Square Integrable function and L_2 - function.

A function $f(x)$ is said to be square integrable.

$$\int_a^b |f(x)|^2 \, dx < \infty$$

A square integrable function $f(x)$ is called L_2 - function if the following condition are satisfied.

(i) $\int_a^b \int_a^b |k(x,t)|^2 \, dx \, dt < \infty \quad x, t \in [a,b]$

(ii) $\int_a^b |k(x,t)|^2 \, dx < \infty, \quad x \in [a,b]$

(iii) $\int_a^b |k(x,t)|^2 \, dt < \infty, \quad t \in [a,b]$

Inner product (or) Scalar Product:-

Let f and g be the two complex L_2 function of real variable x, then Inner product (or) scalar product is denoted by (f, g) and it is defined by,

$$(f, g) = \int_a^b f(x) \overline{g(x)} \, dx$$

Where bar denotes the complex Conjugate.

Define: orthogonal

Two functions f and g are said to be orthogonal if $(f, g) = 0$.

i.e) $\int_a^b f(x)\overline{g(x)}\,dx = 0$

Define: Norm

The Norm of a function $f(x)$ is given by,

$$\|f(x)\| = \left[\int_a^b f(x)\overline{f(x)}\,dx\right]^{1/2} = \left[\int_a^b |f(x)|^2 dx\right]^{1/2}$$

Define: Normalized

A function $f(x)$ is said to be Normalized if $\|f(x)\| = 1$

Define: complex hilbert space:

let H be a complex banach space Then H is called a hilbert space if a complex number (x, y) called the inner product of x and y. is associate to each of the two vectors x and y in such a way that

(i) $\overline{(x, y)} = (y, x)$

(ii) $(\alpha x + \beta y)z = \alpha(x, z) + \beta(y, z)$

(iii) $(x, x) = \|x\|^2$

$\forall$ $x, y, z \in H$ and α, β are scalar.

Example of Hilbert space:-

Consider the banach space consisting of n tuples of complex numbers with the norm of curves $x = (x_1, x_2, \ldots, x_n)$ defined by

$$\|x\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$$

We shall show that if the inner product of two vectors $x = (x_1, \ldots, x_n)$ $y = (y_1, y_2, \ldots, y_n)$ is defined by $(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}$, then it is a hilbert space for all arbitrary vectors.

proof:

Let $x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$

$z = (z_1, z_2, \ldots, z_n)$

and α, β are scalars.

(i) $\overline{(x, y)} = \overline{\sum_{i=1}^{n} x_i \overline{y_i}}$

$= \overline{x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}}$

$= \overline{x_1} y_1 + \overline{x_2} y_2 + \cdots + \overline{x_n} y_n$

$= \sum_{i=1}^{n} \overline{x_i} y_i$

$= \sum_{i=1}^{n} y_i \overline{x_i}$

$\overline{(x, y)} = (y, x)$

(ii) $\alpha x+\beta y=\alpha(x_1, x_2, \dots, x_n)+\beta(y_1, y_2, \dots, y_n)$

$=(\alpha x_1+\beta y_1), (\alpha x_2+\beta y_2), \dots (\alpha x_n+\beta y_n)$

$((\alpha x+\beta y), z)=(\alpha x_1+\beta y_1)\overline{z_1}+(\alpha x_2+\beta y_2)\overline{z_2}$
$\dots\dots+(\alpha x_n+\beta y_n)\overline{z_n}.$

$=\alpha x_1\overline{z_1}+\beta y_1\overline{z_1}+\alpha x_2\overline{z_2}+\beta y_2\overline{z_2}+\dots$
$\dots+\alpha x_n\overline{z_n}+\beta y_n\overline{z_n}$

$=\alpha(x_1\overline{z_1}+x_2\overline{z_2}+\dots+x_n\overline{z_n})$
$+\beta(y_1\overline{z_1}+y_2\overline{z_2}+\dots+y_n\overline{z_n})$

$=\alpha\left(\sum_{i=1}^{n} x_i\overline{z_i}\right)+\beta\left(\sum_{i=1}^{n} y_i\overline{z_i}\right)$

$((\alpha x+\beta y), z)=\alpha(x, z)+\beta(y, z)$

$(\alpha x+\beta y, z)=\alpha(x, z)+\beta(y, z)$

(iii) $(x, x)=\sum_{i=1}^{n} x_i\overline{x_i}$

$=\sum_{i=1}^{n}|x_i|^2$

$(x, x)=\|x\|^2$

Hence the given space is the hilberd space

Orthogonal system of function :-

We know that a finite (or) infinite set $\{f(x)\}$ defined on an interval $a \le x \le b$ is said to be orthogonal set if $(f_i, f_j) = 0$

i.e) $\int_a^b f_i(x) f_j(x)\, dx = 0$, $i \neq j$

If none of these element of the set is a vector, then it is called proper orthogonal vector.

The set $\{f_i(x)\}$ is said to be orthonormal if $(f_i, f_j) = \int_a^b f_i(x) f_j(x)\, dx$

$$= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Hermition :-

The integral operator $k = \int_a^b k(x,t)\, dt$ is said to be Hermition (or) self adjoint, if it satisfies the condition.

$(kf, g) = (f, kg)$

Hilberd schmidt kernal :-

A kernal $K(x,t)$ is said to be Hilberd schmidt kernal if it is Hermition and square integrable.

1. Gram Schmidt orthogonalized process:
Let $\{f_1, f_2, \cdots f_k, \cdots\}$ be the set of given functions. We can construct an orthonormal set $\{g_1, g_2, \cdots, g_k, \cdots\}$ by gram schmidt process as follows.

Let $g_1 = \frac{f_1}{\|f_1\|}$

To find g_2, we can defined.

$w_2(x) = f_2(x) - (f_2, f_1) g_1$.

The function w_2 is orthogonal to g_1.

$\therefore g_2$ can be constructed by setting $g_2 = \frac{w_2}{\|w_2\|}$

proceeding in the same way, we get.

$$w_k(x) = f_k(x) - \sum_{i=1}^{k-1} (f_k, f_i) g_i$$

and $g_k = \frac{w_k}{\|w_k\|}$

Also if we are given set of orthonormal function, we can convert it into an orthonormal system, it is possible to construct the theory of furier series.

Suppose we want to find the best approximation of an arbitrary function $f(x)$

interms of an linear combination of an Linear combination of an orthonormal set $(g_1, g_2, \dots, g_n)$ for any $(\alpha_1, \alpha_2, \dots, \alpha_n)$

$$\|f - \sum_{i=1}^{n} \alpha_i g_i\|^2 = \|f\|^2 + \sum_{i=1}^{n} |(f, g_i) - \alpha_i|^2$$
$$- \sum_{i=1}^{n} |(f, g_i)|^2 \to ①$$

clearly the minimum can be attained by setting $\alpha_i = (f, g_i) a_i (say)$

Here the number a_i are known as furier coefficients of the function $f(x)$ relative to the orthonormal system g_i.

$$① \Rightarrow \|f - \sum_{i=1}^{n} \alpha_i g_i\|^2 = \|f\|^2 - \sum_{i=1}^{n} |a_i|^2 \to ②$$

since L.H.S of ② in non-negative. we get.

$$\sum_{i=1}^{n} |a_i|^2 \leq \|f\|^2$$

suppose we are given an infinite orthonormal system $g_i(x)$ in I_2, and, a sequence of constant α_i then the converges of the series $\sum_{k=1}^{\infty} |\alpha_k|^2$.

2. Schwartz inequality:

Statement:-

If $f(x)$ and $g(x)$ be any two function in a Hilbert space, then $|(f,g)| \leq \|f\| \, \|g\|$.

Proof:-

Case (i)

If $g=0$ then $\|g\|=0$ and

$|(f,g)|=0$

$\therefore$ In this case both side vanish and the result is trivially true.

case (ii)

let $g \neq 0$, for any scalar λ.

we have,

$(f+\lambda g, f+\lambda g) \geq 0$

$\Rightarrow (f, f+\lambda g) + \lambda (g, f+\lambda g) \geq 0$

$(f,f) + \bar{\lambda}(f,g) + \lambda (g,f) + \lambda\bar{\lambda}(g,g) \geq 0$

$\|f\|^2 + \bar{\lambda}(f,g) + \lambda(g,f) + |\lambda|^2 \|g\|^2 \geq 0 \quad \to (1)$

since $g \neq 0, \|g\| \neq 0$

put $\lambda = -\dfrac{(f,g)}{\|g\|^2}$ in (1), we get

$$\|f\|^2 - \frac{\overline{(f,g)}}{\|g\|^2}(f,g) - \frac{(f,g)}{\|g\|^2}(g,f)$$

$$+ \frac{|(f,g)|^2}{\|g\|^4}\|g\|^2 \geq 0$$

$$\|f\|^2 - \frac{|(f,g)|^2}{\|g\|^2} - \frac{(f,g)}{\|g\|^2}\overline{(f,g)} + \frac{|(f,g)|^2}{\|g\|^2} \geq 0$$

$$\|f\|^2 - \frac{|(f,g)|^2}{\|g\|^2} \geq 0$$

$$\|f\|^2 \|g\|^2 - |(f,g)|^2 \geq 0$$

$$-|(f,g)|^2 \geq -\|f\|^2 \|g\|^2$$

$$|(f,g)| \leq \|f\| \, \|g\|$$

Minkowski inequality:-

3. Statement:-

If $f(x)$ and $g(x)$ be any two L_2 function in a Hilberd space H, then $|(f+g)| \leq \|f\| + \|g\|$.

proof:-

case (i)

If $f+g=0$ then $\|(f+g)\|=0$ and $|(f+g)|=0$

In this case both sides vanish and the result is trivially true.

case (ii)

let $$\|f+g\|^2 = \sum_{i=1}^{n} |f_i+g_i|^2$$

$$\leq \sum_{i=1}^{n} |f_i+g_i| \, |f_i+g_i|$$

$$\leq \sum_{i=1}^{n} |f_i+g_i| \left[|f_i| + |g_i| \right]$$

$$\leq \sum_{i=1}^{n} |f_i+g_i| \, |f_i| + \sum_{i=1}^{n} |f_i+g_i| \, |g_i|$$

$$\leq \sum_{i=1}^{n} |(f_i+g_i) f_i| + \sum_{i=1}^{n} |(f_i+g_i) g_i|$$

$$\leq \|f+g\| \, \|f\| + \|f+g\| \, \|g\|$$

$$\|f+g\|^2 \leq \|f+g\| \left[\|f\| + \|g\| \right]$$

4. Bessell's inequality :-

Statement:

For every square integrable function $f(x)$

$$\sum_{i=1}^{n} |c_i|^2 \le \sum_{i=1}^{n} |f, \phi_i|^2 \le \|f\|^2$$

where $f(x)$ is real and continuous and $\phi_i(x)$ is real and continuous and consisting normalized orthogonal set.

Proof :-

Consider,

$$\int_a^b |f(x) - \sum_{i=1}^{n} c_i \phi_i(x)|^2 dx = \int_a^b |f(x)|^2 dx$$

$$+ \sum_{i=1}^{n} \int_a^b |c_i|^2 |\phi_i(x)|^2 dx - \sum_{i=1}^{n} \int_a^b \overline{f(x)}\, c_i \phi_i(x) dx$$

$$- \sum_{i=1}^{n} \int_a^b f(x)\, \overline{c_i \phi_i(x)}\, dx \rightarrow (1)$$

Now since,

$$\left.\begin{aligned} &\int_a^b |\phi_i(x)|^2 dx = 1 \\ &\int_a^b \overline{f(x)}\, \phi_i(x)\, dx = \overline{c_i} \\ &\int_a^b f(x) \overline{\phi_i(x)}\, dx = c_i \end{aligned}\right\} \rightarrow (2)$$

(1) $\Rightarrow$

$$\int_a^b |f(x) - \sum_{i=1}^{n} c_i \phi_i(x)|^2 dx = \int_a^b |f(x)|^2 dx$$

$$+ \sum_{i=1}^{n} |c_i|^2 - \sum_{i=1}^{n} |c_i|^2 - \sum_{i=1}^{n} |c_i|^2$$

$= \int_a |f(x)|^2 dx - \sum_{i=1}^{n} |c_i|^2 \rightarrow (3)$

The I.E of (3) has non-negative value for any 'n', we have

$0 < \int_a^b |f(x)|^2 dx - \sum_{i=1}^{n} |c_i|^2$

$\sum_{i=1}^{n} |c_i|^2 \leq \int_a^b |f(x)|^2 dx \leq \|f\|^2$

It follows that series $\sum_{i=1}^{n} |c_i|^2$ is always convergent and its sum satisfies the inequality.

5. State and prove hilbert schmidt theorem.

Statement:-

Any function $f(x)$ which can be expressed in the form,

$f(x) = \int_a^b k(x,t) h(t) dt$

is almost everywhere equal to the sum of its usual system with regarded to the orthogonal system. $\phi_n(x)$ of eigen function of the symmetric kernal $k(x,t)$

The kernal $k(x,t)$ is integrable together with the square of its modless with regard to both of its variables x and t. The integral $\int |k(x,t)|^2 dx$ is bounded and $h \in L_2[a,b]$

proof:-
We have to prove that,
$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \rightarrow (1)$
Where the coefficient of ϕ_n are the furier coefficient of the function $f(x)$ with respect to the system $\{\phi_n(x)\}$.

$$f_n = \int_a^b f(x) \overline{\phi_n(x)}\, dx \rightarrow (2)$$

We observe that,
(i) The furier series $\sum_{n=1}^{\infty} f_n \phi_n(x)$ is convergent in the domain $[a,b]$
(ii) It's sum is the function $f(x)$.
consider the homogenous integral equation $f(x) = \int_a^b k(x,t) h(t)\, dt$ $\rightarrow (3)$

(2) =>

$$f_n = \int_a^b \left\{ \int_a^b k(x,t) \overline{\phi_n(x)}\, dx \right\} h(t)\, dt$$

Now since,

$$\int_a^b k(x,t) \overline{\phi_n(x)}\, dx = \int_a^b \overline{k(x,t) \phi_n(x)}\, dx$$

$$\therefore \int_a^b k(x,t) \overline{\phi_n(x)}\, dx = \frac{1}{\lambda_n} \overline{\phi_n(t)}$$

$$\therefore f_n = \int_a^b \frac{\overline{\phi_n(t)}}{\lambda_n} h(t)\, dt$$

$$f_n = \frac{h_n}{\lambda_n} \rightarrow (4)$$

... h_n are the furier coefficient of the given function h with regard to the system $\{\phi_n(x)\}$.

$$h_n = \int_a^b h(x)\overline{\phi_n(x)}\,dx \to (5)$$

$$\sum_{n=1}^{\infty} f_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n}\phi_n(x) \to (6)$$

The series is convergent if

$$\sum_{k=n+1}^{n+p} \left|\frac{h_k}{\lambda_k}\phi_k(x)\right|^2 \le \sum_{k=n+1}^{n+p} |h_k|^2 \sum_{k=n+1}^{n+p} \frac{|\phi_k(x)|^2}{\lambda_k} \to (7)$$

Now $\sum_{k=n+1}^{n+p} |h_k|^2 < \epsilon$ for any arbitrary.

Choosen the +ve small number ϵ.

The $\sum_{k=n+1}^{n+p} \frac{|\phi_k(x)|^2}{\lambda_k}$ is bounded

since $\int_a^b |k(x,t)|^2\,dx$ is bounded.

$\therefore$ The furier series eqn (6) of the function $f(x)$ with respect to the system $\phi_n(x)$ is absolutely convergent.

Let $S(x)$ denote the sum of the series eqn (6).

$$\therefore S(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n}\phi_n(x) \to (8)$$

Now consider $p(x) = f(x) - S(x) \to (9)$

then the function $f(x), S(x)$ have the same furier coefficient with regard to the system $\phi_n(x)$ multiply (9) by $\overline{\phi_n(x)}$ and integrate with respect to x on the interval $[a, b]$, we get

$$\int_a^b p(x)\overline{\phi_n(x)}\,dx = \int_a^b f(x)\overline{\phi_n(x)}\,dx - \int_a^b s(x)\overline{\phi_n(x)}\,dx$$

$$\int_a^b p(x)\overline{\phi_n(x)}\,dx = 0.$$

which implies that $p(x)$ is orthogonal to all the eigen function $\phi_n(x)$ of the kernal $k(x,t)$ But we know that the function $p(x)$ is orthogonal to the kernal $k(x,t)$.

i.e) $\int_a^b k(x,t)\overline{p(t)}\,dt = 0.$

which implies to function f is orthogonal to p.

Let $\int_a^b |p(x)|^2\,dx = \int_a^b \overline{p(x)}\,p(x)\,dx$

$$= \int_a^b \overline{p(x)}\,[f(x) - s(x)]\,dx$$

$$= \int_a^b \overline{p(x)}\,f(x)\,dx - \int_a^b \overline{p(x)}\,s(x)\,dx$$

$$= 0 - \int_a^b \overline{p(x)}\,s(x)\,dx$$

($\because$ p & f are orthogonal

$$\int_a^b |p(x)|^2 dx = -\sum_{n=1}^{\infty} \frac{h_n}{\lambda_n} \int_a^b p(x)\phi_n(x)dx$$

$$\int_a^b |p(x)|^2 dx = 0.$$

$$p(x) = 0$$

$$f(x) - S(x) = 0$$

$$f(x) = S(x)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n} \phi_n(x)$$

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x).$$

6. Hilbert theorem:-

Statement:

Every symmetric kernal with a norm not equal to zero has atleast one eigen value.

proof:

consider the non_homogenous Fredholm I.E of second kind.

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt \quad \to (1)$$

The soln of the I.E is in the form of a fourier series in λ is given by,

$$\phi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b k_n(x,t)f(t)dt \quad \to (2)$$

x'ly (2) by $\overline{f(x)}$ and integrating on a interval [a,b] we get,

$$\int_a^b \phi(x) f(x)\, dx = \sum_{n=0} A_n \lambda^n \rightarrow (3)$$

where,

$$A_n = \int_a^b \int_a^b k_n(x,t) f(\overline{x}) f(t)\, dt\, dx \quad (4)$$

with $n=0$, $k_0 = 1$, $k_1(x,t) = k(x,t)$

the recurrence relation for iterated kernal are defined by,

$$k_{n+1}(x,t) = \int_a^a k_n(x,s) k_1(s,t)\, ds \quad (5)$$

$$k_{2n}(x,t) = \int_a^b k_n(x,s) k_n(s,t)\, ds,$$

Now since,

$k_n(x,t) = k_n(t,x)$ ∴ The kernal is symmetric,

we have,

$$A_{2n} = \int_a^b \int_a^b k_n(x,s) f(x)\, dx \int_a^b k_n(\overline{t,s}) f(t)\, dt\, ds$$

$$A_{2n} = \int_a^b \left| \int_a^b k_n(x,t) f(\overline{x})\, dx \right|^2 ds \quad (8)$$

Therefore we can say that all the coefficient of series (3) with given subscript are non-negative real number.

ie) $A_{2n} > 0$

Also from (1) & (2), we get.

$$A_{2n} = \int_a^b \left[\int_a^b k_{n+1}(x,s)\overline{f(x)}\,dx\right] \left[\int_a^b \overline{k_{n+1}(t,s)}\, f(t)\,dt\right] ds$$

Now using Schwartz inequality

$$A_{2n}^2 \le \int_a^b \left|\int_a^b k_{n-1}(x,s)\overline{f(x)}\,dx\right|^2 ds \int_a^b \left|\int_a^b k_{n+1}(x,s) f(x)\,dx\right|^2 ds$$

By using eqn (6)

$\Rightarrow A_{2n}^2 \le A_{2n-2} \cdot A_{2n+2}$ hold for every $n \ge 2$. (7)

particularly using eqn (10), we have

$$A_2 = \int_a^b \int_a^b k_2(x,t)\overline{f(x)}\, f(t)\,dt\,dx$$

$$A_2 = \int_a^b \left|\int_a^b k(x,t) f(x)\,dx\right|^2 ds \quad (8)$$

Now,

$$A_4 = \int_a^b \int_a^b k_4(x,t)\overline{f(x)}\, f(t)\,dt\,dx$$

$$A_4 = \int_a^b \left|\int_a^b k_2(x,t)\, f(x)\,dx\right|^2 ds \quad (9)$$

Since the kernel $k(x,t)$ has a non-zero norm.

$\therefore$ There exist a function f belong to L_2 such that $A_2 > 0$.

Now we shall prove that $A_4 > 0$

If $A_4 = 0$ then eqn ⑨ gives

$$\int_a^b k_2(x,t) f(x) \, dx = 0$$

Almost every where with regarded to t in the domain (a, b)

$\therefore$ From ⑧ we get

$$A_2 = 0$$

which gives the contradiction

$\therefore$ All the coefficient with even induces are positive and satisfies the inequality

$$\frac{A_{2n+2}}{A_{2n}} \geq \frac{A_{2n}}{A_{2n-2}} \quad \text{—} ⑩$$

The above inequality forms a non-decreasing sequence

The series $\sum A_n \lambda^n$ cannot be convergent for every value of λ unless $A = 0$.

If $A_4 > 0$, then from eqn ⑦ we observe that the same will be true $A_6, A_8, \ldots$

The ration A_{2n} / A_{2n-2} will be increasing, the series considered by taking the terms of even order in the series $\Sigma A_n \lambda^n$ cannot be convergent for every value of λ.

If A_4 is zero then it is necessary and sufficient that $f(x)$ be orthogonal to the kernal $K_2(x,t)$.

Now we shall find the interval in which atleast one eigen value λ of the kernal $K(x,t)$ is contained.

The terms of the series eqn satisfies the inequality.

$$\frac{A_{2n+2}}{A_{2n}} \frac{\lambda^{2n+2}}{\lambda^{2n}} \geq \frac{A_4}{A_2} \cdot \frac{\lambda^4}{\lambda^2} = \frac{A_4}{A_2} \cdot \lambda^2$$

The series is divergent if.

$$\frac{A_4}{A_2} |\lambda|^2 > 1$$

$$\Rightarrow |\lambda| > \left(\frac{A_2}{A_4}\right)^{\frac{1}{2}}$$

Hence the eigen value of the kernal $K(x,t)$ is continuous in the interval $\left\{-\left(\frac{A_2}{A_4}\right)^{\frac{1}{2}}, \left(\frac{A_2}{A_4}\right)^{\frac{1}{2}}\right\}$ which is real.

Fredholm I.E using Schmidt solution for $f(x) = \lambda\int_0^1 e^x e^t f(t)\,dt$.

Solution:

Given that

$$f(x) = \lambda\int_0^1 e^x e^t f(t)\,dt \rightarrow (1)$$

$$f(x) = \lambda e^x \int_0^1 e^t f(t)\,dt \rightarrow (2)$$

Assume that

$$C_1 = \int_0^1 e^t f(t)\,dt \rightarrow (3)$$

(2) $\Rightarrow$

$$f(x) = \lambda e^x C_1 \rightarrow (4)$$

$$f(t) = \lambda e^t C_1$$

(3) $\Rightarrow$

$$C_1 = \int_0^1 e^t f(t)\,dt$$

$$= \int_0^1 e^t \lambda e^t C_1\,dt$$

$$= \lambda C_1 \int_0^1 e^{2t}\,dt$$

$$= \lambda C_1 \left[\frac{e^{2t}}{2}\right]_0^1$$

$$= \lambda C_1 \left[\frac{e^2}{2} - \frac{1}{2}\right]$$

$$C_1 = \frac{\lambda C_1}{2}\left[e^2 - 1\right]$$

$$c_1 - \frac{\lambda c_1}{2}(e^2-1) = 0$$

$$\cancel{\Rightarrow \frac{2c_1 - \lambda c_1(e^2-1)}{2} = 0}$$

$$\cancel{c_1\left[\frac{2-\lambda(e^2-1)}{2}\right] = 0}$$

$$\cancel{c_1 = 2}$$

$$c_1\left[1 - \frac{\lambda}{2}(e^2-1)\right] = 0$$

case (i)

If $c_1 = 0$

(4) $\Rightarrow f(x) = 0$

case (ii)

If $c_1 \neq 0$

$$1 - \frac{\lambda}{2}(e^2-1) = 0$$

$$\frac{\lambda}{2}(e^2-1) = 1$$

$$\lambda = \frac{2}{e^2-1}$$

(4) $\Rightarrow f(x) = \lambda e^x c_1$

$$= \frac{2c_1}{e^2-1} e^x$$

$$\therefore \frac{2c_1}{e^2-1} = 1$$

$$f(x) = e^x$$

2. Using Hilbert Schmidt theorem solve the following I.E

$u(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2) u(t) dt$

Solution:-

Given that,

$u(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2) u(t) dt$ $\to$ ①

The standard form is given by,

$u(x) = f(x) + \lambda \int_a^b k(x,t) u(t) dt$ $\to$ ②

Comparing eqn ① & ②

$f(x) = (x+1)^2$, $\lambda = 1$, $k(x,t) = (xt + x^2t^2)$ $\to$ ③

We can find the eigen value and the corresponding normalized Eigen function.

$u(x) = \lambda \int_{-1}^{1} (xt + x^2t^2) u(t) dt \to$ ④

$u(x) = \lambda x \int_{-1}^{1} t u(t) dt + \lambda x^2 \int_{-1}^{1} t^2 u(t) dt$ $\to$ ⑤

Assume that,

$$\left.\begin{aligned} c_1 &= \int_{-1}^{1} t\, u(t) dt \\ c_2 &= \int_{-1}^{1} t^2 u(t) dt \end{aligned}\right\} \to ⑥$$

⑤ $\Rightarrow$

$u(x) = \lambda x c_1 + \lambda x^2 c_2 \to$ ⑦

$u(t) = \lambda t c_1 + \lambda t^2 c_2$

⑥ $\Rightarrow$

$c_1 = \int_{-1}^{1} t^2 (\lambda t c_1 + \lambda t^2 c_2) dt$

$= \lambda c_1 \int_{-1}^{1} t^2 dt + \lambda c_2 \int_{-1}^{1} t^3 dt$

$= \lambda c_1 \left[\frac{t^3}{3}\right]_{-1}^{1} + \lambda c_2 \left[\frac{t^4}{4}\right]_{-1}^{1}$

$= \lambda c_1 \left[\frac{1}{3} + \frac{1}{3}\right] + 0$

$c_1 = \frac{2\lambda c_1}{3}$

$\left[c_1 - \frac{2\lambda c_1}{3}\right] = 0 \rightarrow (8)$

(5) $\Rightarrow$

$c_2 = \int_{-1}^{1} t^2(\lambda t c_1 + \lambda t^2 c_2) dt$

$= \lambda c_1 \int_{-1}^{1} t^3 dt + \lambda c_2 \int_{-1}^{1} t^4 dt$

$= \lambda c_1 \left[\frac{t^4}{4}\right]_{-1}^{1} + \lambda c_2 \left[\frac{t^5}{5}\right]_{-1}^{1}$

$c_2 = \lambda c_2 \left[\frac{1}{5} + \frac{1}{5}\right]$

$c_2 = \frac{2\lambda c_3}{5}$

$\left[c_2 - \frac{2\lambda c_3}{5}\right] = 0 \rightarrow (9)$

Hence the eqn (8) & (9) are homogenous equation.

$$D(\lambda) = \begin{vmatrix} 1 - 2\lambda/3 & 0 \\ 0 & 1 - \frac{2\lambda}{5} \end{vmatrix}$$

$$0 = \left(1 - \frac{2\lambda}{3}\right)\left(1 - \frac{2\lambda}{5}\right)$$

$$1 - \frac{2\lambda}{3} = 0, \quad 1 - \frac{2\lambda}{5} = 0$$

$$\lambda = \frac{3}{2}, \frac{5}{2}$$

To find Eigen function,

when $\lambda = \frac{3}{2}$

Case (i),

$$(8) \Rightarrow c_1\left(1 - \frac{2}{3} \times \frac{3}{2}\right) = 0$$

$$c_1 = 0$$

$$(9) \Rightarrow c_2\left(1 - \frac{2}{5} \times \frac{3}{2}\right) = 0$$

$$c_2\left(1 - \frac{3}{5}\right) = 0$$

$$c_2\left(\frac{2}{5}\right) = 0$$

$$c_2 = 0$$

Assume that $c_1 \neq 0$, $c_2 = 0$

$$(7) \Rightarrow u_1(x) = \lambda x c_1$$

$$u_1(x) = \frac{3}{2} c_1 x$$

$$u_1(x) = x \quad \left(\because \frac{3c_1}{2} = 1\right)$$

Now the corresponding normalized eigen function $\phi_1(x)$ is given by,

$$\phi_1(x) = \frac{u_1(x)}{\|u_1(x)\|}$$

$$= \frac{x}{\|x\|}$$

$$\phi_1(x) = \frac{x}{\sqrt{2/3}}$$

$$\phi_1(x) = \frac{\sqrt{3}\,x}{\sqrt{2}}$$

$$\|f(x)\| = \left[\int_a^b |f(x)|^2 dx\right]^{1/2}$$

$$\|x\| = \left[\int_{-1}^{1} x^2 dx\right]^{1/2}$$

$$= \left[\left(\frac{x^3}{3}\right)\right]^{1/2}$$

$$\|x\| = \sqrt{2/3}$$

Case (ii)

$$\lambda = 5/2$$

$$⑧ \Rightarrow c_1\left(1 - \frac{2}{5} \times \frac{5}{2}\right) = 0$$

$$c_1 = 0$$

$$⑨ \Rightarrow c_2\left(1 - \frac{2}{5} \times \frac{5}{2}\right) = 0$$

Assume that $c_1 = 0$, $c_1 \neq 0$

$$⑦ \Rightarrow u_2(x) = \lambda x^2 c_2$$

$$= \frac{5c_1}{2} x^2 \qquad \therefore \frac{5c_1}{2} = 1$$

$$u_2(x) = x^2$$

Now the corresponding normalized eigen function $\phi_2(x)$ is given by,

$$= \frac{x^2}{\|x^2\|}$$

$$\phi_2(x) = \frac{x^2}{\sqrt{2/5}}$$

$$\phi_2(x) = \frac{\sqrt{5}\,x^2}{\sqrt{2}}$$

$$\|x^2\| = \left[\int_{-1}^{1} x^4\,dx\right]^{1/2}$$

$$= \left[\left(\frac{x^5}{5}\right)\right]^{1/2}$$

$$\|x^2\| = \sqrt{2/5}$$

Let $f_n = \int_a^b f(x)\,\phi_n(x)\,dx$

$$f_1 = \int_{-1}^{1} (x+1)^2 \frac{\sqrt{3}\,x}{\sqrt{2}}\,dx$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \int_{-1}^{1} (x^2+2x+1)\,x\,dx$$

$$= \frac{\sqrt{3}}{2} \int_{-1}^{1} (x^3+2x^2+x)\,dx$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \left[2\left(\frac{2}{3}\right)\right]$$

$$f_1 = \frac{\sqrt{3}}{\sqrt{2}} \left(\frac{4}{3}\right)$$

$$f_1 = \frac{4\sqrt{3}}{3\sqrt{2}}$$

$$f_2 = \int_{-1}^{1} (x+1)^2 \frac{\sqrt{5}x^2}{\sqrt{2}}\, dx$$

$$= \frac{\sqrt{5}}{\sqrt{2}} \int_{-1}^{1} (x^2+2x+1)x^2\, dx$$

$$= \frac{\sqrt{5}}{\sqrt{2}} \int_{-1}^{1} (x^4+2x^3+x^2)\, dx$$

$$= \frac{\sqrt{5}}{\sqrt{2}} \left[\frac{x^5}{5} + 2\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{\sqrt{5}}{\sqrt{2}} \left[\frac{2}{5} + \frac{2}{3}\right]$$

$$= \frac{\sqrt{5}}{\sqrt{2}} \frac{16}{15}$$

$$f_2 = \frac{16}{15} \frac{\sqrt{5}}{\sqrt{2}}$$

Since $\lambda = 1$, $\lambda_1 = 3/2$, $\lambda_2 = 5/2$

Then the soln of given eqn is given by,

$$u(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

$$u(x) = f(x) + \lambda \left[\frac{f_1}{\lambda_1 - \lambda} \phi_1(x) + \frac{f_2}{\lambda_2 - \lambda} \phi_2(x)\right]$$

$$= (x+1)^2 + \left[\frac{4\sqrt{3}/3\sqrt{2}}{3/2 - 1} \cdot \frac{\sqrt{3}x}{\sqrt{2}} + \frac{16\sqrt{5}/15\sqrt{3}}{5/2 - 1} \cdot \frac{\sqrt{5}x^2}{\sqrt{2}}\right]$$

$$= (x+1)^2 + \left[\frac{4/3 \cdot 3/2\, x}{1/2} + \frac{\frac{16 \cdot 5x^2}{15 \cdot 2}}{3/2}\right]$$

$$u(x) = (x+1)^2 + 4x + \frac{16}{9}x^2$$

Note:-

(i) If $\lambda \neq \lambda_1, \lambda \neq \lambda_2$ then the solution is given by,

$$y(x) = f(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

(ii) If $\lambda = \lambda_k$ and $f_k = 0$ then the solution is given by,

$$y(x) = f(x) + A\phi_k(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

(iii) If $\lambda = \lambda_k$ and $f_k \neq 0$, it has no solution.

1. Find the solution of the I.E

$$u(x) = (x^2+1) + \frac{3}{2}\int_{-1}^{1} (xt + x^2t^2) u(t)\, dt$$

by using hilbert's schmidt theorem.

Solution:

Given that

$$u(x) = (x^2+1) + \frac{3}{2}\int_{-1}^{1} (xt + x^2t^2) u(t)\, dt \quad \rightarrow (1)$$

The standard form is given by,

$$u(x) = f(x) + \lambda \int_a^b k(x,t) u(t)\, dt \quad \rightarrow (2)$$

From (1) & (2), we get

$$f(x) = (x^2+1), \lambda = 3/2, k(x,t) = (xt + x^2t^2) \quad \rightarrow (3)$$

We can find the Eigen value and the corresponding normalized Eigen function,

$$u(x) = \lambda \int_{-1}^{1} (xt + x^2t^2) u(t)\, dt \quad \rightarrow (4)$$

$$u(x) = \lambda x \int_{-1}^{1} t\, u(t)\, dt + \lambda x^2 \int_{-1}^{1} t^2 u(t)\, dt$$

$\rightarrow (5)$

ASSU...

$$\left.\begin{aligned} c_1 &= \int_{-1}^{1} t\,u(t)\,dt \\ c_2 &= \int_{-1}^{1} t^2 u(t)\,dt \end{aligned}\right\} \rightarrow (6)$$

(5) $\Rightarrow$

$$u(x) = \lambda x c_1 + \lambda x^2 c_2 \rightarrow (7)$$

$$u(t) = \lambda t c_1 + \lambda t^2 c_2$$

(6) $\Rightarrow$

$$c_1 = \int_{-1}^{1} t(\lambda t c_1 + \lambda t^2 c_2)\,dt.$$

$$= \lambda c_1 \int_{-1}^{1} t^2\,dt + \lambda c_2 \int_{-1}^{1} t^3\,dt$$

$$= \lambda c_1 \left[\frac{t^3}{3}\right]_{-1}^{1} + \lambda c_2 \left[\frac{t^4}{4}\right]_{-1}^{1}$$

$$c_1 = \frac{2\lambda c_1}{3}$$

$$c_1 - \frac{2\lambda c_1}{3} = 0 \rightarrow (8)$$

(6) $\Rightarrow$ $$c_2 = \int_{-1}^{1} t^2(\lambda t c_1 + \lambda t^2 c_2)\,dt$$

$$= \lambda c_1 \int_{-1}^{1} t^3\,dt + \lambda c_2 \int_{-1}^{1} t^4\,dt$$

$$= \lambda c_1 \left[\frac{t^4}{4}\right]_{-1}^{1} + \lambda c_2 \left[\frac{t^5}{5}\right]_{-1}^{1}$$

$$= \lambda c_1 (0) + \lambda c_2 \frac{2}{5}$$

$$c_2 = \frac{2\lambda c_2}{5}$$

$$c_2 - \frac{2\lambda c_2}{} = 0 \rightarrow (9)$$

Hence the eqn (8) & (9) are Homogenous equation.

$$D(\lambda) = \begin{vmatrix} 1-\frac{2\lambda}{3} & 0 \\ 0 & 1-\frac{2\lambda}{5} \end{vmatrix}$$

$$0 = \left(1-\frac{2\lambda}{3}\right)\left(1-\frac{2\lambda}{5}\right)$$

$$1-\frac{2\lambda}{3} = 0, \quad 1-\frac{2\lambda}{5} = 0$$

$$\lambda = \frac{3}{2}, \frac{5}{2}$$

$\therefore$ The Eigen values are $\frac{3}{2}, \frac{5}{2}$

To find Eigen function when $\lambda = \frac{3}{2}$

Case (i)

(8) $\Rightarrow$

$$c_1\left(1-\frac{2}{3}\times\frac{3}{2}\right) = 0$$

$$c_1 = 0$$

(9) $\Rightarrow$

$$c_2\left(1-\frac{2}{5}\times\frac{3}{2}\right) = 0$$

$$c_2\left(1-\frac{3}{5}\right) = 0$$

$$c_2 = 0$$

Assume that $c_1 \neq 0, c_2 = 0$

(7) $\Rightarrow$ $u_1(x) = \lambda x c_1$

$$u_1(x) = \frac{3}{2} a x$$

$$u_1(x) = x$$

Now the corresponding normalized eigen function $\phi_1(x)$ is given by,

$$\phi_1(x) = \frac{u_1(x)}{\|u_1(x)\|} \qquad \|f(x)\| = \left[\int_a^b |f(x)|^2 dx\right]$$

$$\|x\| = \left[\int_{-1}^{1} x^2 dx\right]^{1/2}$$

$$= \frac{x}{\|x\|}$$

$$= \left[\left(\frac{x^3}{3}\right)\right]^{1/2}$$

$$= \frac{x}{\sqrt{2/3}} \qquad \|x\| = \sqrt{2/3}$$

$$\phi_1(x) = \frac{\sqrt{3}\,x}{\sqrt{2}}$$

Case (ii)

$$\lambda = 5/2$$

(5) $\Rightarrow$ $c_1\left(1 - 2/3 \times 5/2\right) = 0$

$$c_1 = 0$$

(9) $\Rightarrow$

$$c_2\left(1 - \frac{2}{5} \times 5/2\right) = 0$$

$$c_2 = 0$$

Assume that $c_1 = 0,\ c_2 \neq 0$

(1) $\Rightarrow$ $u_2(x) = \lambda x^2 c_2$

$$= \frac{5c_1}{2} x^2 \qquad \therefore \frac{5c_1}{2} = 1$$

$$u_2(x) = x^2$$

Now the corresponding normalized eigen function $\phi_2(x)$ is given by

$$\phi_2(x) = \frac{u_2(x)}{\|u_2(x)\|}$$

$\|f(x)\| = \left[\int_a^b |f(x)|^2\, dx\right]^{\frac{1}{2}}$

$\|x^2\| = \left[\int_{-1}^{1} x^4 dx\right]^{\frac{1}{2}}$

$= \left[\left(\frac{x^5}{5}\right)\right]^{\frac{1}{2}}$

$\|x^2\| = \sqrt{2/5}$

$$\phi_2(x) = \frac{x^2}{\|x^2\|}$$

$$= \frac{x^2}{\sqrt{2/5}}$$

$$\phi_2(x) = \frac{\sqrt{5}\, x^2}{\sqrt{2}}$$

$$f_n = \int_a^b f(x)\phi_n(x)\, dx$$

$$f_1 = \int_{-1}^{1} (x^2+1)\frac{\sqrt{3}\, x}{\sqrt{2}}\, dx$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\int_{-1}^{1} (x^3+1)\, dx$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$f_1 = 0$$

$$f_2 = \int_{-1}^{1} (x^2+1)\frac{\sqrt{5}\, x^2}{\sqrt{2}}\, dx$$

$$= \frac{\sqrt{5}}{\sqrt{2}}\int_{-1}^{1} x^4 + x^2 dx$$

$$= \frac{\sqrt{5}}{\sqrt{2}}\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_{-1}^{1} = \frac{\sqrt{5}}{\sqrt{2}}\frac{16}{15}$$

$$= \frac{16\sqrt{5}}{15\sqrt{2}}$$

Here ...

∴ The solution given by,

$$y(x) = f(x) + A\phi_k(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda}\phi_m(x)$$

$$u(x) = (x^2+1) + A\phi_1(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda}\phi_m(x)$$

$$= (x^2+1) + A\frac{\sqrt{3}x}{\sqrt{2}} + \frac{3}{2}\left[\frac{f_1}{\lambda_1 - \lambda}\phi_1(x) + \frac{f_2}{\lambda_2 - \lambda}\phi_2(x)\right]$$

$$= (x^2+1) + \frac{A\sqrt{3}x}{\sqrt{2}} + \frac{3}{2}\left[0 + \frac{16\sqrt{5}/15\sqrt{2}}{5/2 \cdot 3/2} \cdot \frac{\sqrt{5}x^2}{\sqrt{2}}\right]$$

$$= x^2+1 + \frac{A\sqrt{3}x}{\sqrt{2}} + \frac{3}{2}\left[\frac{16 \cdot 5 \cdot x^2}{3 \cdot 15 \cdot 2}\right]$$

$$u(x) = x^2+1 + A\frac{\sqrt{3}x}{\sqrt{2}} + 4x^2$$

$$u(x) = 5x^2 + \frac{A\sqrt{3}x}{\sqrt{2}} + 1$$

2. Solve the following symmetric equation with the help of Hilbert Schmidt theorem,

$$u(x) = 1 + \lambda\int_0^{\pi} \cos(x+t)u(t)\,dt$$

Solution:-

Given that,

$$u(x) = 1 + \lambda\int_0^{\pi} \cos(x+t)u(t)\,dt \rightarrow (1)$$

The standard form is given by,

$$u(x) = f(x) + \lambda\int_a^b k(x,t)u(t)\,dt \rightarrow (2)$$

From ① & ② we get

$f(x)=1, \quad \lambda=\lambda, \quad k(x,t)=\cos(x+t) \rightarrow ③$

We can find the eigen value and the corresponding normalized Eigen function,

$$u(x)=\lambda\int_0^{\pi}\cos(x+t)u(t)dt \rightarrow ④$$

$$u(x)=\lambda\int_0^{\pi}\cos x\cos t-\sin x\sin t\, u(t)dt$$

$$u(x)=\lambda\cos x\int_0^{\pi}\cos t\,u(t)dt-\lambda\sin x\int_0^{\pi}\sin t\,u(t)dt \rightarrow ⑤$$

Assume that

$$\left.\begin{aligned} c_1 &= \int_0^{\pi}\cos t\,u(t)dt \\ c_2 &= \int_0^{\pi}\sin t\,u(t)dt \end{aligned}\right\} \rightarrow ⑥$$

$$⑤ \Rightarrow u(x)=\lambda\cos x\,c_1-\lambda\sin x\,c_2 \rightarrow ⑦$$

$$u(t)=\lambda c_1\cos t-\lambda c_2\sin t$$

⑥ $\Rightarrow$

$$c_1=\int_0^{\pi}\cos t\,(\lambda c_1\cos t-\lambda c_2\sin t)dt$$

$$=\lambda c_1\int_0^{\pi}\cos^2 t\,dt-\lambda c_2\int_0^{\pi}\cos t\sin t\,dt$$

$$=\lambda c_1\int_0^{\pi}\frac{1+\cos 2t}{2t}dt-\lambda c_2\int_0^{\pi}\frac{\sin 2t}{2}dt$$

$$=\frac{\lambda c_1}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{\pi}-\frac{\lambda c_2}{2}\left[\frac{-\cos 2t}{2}\right]_0^{\pi}$$

$$= \frac{\lambda c_1}{2}(\pi) + \frac{\lambda c_2}{2}\left[\frac{1}{2} - \frac{1}{2}\right]$$

$$c_1 = \frac{\lambda c_1 \pi}{2}$$

$$c_1\left[1 - \frac{\lambda\pi}{2}\right] = 0 \rightarrow (8)$$

(6) $\Rightarrow$

$$c_2 = \int_0^\pi \sin t(\lambda c_1 \cos t - \lambda c_2 \sin t)\,dt$$

$$= \lambda c_1 \int_0^\pi \sin t \cos t\,dt - \lambda c_2 \int_0^\pi \sin^2 t\,dt$$

$$= \lambda c_1 \int_0^\pi \frac{\sin 2t}{2}\,dt - \lambda c_2 \int_0^\pi \left[1 - \frac{\cos 2t}{2}\right]dt$$

$$= \frac{\lambda c_1}{2}\left[-\frac{\cos 2t}{2}\right]_0^\pi - \frac{\lambda c_2}{2}\left[t - \frac{\sin 2t}{2}\right]_0^\pi$$

$$c_2 = -\frac{\lambda c_2 \pi}{2}$$

$$c_2\left[1 + \frac{\lambda\pi}{2}\right] = 0 \rightarrow (9)$$

Hence the eqn (8) & (9) are homogeneous equation.

$$P(\lambda) = \begin{vmatrix} 1 - \frac{\lambda\pi}{2} & 0 \\ 0 & \frac{1+\lambda\pi}{2} \end{vmatrix}$$

$$0 = \left(1 - \frac{\lambda\pi}{2}\right)\left(1 + \frac{\lambda\pi}{2}\right)$$

$$1 - \frac{\lambda \pi}{2} = 0 \quad ; \quad \frac{1 + \lambda \pi}{2} = 0$$

$$-\frac{\lambda \pi}{2} = -1 \qquad \frac{\lambda \pi}{2} = -1$$

$$\lambda = \frac{2}{\pi} \qquad \lambda = -\frac{2}{\pi}$$

$\therefore$ The eigen values are $\frac{2}{\pi}, -\frac{2}{\pi}$

To find Eigen function.

When $\lambda = \frac{2}{\pi}$

Case (1)

(8) $\Rightarrow$

$$c_1 \left(1 - \frac{2}{\pi} \frac{\pi}{2}\right) = 0$$

$$c_1 = 0$$

(9) $\Rightarrow$

$$c_2 \left(1 + \frac{\lambda \pi}{2}\right) = 0$$

$$c_2 \left(1 + \frac{2}{\pi} \frac{\pi}{2}\right) = 0$$

$$c_2 = 0$$

Assume that $c_1 \neq 0$, $c_2 = 0$

(7) $\Rightarrow$ $u_1(x) = \lambda c_1 \cos x$

$$= \frac{2}{\pi} c_1 \cos x \quad \because \frac{2 c_1}{\pi} = 1$$

$$u_1(x) = \cos x$$

... normalized eigen function $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{u_1(x)}{\|u_1(x)\|}$$

$$\phi_1(x) = \frac{\cos x}{\|\cos x)} \qquad \|\cos x\| = \left[\int_0^{\pi} \cos^2 x\, dx\right]$$

$$= \frac{\cos x}{\sqrt{\pi/2}} \qquad = \left\{\int_0^{\pi} \left[\frac{1+\cos 2t}{2}\right]\right.$$

$$= \left\{\frac{1}{2}\left(x + \frac{\sin 2t}{2}\right]\right.$$

$$\phi_1(x) = \frac{\sqrt{2}\cos x}{\sqrt{\pi}}$$

$$= \left\{\frac{1}{2}\pi\right\}^{1/2}$$

$$= \sqrt{\pi/2}$$

Case (ii)

$$\lambda = -2/\pi$$

(8) $\Rightarrow$

$$c_1\left(1 - \frac{\lambda\pi}{2}\right) = 0$$

$$c_1\left(1 + \frac{2}{\pi}\frac{\pi}{2}\right) = 0$$

$$c_1 = 0$$

(9) $\Rightarrow$

$$c_2\left(1 + \frac{\lambda\pi}{2}\right) = 0$$

$$c_2\left(1 - \frac{2}{\pi}\,\pi/2\right) = 0$$

$$c_2 = 0$$

Assume that $c_1 = 0$, $c_1 \neq 0$

$(D) \Rightarrow u_2(x) = -\lambda \sin x\, c_2$

$$= -\left(\frac{2}{\pi}\right)\sin x\, c_2$$

$$= \frac{2c_2}{\pi}\sin x \qquad \because \frac{2c_2}{\pi} = 1$$

$$u_2(x) = \sin x$$

now, the corresponding normalized eigen function $\phi_2(x)$ is given by,

$$\phi_2(x) = \frac{u_2(x)}{\|u_2(x)\|}$$

$$\phi_2(x) = \frac{\sin x}{\|\sin x\|} \qquad \|\sin x\| = \left[\int_0^{\pi} \sin^2 x\right]^{\frac{1}{2}}$$

$$= \frac{\sin x}{\sqrt{\pi/2}} \qquad = \left\{\int_0^{\pi}\left[1 - \frac{\cos 2x}{2}\right]\right\}^{\frac{1}{2}}$$

$$\phi_2(x) = \frac{\sqrt{2}\sin x}{\sqrt{\pi}} \qquad = \left\{\frac{1}{2}\left[x - \frac{\sin x}{2}\right]\right\}^{\frac{1}{2}}$$

Let,

$$= \left\{\frac{1}{2}\pi\right\}^{\frac{1}{2}}$$

$$f_n = \int_a^b f(x)\phi_n(x)\,dx$$

$$= \sqrt{\pi/2}$$

$$f_1 = \int_0^{\pi} \frac{\sqrt{2}\cos x}{\sqrt{\pi}}\,dx$$

$$f_1 = \frac{\sqrt{2}}{\sqrt{\pi}}\left[\sin x\right]_0^{\pi}$$

$$f_1 = 0$$

$$f_2 = \int_0^{\pi} \frac{\sqrt{2}\sin x}{\sqrt{\pi}} \, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\pi} \sin x \, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[-\cos x\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} (2)$$

$$f_2 = \frac{2\sqrt{2}}{\sqrt{\pi}}$$

case (i)

If $\lambda \neq \lambda_1$, and $\lambda \neq \lambda_2$ then there exists a unique solution of eqn (1)

$$u(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

$$u_2(x) = \sin x$$

Now, the corresponding normalized eigen function $\phi_2(x)$ is given by,

$$\phi_2(x) = \frac{u_2(x)}{\|u_2(x)\|}$$

$$\|\sin x\| = \left\{\int_0^{\pi} \sin^2 x \, dx\right\}$$

$$= \left\{\int_0^{\pi} \frac{1-\cos 2x}{2} dx\right\}^{1/2}$$

$$= \frac{\sin x}{\|\sin x\|}$$

$$= \left\{\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}\right\}$$

$$= \frac{\sin x}{\sqrt{\pi}\sqrt{2}}$$

$$\phi_2(x) = \frac{\sqrt{2}\sin x}{\sqrt{\pi}}$$

$$= \sqrt{\pi/2}$$

let $f_n = \int_a^b f(x) \phi_n(x) dx$

$$f_1 = \int_0^{\pi} \frac{\sqrt{2}\cos x}{\sqrt{\pi}} dx,$$