_{i=1}^{n} c_i \alpha_{1i}$$

$$c_1 = \lambda \left[c_1\alpha_{11} + c_2\alpha_{12} + \cdots + c_n\alpha_{1n} \right]$$

lly

$$c_2 = \lambda \left[c_1\alpha_{21} + c_2\alpha_{22} + \cdots + c_n\alpha_{2n} \right]$$

$$\vdots$$

$$c_n = \lambda \left[c_1\alpha_{n1} + c_2\alpha_{n2} + \cdots + c_n\alpha_{nn} \right]$$

i.e.

$$\left.\begin{aligned}
(1-\lambda\alpha_{11})c_1 - \lambda\alpha_{12}c_2 - \cdots - \lambda\alpha_{1n}c_n &= 0 \\
-\lambda\alpha_{21}c_1 + (1-\lambda\alpha_{22})c_2 - \cdots - \lambda\alpha_{2n}c_n &= 0 \\
&\vdots \\
-\lambda\alpha_{n1}c_1 - \lambda\alpha_{n2}c_2 - \cdots + (1-\lambda\alpha_{nn})c_n &= 0
\end{aligned}\right\} \rightarrow (10)$$

The determinate of the equation (10) can be written as

$$D(\lambda) = \begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & \cdots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & \cdots & -\lambda\alpha_{2n} \\ \vdots & & & \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \cdots & 1-\lambda\alpha_{nn} \end{vmatrix}$$

case (i):

If $D(\lambda) \neq 0$, the system of eqn (6) as trivial solution given by

$c_1 = c_2 = \cdots\cdots = c_n = 0$

using (5), we observe that kanc as only zero are trivial solution.

i.) $u(x) = 0$

case (ii)

If $D(\lambda) = 0$ atleast one of the c_i's can be assigned arbitrary constant and the remaining c_i's can be determine.

Hence if $D(\lambda) = 0$, there exist Infinitely many solution of Integral eqn (1).

1) Define orthogonal:

Two function $f_1(x)$ and $f_2(x)$ are continuous on the interval $[a, b]$ are said to be orthogonal if $\int_a^b f_1(x) f_2(x)\, dx = 0$

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Eigen value and Eigen function

The values of λ for which $D(\lambda)=0$ are called Eigen values and the non-trivial solution of integral equation

$$u(x) = \lambda \int_a^b k(x,t)\, u(t)\, dt$$

is called a corresponding Eigen function of λ.

Theorem :1

If $k(x,t)$ is symmetric, $f_0(x)$ and $f_1(x)$ are Eigen function of $k(x,t)$ corresponding to the Eigen value λ_0 and λ_1 respectively. $(\lambda_0 \neq \lambda_1)$ then $f_0(x)$ and $f_1(x)$ are orthogonal in $[a,b]$

ie) $\int_a^b f_0(x) f_1(x)\, dx = 0.$

proof:

Given that,

$f_0(x)$, $f_1(x)$ are Eigen function corresponding to the Eigen values λ_0 and λ_1 $(\lambda_0 \neq \lambda_1)$ respectively of homogenous fredholm Integral equation of second kind

$$u(x) = \lambda \int_a^b k(x,t)\, u(t)\, dt \rightarrow ①$$

since λ_0 and λ_1 are Eigen values of the corresponding Eigen functions $f_0(x)$ and $f_1(x)$.

① $\Rightarrow$

$$f_0(x) = \lambda_0 \int_a^b k(x,t)\, f_0(t)\, dt \rightarrow ②$$

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$$f_1(x) = \lambda_1 \int_a^b K(x,t) f_1(t)\, dt \rightarrow ③$$

Given that

$K(x,t)$ is symmetric

i.e. $K(x,t) = K(t,x) \rightarrow ④$

x'^{ly} by $f_1(x)$ both sides of eqn ②

we get

$$f_0(x) f_1(x) = \lambda_0 \int_a^b K(x,t) f_0(t) f_1(x)\, dt$$

$$= \lambda_0 f_1(x) \int_a^b K(x,t) f_0(t)\, dt \rightarrow ⑤$$

on $\int$ing w.r to 'x' from a to b

$$\int_a^b f_0(x) f_1(x)\, dx = \lambda_0 \int_a^b f_1(x) \left[\int_a^b K(x,t) f_0(t)\, dt \right] dx$$

$$= \lambda_0 \int_a^b f_0(t) \left[\int_a^b K(t,x) f_1(x)\, dx \right] dt \rightarrow ⑥$$

from eqn ③

$$f_1(t) = \lambda_1 \int_a^b K(t,x) f_1(x)\, dx$$

$$\frac{f_1(t)}{\lambda_1} = \int_a^b K(t,x) f_1(x)\, dx$$

⑥ $\Rightarrow$

$$\int_a^b f_0(x) f_1(x)\, dx = \lambda_0 \int_a^b f_0(t) \frac{f_1(t)}{\lambda_1}\, dt$$

$$\int_a^b f_0(x) f_1(x)\, dx = \frac{\lambda_0}{\lambda_1} \int_a^b f_0(t) f_1(t)\, dt$$

$$\lambda_1 \int_a^b f_0(x) f_1(x)\, dx = \lambda_0 \int_a^b f_0(t) f_1(x)\, dx$$

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$$(\lambda_1-\lambda_0)\int_a^b \phi_0(x) \phi_1(x)\,dx = 0$$

$\because \lambda_1 \neq \lambda_0$

$$\Rightarrow \int_a^b \phi_0(x)\phi_1(x)\,dx = 0.$$

$\therefore$ The function are orthogonal.

Theorem : 2

The Eigen values of symmetric Kernal are real

proof.

Let $u(x) = \lambda \int_a^b k(x,t)\,u(t)\,dt \rightarrow (1)$

be a homogenous fredholm Integral equation of second kind

To prove :

The Eigen values are real

Let if possible eqn (1) has an Eigen value $\lambda_0 = \alpha + i\beta \rightarrow (2)$

Let $\phi_0(x) = u + iv \rightarrow (3)$ be the corresponding Eigen function of λ_0.

The complex conjugate of $\overline{\lambda_0}$ would have a Eigen value corresponding to the Eigen function. $\overline{\phi_0(x)}$

$\therefore$ we have $\overline{\lambda_0} = \alpha - i\beta \rightarrow (4)$

$\overline{\phi_0}(x) = u - iv \rightarrow (5)$

From equation (1) we can deduce

$(1) \Rightarrow$

$$\phi_0(x) = \lambda_0 \int_a^b k(x,t)\phi_0(t)\,dt \rightarrow (6)$$

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$$\overline{\phi_0(x)} = \overline{\lambda_0} \int_a^b K(x,t)\, \overline{\phi_0(t)}\, dt \rightarrow (7)$$

Given that

$K(x,t)$ is symmetric

i.e., $K(x,t) = K(t,x) \rightarrow (8)$

x'ly by $\overline{\phi_0(x)}$ both sides of eqn (6)

we get

$$\phi_0(x)\overline{\phi_0(x)} = \lambda_0 \int_a^b K(x,t)\, \overline{\phi_0(x)}\, \phi_0(t)\, dt$$

$$= \lambda_0 \overline{\phi_0(x)} \int_a^b K(x,t)\, \phi_0(t)\, dt \rightarrow (8)$$

on $\int$ing w.r.t 'x' from a to b.

$$\int_a^b \phi_0(x)\overline{\phi_0(x)}\, dx = \lambda \int_a^b \phi_0(t) \left[\int_a^b K(x,t)\, \overline{\phi_0(x)}\, dt \right] dx \rightarrow (9)$$

From eqn (7)

$$\overline{\phi_0(t)} = \overline{\lambda_0} \int_a^b K(t,x)\, \overline{\phi_0(x)}\, dx$$

$$\frac{\overline{\phi_0(t)}}{\overline{\lambda_0}} = \int_a^b K(t,x)\, \overline{\phi_0(x)}\, dx \rightarrow (10)$$

$$(9) \Rightarrow \int_a^b \phi_0(x)\overline{\phi_0(x)}\, dx = \lambda_0 \int_a^b \phi_0(t)\, \frac{\overline{\phi_0(t)}}{\overline{\lambda_0}}\, dt$$

$$\int_a^b \phi_0(x)\overline{\phi_0(x)}\, dx = \frac{\lambda_0}{\overline{\lambda_0}} \int_a^b \phi_0(t)\, \overline{\phi_0(t)}\, dt$$

$$\overline{\lambda_0} \int_a^b \phi_0(x)\overline{\phi_0(x)}\, dx = \lambda_0 \int_a^b \phi_0(x)\, \overline{\phi_0(x)}\, dx$$

$$(\overline{\lambda_0} - \lambda_0) \int_a^b \phi_0(x)\, \overline{\phi_0(x)}\, dx = 0 \rightarrow (11).$$

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Let $\lambda_0 = \alpha + i\beta$, $\overline{\lambda_0} = \alpha - i\beta$

$\overline{\lambda_0} - \lambda_0 = \alpha - i\beta - (\alpha + i\beta)$

$= \alpha - i\beta - \alpha - i\beta$

$\overline{\lambda_0} - \lambda_0 = -2i\beta \rightarrow (12)$.

Also $\phi_0(x) \cdot \overline{\phi_0(x)} = (u + iv)(u - iv)$

$= u^2 + v^2 \rightarrow (13)$

Sub (12) & (13) in (11),

$$-2i\beta \int_a^b (u^2+v^2)\, dx = 0.$$

$$\Rightarrow 2i\beta \int_a^b u^2+v^2\, dx = 0.$$

Since $\phi_0(x)$ is an Eigen function corresponding to the Eigen value λ_0.

$\Rightarrow \phi_0(x) \neq 0$

$\Rightarrow u^2 + v^2 \neq 0$

$\therefore 2i\beta = 0$

$\Rightarrow \beta = 0$

$\therefore$ The Eigen values of symmetric Kernal are real.

Problem:

1. Show that the homogenous I.E $u(x) = \lambda \int_0^1 (3x-2)t\, u(t)\, dt$ has no Eigen values and Eigen functions.

Solution:

Given that

$$u(x) = \lambda \int_0^1 (3x-2)\, t\, u(t)\, dt \rightarrow (1)$$

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$$u(x) = \lambda(3x-2)\int_0^1 t\,u(t)\,dt \rightarrow ②$$

Assume that

$$c_1 = \int_0^1 t\,u(t)\,dt \rightarrow ③.$$

② ⇒

$$u(x) = \lambda(3x-2)\,c_1 \rightarrow ④$$

$$u(t) = \lambda(3t-2)\,c_1$$

③ ⇒

$$c_1 = \int_0^1 t\lambda(3t-2)\,c_1\,dt$$

$$c_1 = \lambda c_1 \int_0^1 (3t^2-2t)\,dt$$

$$c_1 = \lambda c_1 \left[t^3 - t^2\right]_0^1$$

$$c_1 = \lambda c_1 \left[(1-1)-(0-0)\right]$$

$$c_1 = 0.$$

∴ ④ ⇒

$$u(x) = 0.$$

∴ Hence the given I.E has no Eigen values and Eigen function

2. Find the Eigen values and the corresponding Eigen function of the homogenous I.E

$$u(x) = \lambda\int_0^1 \sin\pi x\cos\pi t\;u(t)\,dt.$$

Solution:-

Given that

$$u(x) = \lambda\int_0^1 \sin\pi x\cos\pi t\;u(t)\,dt \rightarrow ①.$$

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$$u(x) = \lambda \sin\pi x \int_0^1 \cos\pi t \; u(t)\,dt \rightarrow (2)$$

Assume that.

$$c_1 = \int_0^1 \cos\pi t \; u(t)\,dt \rightarrow (5).$$

(2) $\Rightarrow$

$$u(x) = \lambda \sin\pi x \; c_1 \rightarrow (4)$$

$$u(t) = \lambda \sin\pi t \; c_1$$

(3) $\Rightarrow$

$$c_1 = \int_0^1 \cos\pi t \; \lambda \sin\pi t \, c_1 \; dt$$

$$c_1 = \lambda c_1 \int_0^1 \cos\pi t \sin\pi t \, dt$$

$$c_1 = \lambda c_1 \int_0^1 \frac{\sin 2\pi t}{2} \, dt.$$

$$= \frac{\lambda c_1}{2} \int_0^1 \sin 2\pi t \, dt$$

$$= \frac{\lambda c_1}{2} \left[\frac{-\cos 2\pi t}{2\pi} \right]_0^1$$

$$= \frac{-\lambda c_1}{4\pi} \left[\cos 2\pi t \right]_0^1$$

$$c_1 = \frac{-\lambda c_1}{4\pi} \left[\cos 2\pi - \cos 0 \right]$$

$$= \frac{-\lambda c_1}{4\pi} \left[1 - 1 \right]$$

$$c_1 = 0$$

(4) $\Rightarrow$

$$u(x) = 0.$$

$\therefore$ Hence the given I.E has no Eigen values and Eigen function.

(2)

3. Solve the homogenous fredholm I.E of second kind

vom.a.

$$u(x) = \lambda \int_0^{2\pi} \sin(x+t)\, u(t)\, dt.$$

find its Eigen value and Eigen function.

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Solution:

Given that

$$u(x) = \lambda \int_0^{2\pi} \sin(x+t)\, u(t)\, dt \rightarrow (1)$$

$$= \lambda \int_0^{2\pi} [\sin x \cos t + \cos x \sin t]\, u(t)\, dt.$$

$$u(x) = \lambda \sin x. \int_0^{2\pi} \cos t\, u(t)\, dt + \lambda \cos x \int_0^{2\pi} \sin t\, u(t)\, dt \rightarrow (2)$$

Assume that

$$\left.\begin{aligned} c_1 &= \int_0^{2\pi} \cos t\, u(t)\, dt \\ c_2 &= \int_0^{2\pi} \sin t\, u(t)\, dt \end{aligned}\right\} \rightarrow (3)$$

$(2) \Rightarrow \quad u(x) = \lambda \sin x\, c_1 + \lambda \cos x\, c_2 \rightarrow (4).$

$u(t) = \lambda c_1 \sin t + \lambda c_2 \cos t$

$(3) \Rightarrow \quad c_1 = \int_0^{2\pi} \cos t\, [\lambda c_1 \sin t + \lambda c_2 \cos t]\, dt$

$$c_1 = \lambda c_1 \int_0^{2\pi} \cos t \sin t\, dt + \lambda c_2 \int_0^{2\pi} \cos^2 t\, dt$$

$\sin A \cos A = \frac{\sin 2A}{2}$

$$= \lambda c_1 \int_0^{2\pi} \frac{\sin 2t}{2}\, dt + \lambda c_2 \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= \frac{\lambda c_1}{2} \int_0^{2\pi} \sin 2t\, dt + \frac{\lambda c_2}{2} \int_0^{2\pi} (1+\cos 2t)\, dt$$

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$$c_1 = \frac{\lambda c_1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$c_1 = -\frac{\lambda c_1}{4}\left[\cos 2t\right]_0^{2\pi} + \frac{\lambda c_2}{4}\left[2t + \sin 2t\right]_0^{2\pi}$$

$$c_1 = -\frac{\lambda c_1}{4}\left[1-1\right] + \frac{\lambda c_2}{4}\left[(4\pi+0)-(0+0)\right]$$

$$= \frac{\lambda c_2}{4}(4\pi)$$

$$c_1 = \lambda \pi c_2$$

$$c_1 - \lambda \pi c_2 = 0 \longrightarrow (5).$$

$$(3) \Rightarrow c_2 = \int_0^{2\pi} \sin t\left[\lambda c_1 \sin t + \lambda c_2 \cos t\right] dt$$

$$= \lambda c_1 \int_0^{2\pi} \sin^2 t \, dt + \lambda c_2 \int_0^{2\pi} \sin t \cos t \, dt$$

$$= \frac{\lambda c_1}{2} \int_0^{2\pi} (1-\cos 2t) dt + \frac{\lambda c_2}{2} \int_0^{2\pi} \sin 2t \, dt$$

$$= \frac{\lambda c_1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \frac{\lambda c_1}{2}\left[2\pi - 0\right] - \frac{\lambda c_2}{4}\left[1-1\right]$$

$$c_2 = \frac{\lambda c_1}{2}(2\pi)$$

$$c_2 = \lambda \pi c_1$$

$$-\lambda \pi c_1 + c_2 = 0 \longrightarrow (6).$$

Since eqn (5) & (6) are homogenous equation.

$$D(\lambda) = \begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix}$$

$$D = 1 - \lambda^2\pi^2$$

$$\lambda^2\pi^2 = 1$$

$$\lambda^2 = \frac{1}{\pi^2}$$

$$\lambda = \pm \frac{1}{\pi}$$

$\therefore$ The Eigen values are $\frac{1}{\pi}, -\frac{1}{\pi}$.

To find eigen function.

case (i)

Let $\lambda = \frac{1}{\pi}$

(5) $\Rightarrow$ $c_1 - \frac{1}{\pi}\pi c_2 = 0$

$$c_1 - c_2 = 0$$

$$c_1 = c_2$$

(4) $\Rightarrow$ $u(x) = \lambda c_1 \sin x + \lambda c_2 \cos x$

$$= \frac{c_1}{\pi}\sin x + \frac{c_1}{\pi}\cos x.$$

$$= \frac{c_1}{\pi}[\sin x + \cos x]$$

$\left(\because \frac{c_1}{\pi} = 1\right)$

$$u(x) = \sin x + \cos x$$

case (ii)

Let $\lambda = -1/\pi$

(5) $\Rightarrow$ $c_1 + \frac{1}{\pi}\pi c_2 = 0.$

$$c_1 + c_2 = 0$$

$C_1 = -C_2$

② →

$u(x) = \lambda C_1 \sin x + \lambda C_2 \cos x$

$= \frac{-C_1}{\pi} \sin x + \frac{C_1}{\pi} \cos x$

$u(x) = \frac{C_1}{\pi} (\cos x - \sin x) \quad \left[\because \frac{C_1}{\pi} = 1\right.$

$u(x) = \cos x - \sin x$

$\therefore$ The Eigen functions are

$u(x) = \sin x + \cos x$

$u(x) = \cos x - \sin x$

2. Find the Eigen value and Eigen function of the I.E

$$u(x) = \lambda \int_{-1}^{1} (5x t^3 + 4x^2 t) u(t) dt$$

Solution:

Given that

$$u(x) = \lambda \int_{-1}^{1} (5x t^3 + 4x^2 t) u(t) dt \rightarrow ①$$

$$u(x) = \lambda 5x \int_{-1}^{1} (t^3) u(t) dt$$
$$+ \lambda 4x^2 \int_{-1}^{1} t \, u(t) dt \rightarrow ②$$

Assume that

$$\left.\begin{aligned} C_1 &= \int_{-1}^{1} t^3 u(t) dt \\ C_2 &= \int_{-1}^{1} t \, u(t) dt \end{aligned}\right\} \rightarrow ③$$

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(2) $\Rightarrow$ $u(x) = \lambda 5x c_1 + \lambda 4 x^2 c_2 \rightarrow (4).$

$u(t) = \lambda 5 t c_1 + \lambda 4 t^2 c_2$

(3) $\Rightarrow$

$$c_1 = \int_{-1}^{1} t^3 [5\lambda t c_1 + \lambda 4 t^2 c_2] \, dt$$

$$= 5\lambda c_1 \int_{-1}^{1} t^4 \, dt + 4\lambda c_2 \int_{-1}^{1} t^5 \, dt.$$

$$= 5\lambda c_1 \left[\frac{t^5}{5}\right]_{-1}^{1} + 4\lambda c_2 \left[\frac{t^6}{6}\right]_{-1}^{1}$$

$$= \lambda c_1 [t^5]_{-1}^{1} + \frac{4\lambda c_2}{6} \underbrace{\cancel{[t^6]_{-1}^{1}}}_{0}$$

$$= \lambda c_1 [1+1]$$

$$= 2\lambda c_1$$

$$c_1(1-2\lambda) = 0 \rightarrow (5).$$

(3) $\Rightarrow$.

$$c_2 = \int_{-1}^{1} t [5\lambda c_1 t + 4\lambda c_2 t^2] \, dt$$

$$= 5\lambda c_1 \int_{-1}^{1} t^2 \, dt + 4\lambda c_2 \int_{-1}^{1} t^3 \, dt$$

$$= 5\lambda c_1 \left[\frac{t^3}{3}\right]_{-1}^{1} + 4\lambda c_2 \left[\frac{t^4}{4}\right]_{-1}^{1}$$

$$c_2 = \frac{5\lambda c_1}{3} (2)$$

$$c_2 = \frac{10\lambda c_1}{3}$$

$$-\frac{10\lambda c_1}{3} + c_2 = 0$$

$$-10\lambda c_1 + 3c_2 = 0 \rightarrow (6).$$

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Since the equation ⑤ & ⑥ are homogenous equation

$$D(\lambda) = \begin{vmatrix} 1-2\lambda & 0 \\ -10\lambda & 3 \end{vmatrix}$$

$$= 3-6\lambda$$

$$0 = 3-6\lambda$$

$$3-6\lambda = 0$$

$$6\lambda = 3$$

$$\lambda = 1/2.$$

∴ The Eigen values are $1/2, 0$.

To find Eigen function.

Case (i)

let $\lambda = 1/2$

⑥ ⇒

$$-10 \times \frac{1}{2} \times C_1 + 3C_2 = 0$$

$$-5C_1 + 3C_2 = 0$$

$$C_2 = \frac{5C_1}{3}$$

④ ⇒

$$u(x) = 5x\lambda C_1 + 4x^2\lambda C_2$$

$$= \frac{5xC_1}{2} + \frac{4x^2 5C_1}{2 \cdot 3}$$

$$= \frac{5C_1}{2}\left[x + \frac{4x^2}{3}\right]$$

$\left(\because \frac{5C_1}{2} = 1\right)$

$$u(x) = x + \frac{4x^2}{3}$$

∴ The Eigen functions are

$$u(x) = x + \frac{4x^2}{3}$$

5 Find the Eigen value and Eigen function for the homogeneous I.E

$$u(x) = \lambda \int_0^1 e^{(x+t)} u(t)\, dt$$

Solution:

Given that

$$u(x) = \lambda \int_0^1 e^{x+t} u(t)\, dt. \rightarrow (1)$$

$$u(x) = \lambda e^{x} \int_0^1 e^{t} u(t)\, dt \rightarrow (2)$$

Assume that

$$c_1 = \int_0^1 e^{t} u(t)\, dt \rightarrow (3).$$

(2) $\Rightarrow$.

$$u(x) = \lambda e^{x} c_1 \longrightarrow (4)$$

$$u(t) = \lambda e^{t} c_1$$

(3) $\Rightarrow$

$$c_1 = \int_0^1 e^{t} (\lambda e^{t} c_1)\, dt$$

$$= \lambda c_1 \int_0^1 e^{2t}\, dt$$

$$= \lambda c_1 \left[\frac{e^{2t}}{2} \right]_0^1$$

$$= \frac{\lambda c_1}{2} [e^2 - e^0]$$

$$= \frac{\lambda c_1}{2} [e^2 - 1]$$

$$2c_1 = \lambda c_1 (e^2 - 1)$$

$$2c_1 - \lambda c_1 (e^2 - 1) = 0$$

$$c_1 (2 - \lambda e^2 + \lambda) = 0$$

$c_1 \left[1 - \lambda/2 \,(e^2-1)\right] = 0 \rightarrow (5)$

Assume that $c_1 \neq 0$.

$$1 - \frac{\lambda}{2}(e^2-1) = 0$$

$$\frac{\lambda}{2}(e^2-1) = 0$$

$$\lambda = \frac{2}{e^2-1}$$

$\therefore$ The Eigen value is $\dfrac{2}{e^2-1}$

To find Eigen function

Let $\lambda = \dfrac{2}{e^2-1}$

$$c_1 \left[1 - \frac{\lambda}{2}(e^2-1)\right] = 0$$

$$c_1 \left[1 - \frac{2}{2(e^2-1)}(e^2-1)\right] = 0$$

$$c_1(1-1) = 0$$

$$c_1 = 0.$$

By our assumption $c_1 \neq 0$

$$u(x) = \lambda e^x c_1$$

$$= \frac{2}{e^2-1} e^x c_1$$

$$u(x) = \frac{2c_1}{e^2-1} e^x \qquad \because \frac{2c_1}{e^2-1} = 1$$

$$u(x) = e^x$$

6. Find the Eigen value and Eigen function for the I.E

$$u(x) = \lambda \int_{-1}^{1} [5xt^3 + 4x^2t + 3xt] u(t) dt$$

solution

Given that

$$u(x) = \lambda \int_{-1}^{1} [5xt^3 + 4x^2t + 3xt] u(t) dt \rightarrow (1)$$

$$u(x) = \lambda 5x \int_{-1}^{1} t^3 u(t) dt + \lambda (4x^2+3x) \int_{-1}^{1} t\, u(t) dt \rightarrow (2)$$

Assume that

$$\left.\begin{aligned} c_1 &= \int_{-1}^{1} t^3 u(t) dt \\ c_2 &= \int_{-1}^{1} t\, u(t) dt \end{aligned}\right\} \rightarrow (3)$$

(2) $\Rightarrow$

$$u(x) = \lambda 5x c_1 + \lambda (4x^2+3x) c_2 \rightarrow (4)$$

$$u(t) = \lambda 5t c_1 + \lambda (4t^2+3t) c_2$$

(3) $\Rightarrow$

$$c_1 = \int_{-1}^{1} t^3 [\lambda 5t c_1 + \lambda (4t^2+3t) c_2] dt$$

$$= \int_{-1}^{1} [\lambda 5t^4 c_1 + \lambda (4t^5 + 3t^4) c_2] dt$$

$$= 5\lambda c_1 + 3\lambda c_2 \int_{-1}^{1} t^4 dt + 4\lambda \int_{-1}^{1} t^5 dt$$

$$= \lambda (5c_1 + 3c_2) \left[\frac{t^5}{5}\right]_{-1}^{1} + 4\lambda \left[\frac{t^6}{6}\right]_{-1}^{1}$$

$$= \frac{\lambda (5c_1 + 3c_2)}{5} [1+1]$$

$$c_1 = \frac{2\lambda (5c_1 + 3c_2)}{5}$$

$$c_1 = \frac{10\lambda c_1}{5} + \frac{6\lambda c_2}{5}$$

$$c_1 = 2\lambda c_1 + \frac{6\lambda c_2}{5}$$

$$c_1 - 2\lambda c_1 = \frac{6\lambda c_2}{5}$$

$$c_1(1-2\lambda) - \frac{6}{5}\lambda c_2 = 0 \rightarrow (5)$$

$$c_2 = \int_{-1}^{1} t\,[\lambda 5 t c_1 + \lambda(4t^2+3t)c_2]\,dt$$

$$= \int_{-1}^{1} [\lambda 5 t^2 c_1 + \lambda(4t^3+3t^2)c_2]\,dt$$

$$= \int (5\lambda c_1 + 3\lambda c_2)\int_{-1}^{1} t^2\,dt + 4\lambda c_2\int_{-1}^{1} t^3\,dt$$

$$= 5\lambda c_1 + 3\lambda c_2\left[\frac{t^3}{3}\right]_{-1}^{1} + 4\lambda c_2\left[\frac{t^4}{4}\right]_{-1}^{1}$$

$$= \frac{5\lambda c_1 + 3\lambda c_2}{3}[1+1]$$

$$= \frac{2\lambda(5c_1+3c_2)}{3}$$

$$c_2 = \frac{10\lambda c_1}{3} + \frac{6\lambda c_2}{3}$$

$$c_2 = \frac{10\lambda c_1}{3} + 2\lambda c_2$$

$$c_2 - 2\lambda c_2 = \frac{10\lambda c_1}{3}$$

$$c_2(1-2\lambda) - \frac{10}{3}\lambda c_1 = 0 \rightarrow (6)$$

since the equation (5) & (6) are homogenous equation

$$D(\lambda) = \begin{vmatrix} (1-2\lambda) & -\frac{6}{5}\lambda \\ -\frac{10}{3}\lambda & (1-2\lambda) \end{vmatrix}$$

$$0 = (1-2\lambda)^2 - \frac{10}{3}\lambda \times \frac{6}{5}\lambda$$

~~since the~~

$$0 = 1 + 4\lambda^2 - 4\lambda - \frac{60\lambda^2}{15}$$

$$0 = 1 + 4\lambda^2 - 4\lambda - 4\lambda^2$$

$$= 1 - 4\lambda$$

$$-1 = -4\lambda$$

$$\frac{1}{4} = \lambda$$

The Eigen values are $1/4, 0$.

To find Eigen function

case (i)

Let $\lambda = 1/4$

(6) $\Rightarrow$

$$c_2(1-2\lambda) - 10/3\,\lambda c_1 = 0$$

$$c_2\left(1 - 2\times\frac{1}{4}\right) - \frac{10}{3}\cdot\frac{1}{4}c_1 = 0$$

$$c_2\left(1 - \frac{1}{2}\right) - \frac{10}{12}\cdot c_1 = 0$$

$$c_2\left(\frac{1}{2}\right) - \frac{5}{6}c_1 = 0$$

$$c_2 = \frac{5}{6}c_1 \cdot 2$$

$$c_2 = \frac{5}{3}c_1$$

(1) $\Rightarrow$

$$u(x) = \lambda 5x c_1 + \lambda(4x^2+3x)c_2$$

$$= \frac{1}{4}5xc_1 + \frac{1}{4}(4x^2+3x)\frac{5c_1}{3} \Rightarrow \frac{5xc_1}{4} + \frac{5x^2c_1}{3} + \frac{5xc_1}{4}$$

$$= \frac{9c_1}{4}\left[x + \right. = \frac{\overset{5}{10}xc_1}{\cancel{4}\,2} + \frac{5x^2c_1}{3}$$

$$5c_1\left[\frac{x}{2} + \frac{x^2}{3}\right]$$

$$u(x) = \frac{x}{2} + \frac{x^3}{3}$$

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A Solution non-homogenous Fredholm Integral Equation with seperable kernal.

Consider the Fredholm I.E of non-homogeneous

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t)\, dt \rightarrow (1)$$

By the defination of seperable kernal, we have

$$K(x,t) = \sum_{i=1}^{n} f_i(x)\, g_i(t) \rightarrow (2)$$

Sub above values in eqn (1), we get

$$u(x) = f(x) + \lambda \int_a^b \sum_{i=1}^{n} f_i(x)\, g_i(t)\, u(t)\, dt$$

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t)\, u(t)\, dt \rightarrow (3)$$

Let us assume that

$$c_i = \int_a^b g_i(t)\, u(t)\, dt \rightarrow (4)$$

(3) $\Rightarrow$

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} f_i(x)\, c_i \rightarrow (5)$$

To find the solution of eqn (1), we multiply eqn (5) successively by $g_1, g_2, \ldots g_n$ and Integrate over the range (a,b)

$$\int_a^b g_1(x)\, u(x)\, dx = \int_a^b g_1(x) f(x)\, dx + \lambda \sum_{i=1}^{n} c_i \int_a^b g_1(x)\, f_i(x)\, dx \rightarrow (6)$$

$$\int_a^b g_2(x)\,u(x)\,dx = \int_a^b g_2(x) f(x)\,dx + \lambda \sum_{i=1}^{n} c_i \int_a^b g_2(x) f_i(x)\,dx \rightarrow (7)$$

$\vdots$

$$\int_a^b g_n(x)\,u(x)\,dx = \int_a^b g_n(x) f(x)\,dx + \lambda \sum_{i=1}^{n} c_i \int_a^b g_n(x) f_i(x)\,dx \rightarrow (8)$$

Let define

$$\alpha_{ji} = \int_a^b g_j(x) f_i(x)\,dx \rightarrow (9).$$

$$\beta_j = \int_a^b g_j(x) f(x)\,dx \rightarrow (10).$$

$$(6) \Rightarrow \quad c_1 = \beta_1 + \lambda \sum_{i=1}^{n} c_i \alpha_{1i}$$

$$c_1 = \beta_1 + \lambda \left[c_1 \alpha_{11} + c_2 \alpha_{12} + \cdots + c_n \alpha_{1n} \right]$$

$$(1 - \lambda \alpha_{11}) c_1 - \lambda c_2 \alpha_{12} - \cdots - \lambda c_n \alpha_{1n} = \beta_1$$

11'y we get

$$-\lambda \alpha_{21} c_1 + (1 - \lambda \alpha_{22}) c_2 - \cdots - \lambda c_n \alpha_{2n} = \beta_2$$

$\vdots$

$$-\lambda \alpha_{n1} c_1 - \lambda \alpha_{n2} c_2 - \cdots \cdots + (1 - \lambda \alpha_{nn}) c_n = \beta_n$$

$\rightarrow (11)$

The determinant of $D(\lambda)$ of this system is given by

$D(\lambda)$

$$D(\lambda) = \begin{vmatrix} (1-\lambda\alpha_{11}) & -\lambda\alpha_{12} & \cdots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & (1-\lambda\alpha_{22}) & \cdots & -\lambda\alpha_{2n} \\ \vdots & & & \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \cdots & (1-\lambda\alpha_{nn}) \end{vmatrix}$$

which is a polynomial eqn of degree n.

Now there are the following cases.

Case (i):

when atleast one right member of the system of eqn (ii) is not zero.

Here we have the following subcases.

(i) If $D(\lambda) \neq 0$

In this case there exist unique non-zero solution of the system of eqn (ii). and therefore eqn (i) as a unique non-zero solution

(ii) If $D(\lambda) = 0$.

The above eqn has either no solution or the posses infinite solution.

Case (ii)

when $f(x) = 0$.

(i) If $D(\lambda) \neq 0$, there exist a unique zero solution

(ii) If $D(\lambda) = 0$, In this case the above system posses infinite non-zero solution.

1. Solve the following I.E

$$u(x) = x + \lambda \int_0^1 (1+x+t)\, u(t)\, dt$$

Solution:

Given that

$$u(x) = x + \lambda \int_0^1 (1+x+t)\, u(t)\, dt \rightarrow (1)$$

$$u(x) = x + \lambda \int_0^1 (1+x)\, u\, dt$$

$$u(x) = x + \lambda (1+x) \int_0^1 u\,dt + \lambda \int_0^1 t\, u(t)\, dt \rightarrow (2)$$

Assume that

$$c_1 = \int_0^1 u(t)\, dt$$

$$c_2 = \int_0^1 t\, u(t)\, dt$$

$\Big\} \rightarrow (3)$

(2) $\Rightarrow$

$$u(x) = x + \lambda (1+x)\, c_1 + \lambda c_2 \rightarrow (4)$$

$$u(t) = t + \lambda (1+t)\, c_1 + \lambda c_2$$

(3) $\Rightarrow$

$$c_1 = \int_0^1 (t + \lambda (1+t)\, c_1 + \lambda c_2)\, dt$$

$$c_1 = \int_0^1 t\, dt + \lambda c_1 \int_0^1 (1+t)\, dt + \lambda c_2 \int_0^1 dt$$

$$c_1 = \left[\frac{t^2}{2}\right]_0^1 + \lambda c_1 \left[t + \frac{t^2}{2}\right]_0^1 + \lambda c_2 \left[t\right]_0^1$$

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$$c_1 = \frac{1}{2} + \lambda c_1\left(1+\frac{1}{2}\right) + \lambda c_2(1)$$

$$c_1 = \frac{1}{2} + \frac{\lambda c_1 3}{2} + \lambda c_2$$

$$2c_1 = 1 + 3\lambda c_1 + 2\lambda c_2$$

$$c_1(2-3\lambda) - 2\lambda c_2 = 1 \longrightarrow (5).$$

(3) $\Rightarrow$

$$c_2 = \int_0^1 t\left[t + \lambda(1+t)c_1 + \lambda c_2\right]dt$$

$$= \int_0^1 t^2 dt + \lambda c_1 \int_0^1 (t+t^2)dt + \lambda c_2 \int_0^1 t\,dt$$

$$= \left[\frac{t^3}{3}\right]_0^1 + \lambda c_1\left[\frac{t^2}{2} + \frac{t^3}{3}\right]_0^1 + \lambda c_2\left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \lambda c_1\left(\frac{1}{2}+\frac{1}{3}\right) + \lambda c_2\left(\frac{1}{2}\right)$$

$$c_2 = \frac{1}{3} + \lambda c_1 \frac{5}{6} + \frac{\lambda c_2}{2}$$

$$6c_2 = 2 + 5\lambda c_1 + 3\lambda c_2$$

$$-5\lambda c_1 + (6-3\lambda)c_2 = 2 \longrightarrow (6)$$

Here eqn (5) & (6) are non-homogenous equation

By crammer's Rule

$$\Delta = D(\lambda) = \begin{vmatrix} 2-3\lambda & -2\lambda \\ -5\lambda & 6-3\lambda \end{vmatrix}$$

$$= (2-3\lambda)(6-3\lambda) - 10\lambda^2$$

$$= 12 - 6\lambda - 18\lambda + 9\lambda^2 - 10\lambda^2$$

$$\Delta = -\lambda^2 - 24\lambda + 12$$

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$$\Delta c_1 = \begin{vmatrix} 1 & -2\lambda \\ 2 & 6-3\lambda \end{vmatrix}$$

$$= 6-3\lambda+4\lambda$$

$$\Delta c_1 = 6+\lambda$$

$$\Delta c_2 = \begin{vmatrix} 2-3\lambda & 1 \\ -5\lambda & 2 \end{vmatrix}$$

$$= 4-6\lambda+5\lambda$$

$$\Delta c_2 = 4-\lambda$$

$$c_1 = \frac{\Delta c_1}{\Delta}$$

$$c_1 = \frac{6+\lambda}{-\lambda^2-24\lambda+12}$$

$$c_2 = \frac{\Delta c_2}{\Delta}$$

$$= \frac{4-\lambda}{-\lambda^2-24\lambda+12}$$

(4) $\Rightarrow$

$$u(x) = x+\lambda(1+x)\,c_1+\lambda c_2$$

$$= x+\frac{\lambda(1+x)(6+\lambda)}{-\lambda^2-24\lambda+12}+\frac{\lambda(4-\lambda)}{-\lambda^2-24\lambda+12}$$

$$= x+\frac{\lambda}{-\lambda^2-24\lambda+12}\left[(1+x)(6+\lambda)+(4-\lambda)\right]$$

$$= x+\frac{\lambda}{-\lambda^2-24\lambda+12}\left[6+\lambda+6x+\lambda x+4-\lambda\right]$$

$$u(x) = x+\frac{\lambda}{-\lambda^2-24\lambda+12}\left[10+6x+\lambda x\right].$$

Q. Solve the I.E

$$u(x) = e^x + \lambda \int_0^1 2e^x e^t \, u(t) \, dt.$$

Solution:

Given

$$u(x) = e^x + \lambda \int_0^1 2e^x e^t \, u(t) \, dt \rightarrow (1)$$

$$u(x) = e^x + 2\lambda e^x \int_0^1 e^t \, u(t) \, dt \rightarrow (2).$$

Assume that

$$c_1 = \int_0^1 e^t \, u(t) \, dt \rightarrow (3)$$

(2) $\Rightarrow$

$$u(x) = e^x + 2\lambda e^x c_1 \rightarrow (4).$$

$$u(t) = e^t + 2\lambda e^t c_1$$

(3) $\Rightarrow$

$$c_1 = \int_0^1 e^t [e^t + 2\lambda e^t c_1] \, dt$$

$$= \int_0^1 e^{2t} \, dt + 2\lambda c_1 \int_0^1 e^{2t} \, dt$$

$$= \left[\frac{e^{2t}}{2}\right]_0^1 + 2\lambda c_1 \left[\frac{e^{2t}}{2}\right]_0^1$$

$$= \frac{e^2 - e^0}{2} + \lambda c_1 [e^2 - e^0]$$

$$c_1 = \frac{e^2 - 1}{2} + \lambda c_1 [e^2 - 1]$$

$$c_1 [1 - \lambda(e^2 - 1)] = \frac{e^2 - 1}{2}$$

$$c_1[1-\lambda e^2+\lambda] = \frac{e^2-1}{2}$$

$$c_1 = \frac{e^2-1}{2[1-\lambda e^2+\lambda]}.$$

④ $\Rightarrow$.

$$u(x) = e^x + 2\lambda\, e^x\left[\frac{e^2-1}{2[1-\lambda e^2+\lambda]}\right]$$

$$u(x) = e^x + \lambda e^x\left[\frac{e^2-1}{(1-\lambda e^2+\lambda)}\right]$$

3. Solve the I.E

$$u(x) = \cos x + \lambda\int_0^{\pi}\sin(x-t)\,u(t)\,dt.$$

Solution:

Given

$$u(x) = \cos x + \lambda\int_0^{\pi}\sin(x-t)\,u(t)\,dt \rightarrow ①$$

$$u(x) = \cos x + \lambda\int_0^{\pi}[\sin x\cos t - \cos x\sin t]\,u(t)\,dt \rightarrow ②.$$

$$u(x) = \cos x + \lambda\sin x\int_0^{\pi}\cos t\,u(t)\,dt - \lambda\cos x\int_0^{\pi}\sin t\,u(t)\,dt. \rightarrow ②.$$

Assume that

$$\left.\begin{aligned} c_1 &= \int_0^{\pi}\cos t\,u(t)\,dt \\ c_2 &= \int_0^{\pi}\sin t\,u(t)\,dt. \end{aligned}\right\} \rightarrow ③.$$

② $\Rightarrow$ $u(x) = \cos x + \lambda\sin x\, c_1 - \lambda\cos x\, c_2 \rightarrow ④.$

$u(t) = \cos t + \lambda\sin t\, c_1 - \lambda\cos t\, c_2$

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③ ⇒

$$c_1 = \int_0^\pi \cos t\,[\cos t + \lambda \sin t\, c_1 - \lambda \cos t\, c_2]\,dt$$

$$c_1 = \int_0^\pi \cos^2 t + \lambda c_1 \cos t \sin t - \lambda \cos^2 t\, c_2]\,dt$$

$$= (1-\lambda c_2)\int_0^\pi \cos^2 t\,dt + \lambda c_1 \int_0^\pi \cos t \sin t\,dt$$

$$= \frac{(1-\lambda c_2)}{2}\left[\frac{t + \sin 2t}{2}\right]_0^\pi + \lambda c_1 \left[\frac{\sin 2t}{2}\right]_0^\pi$$

$$= \frac{(1-\lambda c_2)}{2}\left[t + \frac{\sin 2t}{2}\right]_0^\pi + \frac{\lambda c_1}{2}\left[\frac{-\cos 2t}{2}\right]_0^\pi$$

$$= \frac{(1-\lambda c_2)}{2}\left[t + \frac{\sin 2t}{2}\right]_0^\pi + \frac{\lambda c_1}{2}\left[\frac{-\cos 2t}{2}\right]_0^\pi$$

$$= \frac{(1-\lambda c_2)}{2}[\pi + 0] + \frac{\lambda c_1}{4}[-1+1]$$

$$= \frac{(1-\lambda c_2)(\pi)}{2} + \frac{\lambda c_1}{4}(0)$$

$$= \frac{(1-\lambda c_2)\pi}{2} + \frac{\lambda c_1}{2}$$

$$2c_1 = (1-\lambda c_2)\pi + \lambda c_1$$

$$c_1(2-\lambda) = (1-\lambda c_2)\pi$$

$$2c_1 = \pi - \lambda \pi c_2$$

$$2c_1 + \lambda \pi c_2 = \pi \longrightarrow (5).$$

④ ⇒

$$c_2 = \int_0^\pi \sin t\,[\cos t + \lambda \sin t\, c_1 - \lambda \cos t\, c_2]\,dt$$

$$= \int_0^\pi \sin t \cos t\,dt + \lambda c_1 \int_0^\pi \sin^2 t\,dt$$

$$+ \lambda c_2 \int_0^\pi -\cos t \sin t\,dt$$

$$= \lambda c_1 \int_0^{\pi} \sin^2 t \, dt + (1-\lambda c_2) \int_0^{\pi} \sin t \cos t \, dt$$

$$= \lambda c_1 \int_0^{\pi} \left[\frac{1-\cos 2t}{2}\right] dt + (1-\lambda c_2) \int_0^{\pi} \frac{\sin 2t}{2} \, dt$$

$$= \frac{\lambda c_1}{2} \left[t - \frac{\sin 2t}{2}\right]_0^{\pi} + \frac{(1-\lambda c_2)}{2} \left[-\frac{\cos 2t}{2}\right]_0^{\pi}$$

$$= \frac{\lambda c_1}{2} (\pi) - \frac{(1-\lambda c_2)}{4} (\cos 2\pi - \cos(0))$$

$$= \frac{\lambda c_1 (\pi)}{2} - \frac{(1-\lambda c_2)}{4} (1-1)$$

$$c_2 = \frac{\lambda c_1 \pi}{2}$$

$$2c_2 - \lambda c_1 \pi = 0 \longrightarrow (6)$$

Here eqn (5) & (6) are non-homogenous equation

By crammer's Rule

$$\Delta = D(\lambda) = \begin{vmatrix} 2 & -\lambda\pi \\ -\lambda\pi & 2 \end{vmatrix}$$

$$= 4 + \lambda^2 \pi^2$$

$$\Delta c_1 = \begin{vmatrix} \pi & \lambda\pi \\ 0 & 2 \end{vmatrix}$$

$$\Delta c_1 = 2\pi$$

$$\Delta c_2 = \begin{vmatrix} 2 & \pi \\ -\lambda\pi & 0 \end{vmatrix}$$

$$\Delta c_2 = \lambda \pi^2$$

$$C_1 = \frac{\Delta C_1}{\Delta}$$

$$= \frac{2\pi}{4+\lambda^2\pi^2}$$

$$C_2 = \frac{\Delta C_2}{\Delta}$$

$$= \frac{\lambda\pi^2}{4+\lambda^2\pi^2}$$

(1) $\Rightarrow$

$$u(x) = \cos x + \lambda \sin x \, C_1 - \lambda \cos x \, C_2$$

$$= \cos x + \lambda \sin x \frac{2\pi}{4+\lambda^2\pi^2} - \lambda \cos x \frac{\lambda\pi^2}{4+\lambda^2\pi^2}$$

$$= \cos x + \frac{\lambda\pi}{4+\lambda^2\pi^2}\left[2\sin x - \cos x \lambda\pi\right]$$

$$u(x) = \cos x + \frac{\lambda\pi}{4+\lambda^2\pi^2}\left[2\sin x - \cos x \lambda\pi\right].$$

4. Solve the I.E $y(x) = \sin x + \lambda \int_0^{2\pi} \cos(x+t)\, y(t)\, dt$

Solution:

Given that

$$y(x) = \sin x + \lambda \int_0^{2\pi} \cos(x+t)\, y(t)\, dt \rightarrow (1)$$

$$= \sin x + \lambda \int_0^{2\pi} \cos x \cos t - \sin x \sin t \; y(t)\, dt$$

$$y(x) = \sin x + \lambda \cos x \int_0^{2\pi} \cos t \, y(t)\, dt - \lambda \sin x \int_0^{2\pi} \sin t \, y(t)\, dt \rightarrow (2)$$

Assume that

$$c_1 = \int_0^{2\pi} \cos t \, y(t)\, dt$$
$$c_2 = \int_0^{2\pi} \sin t \, y(t)\, dt \rightarrow (3)$$

(2) $\Rightarrow$

$$y(x) = \sin x + \lambda \cos x \, c_1 - \lambda \sin x \, c_2 \rightarrow (4)$$
$$y(t) = \sin t + \lambda \cos t \, c_1 - \lambda \sin t \, c_2.$$

(3) $\Rightarrow$

$$c_1 = \int_0^{2\pi} \cos t \left[\sin t + \lambda \cos t \, c_1 - \lambda \sin t \, c_2\right] dt$$

$$= \int_0^{2\pi} \cos t \sin t \, dt + \lambda c_1 \int_0^{2\pi} \cos^2 t \, dt - \lambda c_2 \int_0^{2\pi} \cos t \sin t \, dt$$

$$= \lambda c_1 \int_0^{2\pi} \cos^2 t \, dt + (1 - \lambda c_2) \int_0^{2\pi} \cos t \sin t \, dt$$

$$= \lambda c_1 \int_0^{2\pi} \left[\frac{1+\cos 2t}{2}\right] dt + (1 - \lambda c_2) \int_0^{2\pi} \frac{\sin 2t}{2} \, dt$$

$$= \frac{\lambda c_1}{2} \left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{(1-\lambda c_2)}{2} \left[\frac{-\cos 2t}{2}\right]_0^{2\pi}$$

$$= \frac{\lambda c_1}{2} (2\pi) - \frac{(1-\lambda c_2)}{4} \left[\cos 4\pi - \cos(0)\right]$$

$$= \frac{\lambda c_1}{2} (2\pi) - \frac{(1-\lambda c_2)}{4} [1-1]$$

$$c_1 = \frac{\lambda c_1}{2} (2\pi)$$

$$c_1 = \lambda c_1 \pi$$

$$c_1 - \lambda c_1 \pi = 0$$

$$c_1 (1 - \lambda \pi) = 0 \rightarrow (5).$$

$$c_2 = \int_0^{2\pi} \sin t\,[\sin t + \lambda \cos t\, c_1 - \lambda \sin t\, c_2]\, dt$$

$$= \int_0^{2\pi} \sin^2 t\, dt + \lambda c_1 \int_0^{2\pi} \sin t \cos t\, dt - \lambda c_2 \int_0^{2\pi} \sin^2 t\, dt$$

$$= \lambda c_1 \int_0^{2\pi} \sin t \cos t\, dt + (1-\lambda c_2) \int_0^{2\pi} \sin^2 t\, dt$$

$$= \lambda c_1 \int_0^{2\pi} \frac{\sin 2t}{2}\, dt + (1-\lambda c_2) \int_0^{2\pi} \left[\frac{1-\cos 2t}{2}\right] dt$$

$$= \frac{\lambda c_1}{2} \left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + \frac{(1-\lambda c_2)}{2} \left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{\lambda c_1}{4} [-\cos 4\pi + \cos(0)] + \frac{(1-\lambda c_2)}{2} [2\pi + 0]$$

$$= \frac{\lambda c_1}{4} [-1+1] + \frac{(1-\lambda c_2)}{2} (2\pi)$$

$$= \frac{(1-\lambda c_2)}{2} (2\pi)$$

$$c_2 = (1-\lambda c_2)(\pi)$$

$$c_2 = \pi - \lambda \pi c_2$$

$$c_2 + \lambda \pi c_2 = \pi$$

$$c_2(1+\lambda\pi) = \pi \longrightarrow (6).$$

Here eqn (5), (6) are non-homogenous equation

By Crammer's Rule

$$\Delta = D(\lambda) = \begin{vmatrix} 1-\lambda\pi & 0 \\ 0 & (1+\lambda\pi) \end{vmatrix}$$

$$= (1-\lambda\pi)(1+\lambda\pi)$$

$$= 1 + \lambda\pi - \lambda\pi - \lambda^2\pi^2$$

$\Delta = 1-\lambda^2\pi^2$

$$\Delta C_1 = \begin{vmatrix} 0 & 0 \\ \pi & (1+\lambda\pi) \end{vmatrix}$$

$\Delta C_1 = 0$

$$\Delta C_2 = \begin{vmatrix} 1-\lambda\pi & 0 \\ 0 & \pi \end{vmatrix}$$

$\Delta C_2 = \pi - \lambda\pi^2$

$C_1 = \frac{\Delta C_1}{\Delta}$

$C_1 = \frac{0}{1-\lambda^2\pi^2}$

$C_1 = 0$

$C_2 = \frac{\Delta C_2}{\Delta}$

$C_2 = \frac{\pi - \lambda\pi^2}{1-\lambda^2\pi^2}$

(A) $\Rightarrow$

$y(x) = \sin x + \lambda \cos x \, C_1 - \lambda \sin x \, C_2$

$= \sin x - \lambda \sin x \, \frac{\pi - \lambda\pi^2}{1-\lambda^2\pi^2}$

$= \sin x - \lambda\pi \left[\frac{\sin x (1-\lambda\pi)}{1-\lambda^2\pi^2}\right]$

$= \sin x \left[1 - (\cancel{\lambda}\pi \right.$

$y(x) = \sin x \left[1 - \lambda\pi \left(\frac{1-\lambda\pi}{1-\lambda^2\pi^2}\right)\right]$

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5 show that the Integral Equation

$$u(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+t)\, u(t)\, dt$$

posses no solution for $f(x) = x$ but it posses infinitely many solution for $f(x) = 1$.

Solution:

Given

case (i)

Let $f(x) = x$.

$$u(x) = x + \frac{1}{\pi}\int_0^{2\pi} \sin(x+t)\, u(t)\, dt \rightarrow (1)$$

$$= x + \frac{1}{\pi}\int_0^{2\pi} [\sin x \cos t - \cos x \sin t]\, u(t)\, dt$$

$$= x + \frac{1}{\pi}\sin x \int_0^{2\pi} \cos t\, u(t)\, dt - \frac{1}{\pi}\cos x \int_0^{2\pi} \sin t\, u(t)\, dt \rightarrow (2)$$

Assume that

$$\left.\begin{aligned} c_1 &= \int_0^{2\pi} \cos t\, u(t)\, dt \\ c_2 &= \int_0^{2\pi} \sin t\, u(t)\, dt \end{aligned}\right\} \rightarrow (3)$$

(2) $\Rightarrow$

$$u(x) = x + \frac{1}{\pi}\sin x\, c_1 - \frac{1}{\pi}\cos x\, c_2 \rightarrow (4)$$

$$u(t) = t + \frac{1}{\pi}\sin t\, c_1 - \frac{1}{\pi}\cos t\, c_2$$

(3) $\Rightarrow$

$$c_1 = \int_0^{2\pi} \cos t\left(t + \tfrac{1}{\pi}\sin t\, c_1 - \tfrac{1}{\pi}\cos t\, c_2\right) dt$$

$$= \int_0^{2\pi} t\cos t + \frac{1}{\pi} c_1 \cos t \sin t + \frac{1}{\pi} c_2 \cos^2 t \, dt$$

$$= \int_0^{2\pi} t\cos t \, dt + \frac{c_1}{\pi} \int_0^{2\pi} \cos t \sin t \, dt$$

$$+ \frac{c_2}{\pi} \int_0^{2\pi} \cos^2 t \, dt.$$

$$= \int_0^{2\pi} t\cos t \, dt + \frac{c_1}{\pi} \int_0^{2\pi} \left[\frac{\sin 2t}{2}\right] dt$$

$$+ \frac{c_2}{\pi} \int_0^{2\pi} \left[\frac{1+\cos 2t}{2}\right] dt$$

$\int u\,dv = uv - u'v_1$

$u = t$ $\quad$ $dv = \cos t$

$u' = 1$ $\quad$ $v = \sin t$

$u'' = 0$ $\quad$ $v_1 = -\cos t$

$$= \left[t\sin t + \cos t\right]_0^{2\pi} + \frac{c_1}{2\pi} \int_0^{2\pi} \sin 2t \, dt$$

$$+ \frac{c_2}{2\pi} \int_0^{2\pi} (1+\cos 2t) dt$$

$$= (1-1) + \frac{c_1}{2}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + \frac{c_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{c_2}{2\pi}(2\pi)$$

$c_1 = c_2$

$c_1 - c_2 = 0 \rightarrow (5)$

$$c_2 = \int_0^{2\pi} \sin t \left[t + \frac{\sin t}{\pi} c_1 + \frac{\cos t \, c_2}{\pi}\right] dt$$

$$= \int_0^{2\pi} t\sin t + \frac{c_1}{\pi} \sin^2 t + \frac{c_2}{\pi} \sin t \cos t \, dt$$

$$= \int_0^{2\pi} t \sin t \, dt + \frac{c_1}{\pi} \int_0^{2\pi} \sin^2 t \, dt + \frac{c_2}{\pi} \int_0^{2\pi} \sin t \cos t \, dt$$

By using Bernoulli's formula.

$\int u\,dv = uv - u'v_1$

$u = t$ $\quad dv = \sin t$

$u' = 1$ $\quad v = -\cos t$

$u'' = 0$ $\quad v_1 = -\sin t.$

$$= \left[-t\cos t + \sin t\right]_0^{2\pi} + \frac{c_1}{\pi}\int_0^{2\pi}\left[\frac{1-\cos 2t}{2}\right]dt + \frac{c_2}{\pi}\int_0^{2\pi}\frac{\sin 2t}{2}\,dt$$

$$= (-2\pi) + \frac{c_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{c_2}{2\pi}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi}$$

$$= (-2\pi) + \frac{c_1}{2\pi}(2\pi) + \frac{c_2}{2\pi}(1-1)$$

$c_2 = (-2\pi) + c_1$

$-c_1 + c_2 = -2\pi$

$c_1 - c_2 = 2\pi \longrightarrow (6).$

Here equation (5) & (6) are non-homogenous equation.

$$\Delta = D(\lambda) = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}$$

$\Delta = 0.$

$\therefore$ The equation has no solution

$\therefore$ It posses no solution for $f(x) = x$.

case (ii) $f(x) = 1$

$$u(x) = 1 + \frac{1}{\pi}\int_0^{2\pi}\sin(x+t)\,u(t)\,dt. \rightarrow (1)$$

$$= 1 + \frac{1}{\pi}\int_0^{2\pi}[\sin x\cos t + \cos x\sin t]\,dt$$

$$= 1 + \frac{1}{\pi}\sin x\int_0^{2\pi}\cos t\,dt + \frac{1}{\pi}\cos x\int_0^{2\pi}\sin t\,dt \rightarrow (2)$$

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Assume that

$$c_1 = \int_0^{2\pi} \cos t \, u(t)\, dt$$
$$c_2 = \int_0^{2\pi} \sin t \, u(t)\, dt \quad \rightarrow (9)$$

(8) $\Rightarrow$

$$u(x) = 1 + \frac{1}{\pi}\sin x \int_0^{2\pi} c_1 + \frac{1}{\pi}\cos x \, c_2 \rightarrow (10)$$
$$u(t) = 1 + \frac{1}{\pi}\sin t \, c_1 + \frac{1}{\pi}\cos t \, c_2$$

(9) $\Rightarrow$

$$c_1 = \int_0^{2\pi} \cos t \left[1 + \frac{1}{\pi}\sin t \, c_1 + \frac{1}{\pi}\cos t \, c_2\right] dt$$
$$= \int_0^{2\pi} \left[\cos t + \frac{c_1}{\pi}\sin t \cos t + \frac{c_2}{\pi}\cos^2 t\right] dt$$
$$= \int_0^{2\pi} \cos t \, dt + \frac{c_1}{\pi}\int_0^{2\pi} \sin t \cos t \, dt + \frac{c_2}{\pi}\int_0^{2\pi} \cos^2 t \, dt$$
$$= \underbrace{\left[\sin t\right]_0^{2\pi}}_{0} + \frac{c_1}{\pi}\int_0^{2\pi} \frac{\sin 2t}{2} dt + \frac{c_2}{\pi}\int_0^{2\pi} \left[\frac{1+\cos 2t}{2}\right] dt$$
$$= \frac{c_1}{2\pi}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + \frac{c_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$c_1 = \frac{c_2}{2\pi}(2\pi)$$

$$c_1 - c_2 = 0 \rightarrow (11).$$

$$c_2 = \int_0^{2\pi} \sin t \left(1 + \frac{1}{\pi}\sin t \, c_1 + \frac{1}{\pi}\cos t \, c_2\right) dt$$
$$= \int_0^{2\pi} \left(\sin t + \frac{c_1}{\pi}\sin^2 t + \frac{c_2}{\pi}\sin t \cos t\right) dt$$
$$= \int_0^{2\pi} \sin t \, dt + \frac{c_2}{2\pi}\int_0^{2\pi} \left[\sin 2t\right]_0 + \frac{c_1}{2\pi}\int_0^{2\pi} \left[1 - \cos 2t\right]$$
$$= \underbrace{\left[-\cos t\right]_0^{2\pi}}_{0} + \frac{c_2}{4\pi}\underbrace{\left[-\cos 2t\right]_0^{2\pi}}_{0} + \frac{c_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$c_2 = \frac{c_1}{2\pi}(2\pi)$$

$$c_1 - c_2 = 0 \rightarrow (12)$$

Here eqn (11) & (12) are homogenous equation

$$\Delta = D(\lambda) = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0$$

~~Therefore (9) is homogenous~~ and $D(\lambda) = 0$

~~The~~ I.E possess infinitely non-zero solution

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6 Solve the I.E $u(x) = f(x) + \lambda \int_0^{2\pi} \sin x \cos t \; u(t)\, dt$

solution:

So Given

$$u(x) = f(x) + \lambda \int_0^{2\pi} \sin x \cos t \; u(t)\, dt \rightarrow ①$$

$$u(x) = f(x) + \lambda \sin x \int_0^{2\pi} \cos t \; u(t)\, dt \rightarrow ②$$

Assume that

$$c_1 = \int_0^{2\pi} \cos t \; u(t)\, dt. \rightarrow ③$$

② ⇒

$$u(x) = f(x) + \lambda \sin x \, c_1 \rightarrow ④$$

$$u(t) = f(t) + \lambda \sin t \, c_1$$

③ ⇒

$$c_1 = \int_0^{2\pi} \cos t \left(f(t) + \lambda \sin t \, c_1 \right) dt$$

$$c_1 = \int_0^{2\pi} f(t) \cos t \, dt + \lambda c_1 \int_0^{2\pi} \sin t \cos t \, dt$$

$$c_1 = \int_0^{2\pi} f(t) \cos t \, dt + \lambda c_1 \int_0^{2\pi} \left[\frac{\sin 2t}{2} \right] dt$$

$$= \int_0^{2\pi} f(t) \cos t \, dt + \frac{\lambda c_1}{2} \left[\cancel{-\frac{\cos 2t}{2}} \right]_0^{2\pi}$$

$$c_1 = \int_0^{2\pi} f(t) \cos t \, dt$$

④ ⇒

$$u(x) = f(x) + \lambda \sin x \int_0^{2\pi} f(t) \cos t \, dt.$$

7. Solve the I.E $u(x) = f(x) + \lambda \int_{-1}^{1} (xt + x^2t^2) u(t)\, dt$ also find its resolvent Kernal

solution:

Given that,

$$u(x) = f(x) + \lambda \int_{-1}^{1} (xt + x^2t^2) u(t)\, dt \rightarrow (1)$$

$$u(x) = f(x) + x\lambda \int_{-1}^{1} t\, u(t)\, dt + x^2 \lambda \int_{-1}^{1} t^2 u(t)\, dt \rightarrow (2)$$

Assume that.

$$\left.\begin{aligned} c_1 &= \int_{-1}^{1} t\, u(t)\, dt \\ c_2 &= \int_{-1}^{1} t^2 u(t)\, dt \end{aligned}\right\} \rightarrow (3)$$

(2) $\Rightarrow$

$$u(x) = f(x) + x\lambda c_1 + x^2 \lambda c_2 \rightarrow (4)$$

$$u(t) = f(t) + t\lambda c_1 + t^2 \lambda c_2$$

(3) $\Rightarrow$ $c_1 = \int_{-1}^{1} t \left[f(t) + t\lambda c_1 + t^2 \lambda c_2 \right] dt$

$$= \int_{-1}^{1} t f(t)\, dt + \lambda c_1 \int_{-1}^{1} t^2\, dt + \lambda c_2 \int_{-1}^{1} t^3\, dt$$

$$c_1 = \int_{-1}^{1} t f(t)\, dt + \lambda c_1 \left[\frac{t^3}{3}\right]_{-1}^{1} + \lambda c_2 \cancel{\left[\frac{t^4}{4}\right]_{-1}^{1}}$$

$$= \int_{-1}^{1} t f(t)\, dt + \frac{\lambda c_1 2}{3}$$

$$c_1\left(1 - \frac{2\lambda}{3}\right) = \int_{-1}^{1} t f(t)\, dt.$$

$$c_1 = \frac{3}{3 - 2\lambda} \int_{-1}^{1} t f(t)\, dt.$$

$\Delta = 240 - 120\lambda - \dots$

$\Delta C_1 = 60 + \lambda$

$\Delta C_2 = 80$

(5) $\Rightarrow$

$$C_2 = \int_{-1}^{1} t^2 \left[f(t) + \lambda t C_1 + \lambda t^2 C_2\right] dt$$

$$= \int_{-1}^{1} t^2 f(t)\,dt + \lambda C_1 \int_{-1}^{1} t^3\,dt + \lambda C_2 \int_{-1}^{1} t^4\,dt$$

$$= \int_{-1}^{1} t^2 f(t)\,dt + \lambda C_1 \left[\frac{t^4}{4}\right]_{-1}^{1} + \lambda C_2 \left[\frac{t^5}{5}\right]_{-1}^{1}$$

$$= \int_{-1}^{1} t^2 f(t)\,dt + \frac{\lambda C_2 2}{5}$$

$$C_2\left(1 - \frac{2\lambda}{5}\right) = \int_{-1}^{1} t^2 f(t)\,dt$$

$$C_2 = \frac{5}{5-2\lambda} \int_{-1}^{1} t^2 f(t)\,dt.$$

(4) $\Rightarrow$

$$u(x) = f(x) + \frac{\lambda x 3}{3-2\lambda} \int_{-1}^{1} t f(t)\,dt + \frac{\lambda x^2 5}{5-2\lambda} \int_{-1}^{1} t^2 f(t)\,dt$$

$$u(x) = f(x) + \int_{-1}^{1} \left[\frac{3\lambda x t}{3-2\lambda} + \frac{5\lambda x^2 t^2}{5-2\lambda}\right] f(t)\,dt.$$

$\therefore$ The Resoluvent kernal $\dfrac{3\lambda x t}{3-2\lambda} + \dfrac{5\lambda x^2 t^2}{5-2\lambda}$

(4) $\Rightarrow (20-5\lambda)c_1 - 4\lambda c_2 = 5$

(5) $\Rightarrow -4\lambda c_1 + (12-3\lambda) c_2 = 4$

8. Solve the I.E $u(x) = x + \lambda \int_0^1 (xt^2 + tx^2) u(t)\,dt$.

Solution:

Given that

$$u(x) = x + \lambda \int_0^1 (xt^2 + tx^2)\, u(t)\,dt \rightarrow (1)$$

$$u(x) = x + \lambda x \int_0^1 t^2 u(t)\,dt + \lambda x^2 \int_0^1 t\,u(t)\,dt \rightarrow (2).$$

Assume that

$$\left.\begin{aligned} c_1 &= \int_0^1 t^2 u(t)\,dt \\ c_2 &= \int_0^1 t\,u(t)\,dt \end{aligned}\right\} \rightarrow (3).$$

(2) $\Rightarrow$

$$u(x) = x + \lambda x c_1 + \lambda x^2 c_2 \rightarrow (4)$$

$$u(t) = t + \lambda t c_1 + \lambda t^2 c_2$$

(3) $\Rightarrow$

$$c_1 = \int_0^1 t^2 (t + \lambda t c_1 + \lambda t^2 c_2)\,dt.$$

$$= \int_0^1 t^3\,dt + \lambda c_1 \int_0^1 t^3\,dt + \lambda c_2 \int_0^1 t^4\,dt$$

$$= \left[\frac{t^4}{4}\right]_0^1 + \lambda c_1 \left[\frac{t^4}{4}\right]_0^1 + \lambda c_2 \left[\frac{t^5}{5}\right]_0^1$$

$$= \frac{1}{4}(1-0) + \frac{\lambda c_1}{4}(1-0) + \frac{\lambda c_2}{5}(1-0)$$

$$c_1 = \frac{1}{4} + \frac{\lambda c_1}{4} + \frac{\lambda c_2}{5}$$

$$c_1 = \frac{1}{4}(1 + \lambda c_1) + \frac{\lambda c_2}{5}$$

$$c_1 - \frac{1}{4}(1 + \lambda c_1) - \frac{\lambda c_2}{5} = 0.$$

$$c_1\left(1 - \frac{1}{4}(1 + \lambda)\right) - \frac{\lambda c_2}{5} = 0.$$

$$c_1\left(1 - \frac{1}{4} - \frac{\lambda}{4}\right) - \frac{\lambda c_2}{5} = 0$$

$$c_1\left(\frac{3}{4} - \frac{\lambda c_1}{4} - \frac{\lambda c_2}{5} = 0.\right.$$

$\cancel{20}(\cancel{4}-\cancel{\lambda})c_1 - \cancel{5} - \cancel{4\lambda c_2} = \cancel{0}$

$(20-5\lambda)c_1 - 4\lambda c_2 = 5. \rightarrow (5)$

$(3) \Rightarrow.\ c_2 = \int_0^1 t(t + \lambda t c_1 + \lambda t^2 c_2)\,dt.$

$= \int_0^1 t^2\,dt + \lambda c_1 \int_0^1 t^2\,dt + \lambda c_2 \int_0^1 t^3\,dt$

$= \left[\frac{t^3}{3}\right]_0^1 + \lambda c_1 \left[\frac{t^3}{3}\right]_0^1 + \lambda c_2 \left[\frac{t^4}{4}\right]_0^1$

$= \frac{1}{3}(1-0) + \frac{\lambda c_1}{3}(1-0) + \frac{\lambda c_2}{4}(1-0)$

$c_2 = \frac{1}{3} + \frac{\lambda c_1}{3} + \frac{\lambda c_2}{4}$

$= \frac{1}{3}(1+\lambda c_1) + \frac{\lambda c_2}{4}$

$= \frac{4(1+\lambda c_1) + 3\lambda c_2}{12}$

$= \frac{4 + 4\lambda c_1 + 3\lambda c_2}{12}$

$12c_2 = 4 + 4\lambda c_1 + 3\lambda c_2$

$12c_2 - 4 - 4\lambda c_1 - 3\lambda c_2 = 0$

$c_2(12-3\lambda) - 4\lambda c_1 = 4. \rightarrow (6)$

Here eqn (5), (6) are non-homogeneous equation

By Crammer's Rule

$D = D(\lambda) = \begin{vmatrix} (20-5\lambda) & -4\lambda \\ -4\lambda & (12-3\lambda) \end{vmatrix}$

$D = (20-5\lambda)(12-3\lambda) - 16\lambda$

$= 240 - 60\lambda - 60\lambda - 16\lambda^2 + 15\lambda^2$

$= 240 - 120\lambda - 16\lambda + 15\lambda^2$

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$\Delta = 240 - 120\lambda - \lambda^2$

$\Delta c_1 = \begin{vmatrix} 5 & -4\lambda \\ 4 & (12-3\lambda) \end{vmatrix}$

$= 60 - 15\lambda + 16\lambda$

$\Delta c_1 = 60 + \lambda$

$\Delta c_2 = \begin{vmatrix} (20-5\lambda) & 5 \\ -4\lambda & 4 \end{vmatrix}$

$= 80 - 20\lambda + 20\lambda$

$\Delta c_2 = 80$

$c_1 = \dfrac{\Delta c_1}{\Delta}$

$c_1 = \dfrac{60+\lambda}{240-120\lambda-\lambda^2}$

$c_2 = \dfrac{\Delta c_2}{\Delta}$

$c_2 = \dfrac{80}{240-120\lambda-\lambda^2}$

(1) ⇒

$u(x) = x + \lambda x \dfrac{60+\lambda}{240-120\lambda-\lambda^2} + \lambda x^2 \dfrac{80}{240-120\lambda-\lambda^2}$

$= x + \dfrac{\lambda x}{240-120\lambda-\lambda^2}\left[60+\lambda + x\,80\right]$

$u(x) = x + \dfrac{\lambda x}{240-120-\lambda^2}\left[60+\lambda+80x\right]$

Solution of Fredholm I.E of 2nd kind by successive substitution

Statement:

Consider the Fredholm I.E of 2nd kind

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \rightarrow (1)$$

i) If $k(x,t) \neq 0$ is real and continuous in a rectangle $R\ (a \le x \le b,\ a \le t \le b)$ also $k(x,t)$ is bounded by M in R

ie) $|k(x,t)| \le M$ in $R \rightarrow (2)$

ii) If $f(x) \neq 0$ is real and continuous in the interval I, for which $a \le x \le b$

ie) $|f(x)| \le N$ in $I \rightarrow (3)$

iii) λ is constant such that

$$|\lambda| \le \frac{1}{M(b-a)} \rightarrow (4)$$

then eqn (1) has a unique continuous soln in I, which is given by

$$u(x) = f(x) + \lambda \int_a^b k(x,t) f(t)\, dt + \lambda^2 \int_a^b k(x,t) \int_a^b k(t,t_1) f_1(t_1)\, dt_1\, dt + \ldots \rightarrow (5)$$

proof:

The given eqn (1) can be written as follows

$$u(x) = f(x) + \lambda \int_a^b k(x,t_1)\, u(t_1)\, dt_1 \rightarrow (6)$$

$$u(t) = f(t) + \lambda \int_a^b k(t,t_1)\, u(t_1)\, dt_1 \rightarrow (7)$$

Sub (7) in (1), we get

$$u(x) = f(x) + \lambda \int_a^b K(x,t) \left[f(t) + \lambda \int_a^b K(t,t_1) u(t_1)\, dt_1 \right] dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1)\, u(t_1)\, dt_1\, dt \rightarrow (8)$$

Now eqn (7) can be written as

$$u(t) = f(t) + \lambda \int_a^b K(t,t_2)\, u(t_2)\, dt_2 \rightarrow (9)$$

$$u(t_1) = f(t_1) + \lambda \int_a^b K(t_1,t_2)\, u(t_2)\, dt_2 \rightarrow (10)$$

Sub (10) in (8)

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) \left[f(t_1) + \lambda \int_a^b K(t_1,t_2)\, u(t_2)\, dt_2 \right] dt_1\, dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f_1(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1)\, dt_1\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) \int_a^b K(t_1,t_2)\, u(t_2)\, dt_2\, dt_1\, dt$$

Proceeding in the same way, we get

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f_1(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1)\, dt_1\, dt + \dots + \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \dots \int_a^b K(t_{n-2},t_{n-1})\, f(t_{n-1})\, dt_{n-1}\, dt + R_{n+1}(x) \rightarrow (12)$$

where $R_{n+1}(x) = \lambda^{n+1} \int_a^b K(x,t) \int_a^b K(t,t_1) \cdots \int_a^b K(t_{n-1},t_n)$

$u(t_n)\, dt_n \ldots dt \rightarrow (13)$

consider the infinite series

$$f(x) + \lambda \int_a^b K(x,t) f_1(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f t_1\, dt_1\, dt + \cdots \rightarrow (14)$$

Let $S_n(x)$ be the general term of the series (14), we get

$$S_n(x) = \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \cdots \int_a^b K(t_{n-2},t_{n-1}) f(t_{n-1})\, dt_{n-1} \cdots dt\, dt \rightarrow (15)$$

$$|S_n(x)| = \left| \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \cdots \int_a^b K(t_{n-2},t_{n-1}) f(t_{n-1})\, dt_{n-1} \cdots dt_1\, dt \right|$$

$|S_n(x)| \leq |\lambda^n|\, N M^n \frac{(x-a)^n}{n!}$ since $a \leq x \leq b$

$$|S_n(x)| \leq |\lambda^n|\, N M^n \frac{(b-a)^n}{n!} \rightarrow (16)$$

The series (14) converges only when

$$|\lambda| M (b-a) < 1 \Rightarrow |\lambda| < \frac{1}{M(b-a)}$$

which holds by assumption (iii)

Thus the infinite series (14) converges absolutely and uniformly when condition (4) holds.

Hence the given I.E has a unique solution given by (2).

Again.

$$|R_{n+1}(x)| = \left|\lambda^{n+1}\int_a^{x\,b} K(x,t)\int_a^{t\,b} K(t,t_1)\ldots\ldots\right.$$

$$\left.\int_a^{t_{n-1}\,b} K(t_{n-1},t_n)\,u(t_n)\,dt_n\ldots dt_1\,dt\right|$$

$$\le |\lambda^{n+1}|\, P\, m^{n+1}(b-a)^{n+1} \qquad \because |u(x)| \le P$$

Now using ④, we get.

$$\lim_{n\to\infty} |R_{n+1}(x)| = 0$$

Hence the function $u(x)$ satisfying ⑫ is continuous function given by the series ⑭.

Solution of volterra I.E of 2nd kind by succesive substitution

Statement:

Consider the volterra I.E of 2nd kind

$$u(x) = f(x) + \lambda\int_a^x K(x,t)\,u(t)\,dt \rightarrow ①.$$

(i) If $K(x,t) \neq 0$ is real and continuous in a rectangle $R\,(a \le x \le b,\ a \le t \le b)$ also $K(x,t)$ is bounded by M in R.

i.e. $|K(x,t)| \le M$ in $R \rightarrow ②$.

(ii) If $f(x) \neq 0$ is real and continuous in the interval I, for which $a \le x \le b$

i.e. $|f(x)| \le N$ in $I \rightarrow ③$.

(iii) λ is constant such that

$$|\lambda| \le \frac{1}{m(b-a)} \rightarrow ④.$$

then eqn ① has the unique continuous solution on I, which is given by

$$u(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt$$

$$+ \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) f(t_1)\, dt_1\, dt + \dots \rightarrow ⑥$$

proof:

The given eqn ① can be written as follows.

$$u(x) = f(x) + \lambda \int_a^x K(x, t_1)\, u(t_1)\, dt_1$$

$$u(t) = f(t) + \lambda \int_a^t K(t, t_1)\, u(t_1)\, dt_1 \rightarrow ⑦$$

Sub ⑦ in ①, we get.

$$u(x) = f(x) + \lambda \int_a^x K(x,t) \left[f(t) + \lambda \int_a^t K(t,t_1)\, u(t_1)\, dt_1 \right] dt$$

$$= f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) u(t_1)\, dt_1\, dt \rightarrow ⑧$$

Now eqn ⑦ can be written as

$$u(t) = f(t) + \lambda \int_a^t K(t,t_2)\, u(t_2)\, dt_2 \rightarrow ⑨$$

$$u(t_1) = f(t_1) + \lambda \int_a^{t_1} K(t_1, t_2)\, u(t_2)\, dt_2 \rightarrow ⑩$$

Sub ⑩ in ⑧

$$u(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) \left[f(t_1) + \lambda \int_a^{t_1} K(t_1,t_2)\, u(t_2)\, dt_2 \right] dt_1\, dt$$

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$$= f(x) + \lambda \int_a^x K(x,t) f_1(t)\, dt + \lambda^2 \int_a^x K(x,t)$$

$$\int_a^t K(t,t_1) f(t_1)\, dt_1\, dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1)$$

$$\int_a^{t_1} K(t_1,t_2)\, u(t_2)\, dt_2\, dt_1\, dt$$

proceeding in the same way, we get

$$u(x) = f(x) + \lambda \int_a^x K(x,t) f_1(t)\, dt + \lambda^2 \int_a^x K(x,t)$$

$$\int_a^t K(t,t_1) f(t_1)\, dt_1\, dt + \cdots + \lambda^n \int_a^x K(x,t)$$

$$\int_a^{t_1} K(t_0, t_1) \cdots \int_a^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1})\, dt_{n-1}$$

$$dt + R_{n+1}(x) \to (12)$$

where

$$R_{n+1}(x) = \lambda^{n+1} \int_a^x K(x,t) \int_a^t K(t,t_1) \cdots \int_a^{t_{n-1}} K(t_{n-1}, t_n)$$

$$u(t_n)\, dt_n \cdots dt \to (13).$$

consider the infinite series

$$f(x) + \lambda \int_a^x K(x,t) f_1(t)\, dt + \lambda^2 \int_a^x K(x,t)$$

$$\int_a^t K(t, t_1) f(t_1)\, dt_1\, dt + \cdots \to (14).$$

Let $S_n(x)$ be the general term of the series (14), we get

$$S_n(x) = \lambda^n \int_a^x K(x,t) \int_a^t K(t,t_1) \cdots$$

$$\int_a^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1})\, dt_{n-1} \cdots dt_1\, dt$$

$$\to (15)$$

$$|S_n(x)| = \left|\lambda^n \int_a^x K(x,t) \int_a^t K(t,t_1) \cdots \int_a^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1})\, dt_{n-1} \cdots dt_1\, dt\right|$$

$$|S_n(x)| \le \lambda^n N M^n \frac{(b-a)^n}{n!} \rightarrow (16)$$

The series (14) converges only when

$|\lambda| M(b-a) < 1 \Rightarrow |\lambda| < \dfrac{1}{M(b-a)}$

which holds by assumption (iii)

Thus the infinite series (14) converges absolutely and uniformly when condition (4) holds.

Hence the given I.E has a unique solution given by (2)

Again

$$|R_{n+1}(x)| = \left|\lambda^{n+1} \int_a^x K(x,t) \int_a^t K(t,t_1) \cdots \int_a^{t_{n-1}} K(t_{n-1}, t_n)\, u(t_n)\, dt_n \cdots dt_1\, dt\right|$$

$$\le |\lambda^{n+1}|\, P\, M^{n+1} \frac{(b-a)^{n+1}}{(n+1)!} \qquad \because |u(x)| \le P$$

Now using (4), we get

$$\lim_{n\to\infty} |R_{n+1}(x)| = 0$$

Hence the function $u(x)$ satisfying (12) is continuous function given by the series (14).

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3. Theorem:

Let ~~$R(x,t,\lambda)$~~ $R(x,t,\lambda)$ be the reciprocal kernal of Fredholm I.E

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\,u(t)\,dt$$, then

prove that

$$R(x,t,\lambda) = k(x,t) + \lambda \int_a^b k(x,s)\,R(s,t,\lambda)\,ds$$

proof:

By defn of resolvant or reciprocal kernal

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) \rightarrow (1)$$

where,

$$k_1(x,t) = k(x,t)$$

$$k_m(x,t) = \int_a^b k(x,s)\,k_{m-1}(s,t)\,ds \rightarrow (2)$$

$$R(x,t,\lambda) = k(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_a^b k(x,s)\,k_{m-1}(s,t)\,dt$$

put $m = m+1$

$$= k(x,t) + \sum_{m=1}^{\infty} \lambda^{m} \int_a^b k(x,s)\,k_m(s,t)\,ds$$

$$= k(x,t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b k(x,s)\,k_m(s,t)\,ds$$

$$R(x,t,\lambda) = k(x,t) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)\right] k(x,s)\,ds$$

$$R(x,t,\lambda) = k(x,t) + \lambda \int_a^b R(s,t,\lambda)\,k(x,s)\,ds.$$

1 Find the iterated kernal of the following function

$K(x,t) = e^x \cos t,\ a=0,\ b=\pi$

Solution:

Given that

$K(x,t) = e^x \cos t,\ a=0,\ b=\pi$

By defn of iterated kernal

$K_1(x,t) = K(x,t)$

$K_n(x,t) = \int_a^b K(x,s) K_{n-1}(s,t)\,ds,\ n=2,3,\ldots \rightarrow (1)$

put $n=2$ in (1)

$K_2(x,t) = \int_a^b K(x,s) K_1(s,t)\,ds$

$= \int_a^b e^x \cos s\, e^s \cos t\, ds$

$= e^x \cos t \int_0^\pi e^s \cos s\, ds$

$= e^x \cos t \left[\frac{e^s}{1+1} (\cos s + \sin s) \right]$

$= e^x \cos t \left[\left(\frac{e^\pi}{2} (-1+0) \right) - \frac{1}{2} (1+0) \right]$

$= e^x \cos t \left[\frac{-e^\pi - 1}{2} \right]$

$K_2(x,t) = -e^x \cos t \left[\frac{e^\pi + 1}{2} \right]$

put $n=3$ in (1)

$K_3(x,t) = \int_a^b K(x,s) K_2(s,t)\,ds$

$= \int_0^\pi e^x \cos s \cdot -e^s \cos t \left(\frac{e^\pi + 1}{2} \right) ds$

$$= -e^x \cos t \left(\frac{e^\pi+1}{2}\right) \int_0^\pi e^s \cos s \, ds.$$

$$= -e^x \cos t \left(\frac{e^\pi+1}{2}\right) \left(-\left(\frac{e^\pi+1}{2}\right)\right)$$

$$K_3(x,t) = e^x \cos t \left(\frac{e^\pi+1}{2}\right)^2$$

$\vdots$

In general

$$K_n(x,t) = (-1)^{n-1} e^x \cos t \left(\frac{e^\pi+1}{2}\right)^{n-1}$$

2 Find the iterated kernal of the function $K(x,t) = \sin(x-2t)$, $0 \le x \le 2\pi$, $0 \le t \le 2\pi$

Solution:

Given that

$K(x,t) = \sin(x-2t)$, $0 \le x \le 2\pi$, $0 \le t \le 2\pi$

general formula for iterated kernal

$$K_1(x,t) = K(x,t)$$

$$K_n(x,t) = \int_a^b K(x,s) K_{n-1}(s,t) \, ds \rightarrow (1)$$

$n = 2, 3, \dots$

$$K_1(x,t) = \sin(x-2t)$$

put $n=2$

$$(1) \Rightarrow K_2(x,t) = \int_0^{2\pi} \sin(x-2s)\sin(s-2t) \, ds$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(x-2s-s+2t) - \cos(x-2s+s-2t)] \, ds$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\cos(x-3s+2t) - \cos(x-s-2t)\right]ds$$

$$= \frac{1}{2}\left[\frac{\sin(x-3s+2t)}{-3} - \frac{\sin(x-s-2t)}{-1}\right]_0^{2\pi}$$

$K_2(x,t) = 0$

put $n=3$

① ⇒, $K_3(x,t) = 0$.

and so on.

3. Find the resolvant kernal of the following function $K(x,t) = (1+x)(1-t)$, $a=-1$, $b=0$

Solution:

Given that

$K(x,t) = (1+x)(1-t) \quad a=-1,\ b=0$

By defn of Resolvant kernal

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \longrightarrow ①$$

General form of iterated kernal

$K_1(x,t) = K(x,t)$

$$K_m(x,t) = \int_a^b K(x,s)\,K_{m-1}(s,t)\,ds \longrightarrow ②$$

$K_1(x,t) = (1+x)(1-t)$

put $m=2$ in eqn ②

$$K_2(x,t) = \int_{-1}^{0} (1+x)(1-s)(1+s)(1-t)\,ds$$

$$= (1+x)(1-t)\int_{-1}^{0}(1-s^2)\,ds$$

$$= (1+x)(1-t)\left[s - \frac{s^3}{3}\right]_{-1}^{0}$$

$$= (1+x)(1-t)\left[0-\left(-1+\frac{1}{3}\right)\right]$$

$$= (1+x)(1-t)\left[-\left(\frac{-3+1}{3}\right)\right]$$

$$K_2(x,t) = \frac{2}{3}(1+x)(1-t)$$

put $m=3$.

$$K_3(x,t) = \int_{-1}^{0} K(x,s)\, K_2(s,t)\, ds.$$

$$= \int_{-1}^{0} (1+x)(1-s)\, {}^2\!/_3 (1+s)(1-t)\, ds.$$

$$= \frac{2}{3}(1+x)(1-t)\int_{-1}^{0}(1-s^2)\, ds$$

$$= \frac{2}{3}(1+x)(1-t)\,(2/3).$$

$$K_3(x,t) = \left(\frac{2}{3}\right)^2 (1+x)(1-t)$$

In general

$$K_m(x,t) = \left(\frac{2}{3}\right)^{m-1}(1+x)(1-t) \rightarrow (3)$$

sub (3) in (1)

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1}\left(\frac{2}{3}\right)^{m-1}(1+x)(1-t)$$

$$= (1+x)(1-t)\sum_{m=1}^{\infty}\left(\frac{2\lambda}{3}\right)^{m-1}$$

$$= (1+x)(1-t)\left[1+\frac{2\lambda}{3}+\left(\frac{2\lambda}{3}\right)^2+\cdots\right]$$

$$= (1+x)(1-t)\left[1-\frac{2\lambda}{3}\right]^{-1}$$

$$= (1+x)(1-t)\left[\frac{3-2\lambda}{3}\right]^{-1}$$

$$R(x,t,\lambda) = (1+x)(1-t)\left[\frac{3}{3-2\lambda}\right].$$

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4. Solve the following I.E $u(x) = 5x/6 + 1/2 \int_0^1 xt\, u(t)\,dt$ by resolvant Kernal

Solution:

Given $u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\, u(t)\, dt \rightarrow (1)$

The standard form is given by

$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \rightarrow (2)$

Comparing eqn (1) & (2).

$f(x) = \frac{5x}{6}$ $\lambda = \frac{1}{2}$ $a = 0,\ b = 1$ $k(x,t) = xt$

The resolvant kernal is given by.

$R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$

By iterated kernal

$k_1(x,t) = k(x,t)$

$k_n(x,t) = \int_a^b k(x,s)\, k_{n-1}(s,t)\, ds \rightarrow (3)$

$k_1(x,t) = xt$

put $n=2$ in (3)

$k_2(x,t) = \int_0^1 k(x,s)\, k_1(s,t)\, ds$

$= \int_0^1 xs.st\, ds.$

$= \int_0^1 xt s^2\, ds$

$= xt\left[\frac{s^3}{3}\right]_0^1$

$= xt\, \frac{1}{3}$

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$K_2(x,t) = \frac{xt}{3}$

put $n=3$ in ③

$K_3(x,t) = \int_0^1 K(x,s)\, K_2(s,t)\, ds$

$= \int_0^1 xs \cdot \frac{st}{3}\, ds$

$= \int_0^1 xt \frac{s^2}{3}\, ds$

$= xt \left[\frac{s^3}{9}\right]_0^1$

$= xt\left[\frac{1}{9}\right]$

$K_3(x,t) = \frac{xt}{3^2}$

In general

$K_n(x,t) = xt \left(\frac{1}{3}\right)^{n-1}$

The resolvant kernal is given by

$R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$

$(1-x)^{-1} = 1+x+x^2+\dots$

$= \sum_{n=1}^{\infty} \lambda^{n-1} xt \left(\frac{1}{3}\right)^{n-1}$

$= xt \left[1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \dots\right]$

$= xt \left[1 - \frac{\lambda}{3}\right]^{-1}$

$= xt \left[\frac{3-\lambda}{3}\right]^{-1}$

$R(x,t,\lambda) = \frac{3xt}{3-\lambda}$

The soln of I.E is given by

$u(x) = f(x) + \lambda \int_a^b R(x,t,\lambda)\, f(t)\, dt$

$= \frac{5x}{6} + \frac{1}{2}\int_0^1 \frac{3xt}{3-\lambda} \cdot \frac{5t}{6}\, dt$

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8. Solve the

$$= \frac{5x}{6} + \frac{5}{4}\int_0^1 \frac{xt^2}{3-\frac{1}{2}}\,dt$$

$$= \frac{5x}{6} + \frac{5}{4}\int_0^1 \frac{2xt^2}{5}\,dt$$

$$= \frac{5x}{6} + \frac{x}{2}\int_0^1 t^2\,dt$$

$$= \frac{5x}{6} + \frac{x}{2}\left[\frac{t^3}{3}\right]_0^1$$

$$= \frac{5x}{6} + \frac{x}{2}\left[\frac{1}{3}\right]$$

$$= \frac{5x}{6} + \frac{x}{6}$$

$$= \frac{5x+x}{6}$$

$$u(x) = \frac{6x}{6}$$

$$u(x) = x.$$

(or)

$$u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 x\,t\,u(t)\,dt \rightarrow (1).$$

$$u(x) = \frac{5x}{6} + \frac{x}{2}\int_0^1 t\,u(t)\,dt \rightarrow (2).$$

Assume that.

$$c_1 = \int_0^1 t\,u(t)\,dt \rightarrow (3).$$

(2) ⇒ $$u(x) = \frac{5x}{6} + \frac{x}{6}c_1 \rightarrow (4)$$

$$u(t) = \frac{5t}{6} + \frac{t\,c_1}{6}$$

(3) ⇒

$$c_1 = \int_0^1 t\,u(t)\,dt.$$

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$$= \int_0^1 t\left(\frac{5t}{6} + \frac{t c_1}{2}\right) dt.$$

$$= \int_0^1 t^2\left(\frac{5}{6} + \frac{c_1}{2}\right) dt$$

$$= \left(\frac{5}{6} + \frac{c_1}{2}\right)\left[\frac{t^3}{3}\right]_0^1$$

$$= \left(\frac{5}{6} + \frac{c_1}{2}\right)\left(\frac{1}{3}\right)$$

$$c_1 - \frac{c_1}{6} = \frac{5}{18}$$

$$c_1\left(1 - \frac{1}{6}\right) = \frac{5}{18}$$

$$c_1\left(\frac{5}{6}\right) = \frac{5}{18}$$

$$c_1 = 1/3.$$

(2) $\Rightarrow u(x) = \frac{5x}{6} + \frac{x}{6}$

$$= \frac{6x}{6} = x.$$

Solve the following I.E.
$u(x) = f(x) + \lambda\int_0^1 e^{x-t}\, u(t)\, dt$ by using resolvent kernal.

Solution.

Given
$u(x) = f(x) + \lambda\int_0^1 e^{(x-t)}\, u(t)\, dt \rightarrow (1)$

The standard form is.
$u(x) = f(x) + \lambda\int_a^b K(x,t)\, u(t)\, dt \rightarrow (2)$

Comparing eqn (1) & (2)
$f(x) = f(x)$, $\lambda = \lambda$, $a = 0$, $b = 1$

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The resolvent kernal is given by

$$R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$$

By iterated kernal

$$K_1(x,t) = K(x,t)$$

$$K_n(x,t) = \int_a^b K(x,s)\, K_{n-1}(s,t)\, ds \rightarrow ③$$

$$K_1(x,t) = e^{x-t}$$

put $n=2$ in eqn ③.

$$K_2(x,t) = \int_0^1 e^{x-s} K_1(s,t)\, ds.$$

$$= \int_0^1 e^{x-s} e^{s-t}\, ds.$$

$$= \int_0^1 e^{x-s+s-t}\, ds$$

$$= \int_0^1 e^{x-t}\, ds.$$

$$= \cancel{e^{x}} \cancel{\left[\frac{e^{-t}}{-1}\right]_0^1}$$

$$= e^{x-t} \int_0^1 ds.$$

$$K_2 = e^{x-t}.$$

put $n=3$ in eqn ③

$$K_3(x,t) = \int_0^1 e^{x-s} e^{s-t}\, ds.$$

$$= \int_0^1 e^{x-t}\, ds$$

$$= e^{x-t} \int_0^1 ds.$$

$$K_3(x,t) = e^{x-t}.$$

∴ The general

$$K_n(x,t) = e^{x-t}$$

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The resolvant kernal is.

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} e^{x-t}$$

$$= e^{x-t}\left[1+\lambda+\lambda^2+\dots\right]$$

$$= e^{x-t}(1-\lambda)^{-1}$$

$$R(x,t,\lambda) = \frac{e^{x-t}}{1-\lambda}$$

$$\therefore u(x) = f(x) + \lambda \int_0^1 \frac{e^{x-t}}{1-\lambda} f(t)\,dt$$

$$u(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 e^{x-t} f(t)\,dt.$$

(or)

Given $u(x) = f(x) + \lambda \int_0^1 e^{x-t} u(t)\,dt. \rightarrow (1)$

$$u(x) = f(x) + \lambda e^{x} \int_0^1 e^{-t} u(t)\,dt \rightarrow (2)$$

Assume that

$$c_1 = \int_0^1 e^{-t} u(t)\,dt \rightarrow (3)$$

(2) $\Rightarrow$ $u(x) = f(x) + \lambda e^{x} c_1 \rightarrow (4)$

$$u(t) = f(t) + \lambda e^{t} c_1$$

$$c_1 = \int_0^1 e^{-t}\left(f(t) + \lambda e^{t} c_1\right) dt$$

$$= \int_0^1 f(t)e^{-t} + \lambda e^{0} c_1\,dt$$

$$c_1 = \int_0^1 f(t)e^{-t}\,dt + \lambda c_1 \int_0^1 dt$$

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$$c_1 = \int_0^1 f(t) e^{-t}\, dt + \lambda c_1$$

$$c_1 - \lambda c_1 = \int_0^1 e^{-t} f(t)\, dt$$

$$c_1(1-\lambda) = \int_0^1 e^{-t} f(t)\, dt$$

$$c_1 = \frac{1}{1-\lambda} \int_0^1 f(t) e^{-t}\, dt$$

(1) $\Rightarrow$

$$u(x) = f(x) + \frac{\lambda e^x}{1-\lambda} \int_0^1 f(t) e^{-t}\, dt$$

$$u(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 e^{x-t} f(t)\, dt$$

$\therefore$ which is required solution.

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Method of Successive approximation
Solution of Fredholm I.E of Second kind by successive approximation

Consider the Fredholm I.E of second kind

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \rightarrow ①$$

Let us take First approximation (zero order approximation)

consider $u_0(x) = f(x) \rightarrow ②$

Also if $u_n(x)$ and $u_{n-1}(x)$ are the n^{th} and $(n-1)^{th}$ order approximation respectively.

Then they are connected by

$$u_n(x) = f(x) + \lambda \int_a^b k(x,t)\, u_{n-1}(t)\, dt \rightarrow ③$$

By definition of iterated kernal

$$k_1(x,t) = k(x,t)$$

$$k_n(x,t) = \int_a^b k(x,s)\, k_{n-1}(s,t)\, ds \rightarrow ④$$

Now from ③

$$u_1(x) = f(x) + \lambda \int_a^b k(x,t)\, u_0(t)\, dt \rightarrow ⑤$$

$$u_1(x) = f(x) + \lambda \int_a^b k(x,t)\, f(t)\, dt \rightarrow ⑥$$

$$③ \Rightarrow u_2(x) = f(x) + \lambda \int_a^b k(x,t)\, u_1(t)\, dt$$

$$= f(x) + \lambda \int_a^b k(x,s)\, u_1(s)\, ds \rightarrow ⑦$$

$$u_2(x) = f(x) + \lambda \int_a^b k(x_0,s) \left[f(s) + \lambda \int_a^b k(s,t)\, f(t)\, dt \right] ds$$

$$u_2(x) = f(x) + \lambda \int_a^b k(x,s) f(s) ds + \lambda^2 \int_a^b k(x,s) \int_a^b k(s,t) f(t) dt\, ds$$

$$= f(x) + \lambda \int_a^b k(x,s) f(s) ds + \lambda^2 \int_a^b \left[\int_a^b k(x,s) k(s,t) ds \right] f(t) dt$$

$$u_2(x) = f(x) + \lambda \int_a^b k_1(x,s) f(s) ds + \lambda^2 \int_a^b k_2(x,t) f(t) dt$$

$$= f(x) + \sum_{m=1}^{2} \lambda^m \int_a^b k_m(x,t) f(t) dt$$

In general

$$u_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b k_m(x,t) f(t) dt.$$

1. Solve the following Fredholm I.E of 2nd kind $u(x) = 2x + \lambda \int_0^1 (x+t) u(t) dt$ by the method of successive approximation by taking $u_0(x) = 1$

Solution:

Given

$$u(x) = 2x + \lambda \int_0^1 (x+t) u(t) dt \rightarrow (1)$$

and also given that $u_0(x) = 1 \rightarrow (2)$

Let $u_n(x) = f(x) + \lambda \int_a^b k(x,t)\, \underbrace{u_{n-1}(t)}_{u_0(x)} dt \rightarrow (3)$

put $n=1$ in eqn (3).

$$u_1(x) = 2x + \lambda \int_0^1 (x+t) dt$$

$$u_1(x) = 2x + \lambda \left[xt + \frac{t^2}{2} \right]_0^1$$

$$= 2x + \lambda \left[x + \tfrac{1}{2} \right].$$

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$u_1(x) = 2x + \lambda(x + 1/2)$

put $n=2$ in eqn ③.

$$u_2(x) = 2x + \lambda \int_0^1 (x+t)(2t + \lambda(t + 1/2))\, dt.$$

$$= 2x + \lambda \int_0^1 (x+t)(2t + \lambda t + \lambda/2)\, dt$$

$$= 2x + \lambda \int_0^1 [2xt + \lambda x t + \lambda x/2 + 2t^2 + \lambda t + \lambda t/2]\, dt$$

$$= 2x + \lambda \left[xt^2 + \frac{\lambda x t^2}{2} + \frac{\lambda x t}{2} + \frac{2t^3}{3} + \frac{\lambda t^2}{2} + \frac{\lambda t^2}{4} \right]_0^1$$

$$= 2x + \lambda \left[x + \frac{\lambda x}{2} + \frac{\lambda x}{2} + \frac{2}{3} + \frac{\lambda}{3} + \frac{\lambda}{4} \right]$$

$$= 2x + \lambda \left(x + \lambda x + 2/3 + 7\lambda/12 \right)$$

$$= 2x + \lambda x + \lambda^2 x + \lambda\, 2/3 + 7\lambda^2/12.$$

$$u_2(x) = 2x + \lambda(x + 2/3) + \lambda^2 (x + 7/12).$$

put $n=3$ in eqn ③.

$$u_3(x) = 2x + \lambda \int_0^1 (x+t)[2t + \lambda(t + 2/3) + \lambda^2 (t + 7/12)]\, dt.$$

$$= 2x + \lambda \int_0^1 (x+t)(2t + \lambda t + 2\lambda/3 + \lambda^2 t + 7\lambda^2/12)\, dt.$$

$$= 2x + \lambda \int_0^1 \left[2tx + \lambda t x + \frac{2\lambda x}{3} + \lambda^2 x t + \frac{7\lambda^2 x}{12} + 2t^2 + \lambda t^2 + \frac{2\lambda t}{3} + \lambda^2 t^2 + \frac{7\lambda^2 t}{12} \right]$$

$$= 2x + \lambda \left[\frac{2xt^2}{2} + \frac{\lambda x t^2}{2} + \frac{2\lambda x t}{3} + \frac{\lambda^2 x t^2}{2} + \frac{7\lambda^2 x t}{12} + \frac{2t^3}{3} + \frac{\lambda t^3}{3} + \frac{\lambda t^2}{3} + \frac{\lambda^2 t^3}{3} + \frac{7\lambda^2 t^2}{24} \right]_0^1$$

$$= 2x + \lambda \Big[2x^2 + \lambda x^2 + \frac{2\lambda x}{3} + \lambda^2 x^2 + \frac{7\lambda^2 x}{12} + x + \frac{\lambda x}{2} + \frac{\lambda}{3} + \frac{\lambda^2 x}{2} + \frac{7\lambda^2}{24} \Big]$$

$$= 2x + \lambda \Big[xt^2 + \frac{\lambda x t^2}{2} + \frac{2\lambda x t}{3} + \frac{\lambda^2 x t^2}{2} + \frac{7\lambda^2 x t}{12} + \frac{2t^3}{3} + \frac{\lambda t^3}{3} + \frac{2\lambda^2 t^3}{6} + \frac{7\lambda^2 t^2}{24} \Big]_0$$

$$= 2x + \lambda \Big[xt^2 + \lambda\Big(\frac{xt^2}{2} + \frac{2xt}{3}\Big) + \lambda^2\Big(\frac{xt^2}{2} + \frac{7xt}{12}\Big) + \frac{2t^3}{3} + \lambda\Big(\frac{t^3}{3} + \frac{2t^3}{6}\Big) + \lambda^2\Big(\frac{t^3}{6} + \frac{7t^2}{24}\Big) \Big]$$

$$= 2x + \lambda \Big[x + \lambda\Big(\frac{x}{2} + \frac{2x}{3}\Big) + \lambda^2\Big(\frac{x}{2} + \frac{7x}{12}\Big) + \frac{2}{3} + \lambda\Big(\frac{1}{3} + \frac{2}{6}\Big) + \lambda^2\Big(\frac{1}{6} + \frac{7}{24}\Big) \Big]$$

$$= 2x + \lambda(x + 2/3) + \lambda^2(7x/6 + 2/3) + \lambda^3(13x/12 + 5/6)$$

proceeding in the same way, we get the next approximation.

2. Solve the following I.E $u(x) = 1 + \lambda \int_0^1 (x+t)\, u(t)\, dt$ by the method of successive approximation upto 3rd order, $u_0(x) = 1$

Solution:

Given $u(x) = 1 + \lambda \int_0^1 (x+t) u(t) dt \rightarrow (1)$

Also given that $u_0(x) = 1 \longrightarrow (2)$.

let $u_n(x) = f(x) + \lambda \int_a^b K(x,t)\, u_{n-1}(t)\, dt \rightarrow (3)$

put $n=1$ in eqn ③.

$$u_1(x) = 1 + \lambda \int_0^1 (x+t)\, dt.$$

$$= 1 + \lambda \left[xt + \frac{t^2}{2} \right]_0^1$$

$$u_1(x) = 1 + \lambda \left[x + \tfrac{1}{2} \right] \quad u_1(t) = 1 + \lambda (t + \tfrac{1}{2})$$

put $n = 2$ in eqn ③.

$$u_2(x) = 1 + \lambda \int_0^1 (x+t)\left(1 + \lambda t + \lambda/2\right) dt.$$

$$= 1 + \lambda \int_0^1 \left[x + \lambda x t + \frac{\lambda x}{2} + t + \lambda t^2 + \frac{\lambda t}{2} \right] dt$$

$$= 1 + \lambda \left[xt + \frac{\lambda x t^2}{2} + \frac{\lambda x t}{2} + \frac{t^2}{2} + \frac{\lambda t^3}{3} + \frac{\lambda t^2}{4} \right]_0^1$$

$$= 1 + \lambda \left[x + \frac{\lambda x}{2} + \frac{\lambda x}{2} + \frac{1}{2} + \frac{\lambda}{3} + \frac{\lambda}{4} \right]$$

$$= 1 + \lambda \left[x + \lambda x + \frac{1}{2} + \frac{7\lambda}{12} \right]$$

$$= 1 + \lambda x + \lambda^2 x + \frac{\lambda}{2} + \frac{7\lambda^2}{12}$$

$$u_2(x) = 1 + \lambda \left(x + \tfrac{1}{2} \right) + \lambda^2 \left(x + \tfrac{7}{12} \right).$$

put $n = 3$ in eqn ③.

$$u_3(x) = 1 + \lambda \int_0^1 (x+t)\left[\left(1 + \lambda\left(t + \tfrac{1}{2}\right) + \lambda^2\left(t + \tfrac{7}{12}\right)\right]\right] dt.$$

$$= 1 + \lambda \int_0^1 \left[x + \lambda\left(xt + \frac{x}{2}\right) + \lambda^2\left(xt + \frac{7x}{12}\right) + t + \lambda\left(t^2 + \frac{t}{2}\right) + \lambda^2\left(t^2 + \frac{7t}{12}\right) \right] dt.$$

$$= 1 + \lambda \left[xt + \lambda\left(\frac{xt^2}{2} + \frac{xt}{2}\right) + \lambda^2\left(\frac{xt^2}{2} + \frac{7xt}{12}\right) + \frac{t^2}{2} + \lambda\left(\frac{t^3}{3} + \frac{t^2}{4}\right) + \lambda^2\left(\frac{t^3}{3} + \frac{7t^2}{24}\right) \right]_0^1$$

$$= 1+\lambda\left[x+\lambda\left(\frac{x}{2}+\frac{x}{2}\right)+\lambda^2\left(\frac{x}{2}+\frac{7x}{12}\right)+\frac{1}{2}+\lambda\left(\frac{1}{3}+\frac{1}{4}\right)+\lambda^2\left(\frac{1}{3}+\frac{7}{24}\right)\right]$$

$$= 1+\lambda\left[x+\lambda x+\lambda^2\left(\frac{x}{2}+\frac{7x}{12}\right)+\frac{1}{2}+\lambda\left(\frac{7}{12}\right)+\lambda^2\left(\frac{15}{24}\right)\right]$$

$$= 1+\lambda x+\lambda^2 x+\lambda^3\left(\frac{6x+7x}{12}\right)+\frac{\lambda}{2}+\lambda^2\left(\frac{7}{12}\right)+\lambda^3\left(\frac{5}{8}\right)$$

$$= 1+\lambda x+\lambda^2 x+\lambda^3\left(\frac{13}{12}\right)+\frac{\lambda}{2}+\lambda^2\left(\frac{7}{2}\right)+\lambda^3\left(\frac{5}{48}\right)$$

$$u(x)= 1+\lambda\left(x+\frac{1}{2}\right)+\lambda^2\left(x+\frac{7}{12}\right)+\lambda^3\left(\frac{13x}{12}+\frac{5}{48}\right)$$

$$u(x)= 1+\lambda\left(x+\frac{1}{2}\right)+\lambda^2\left(x+\frac{7}{12}\right)+\lambda^3\left(\frac{13x}{12}+\frac{5}{8}\right)$$

∴ which is required solution.