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Classical dynamics

Objective type question

OBJECTIVE TYPE QUESTIONS

1. If a generalised co-ordinate has the dimensions of momentum, the generalised velocity will have the dimension of—
   (A) Velocity (B) Acceleration
   (C) Force (D) Torque

2. The product of any generalised momentum and the associated (or conjugate) co-ordinate must have the dimensions of—
   (A) Energy
   (B) Angular momentum
   (C) Linear momentum
   (D) Force

3. Whatever dimension a generalised co-ordinate has, the product of the generalised force and generalised displacement (co-ordinate) must have the dimension of—
   (A) Force (B) Torque
   (C) Work (D) None of these

4. Equation of motion for bead sliding on a uniformly rotating wire in a force free space is—
   (A) $\ddot{r} = r\omega^2$
   (B) $\ddot{\theta} + \frac{mg\,l\theta}{I} = 0$
   (C) $2mr\dot{r}\,\dot{\theta} + mr^2\,\ddot{\theta} = 0$
   (D) $\ddot{\theta} + \frac{g}{l}\,\theta = 0$

5. A hoop rolling down on an inclined plane without slipping, its velocity at the bottom of the inclined plane—
   (A) $\left(\frac{4gl\sin\phi}{3}\right)^{\frac{1}{2}}$ (B) $\frac{2gl\sin\phi}{3}$
   (C) $\left(\frac{2gl\sin\phi}{3}\right)^{\frac{1}{2}}$ (D) $\frac{4gl\sin\phi}{3}$

6. For an electrical circuit comprising an inductance L and capacitance C, charged to q coulombs and the current flowing in the circuit is i amperes, Lagrangian can be represented as—
   (A) $L\dot{q}^2 - \frac{q^2}{C}$ (B) $\frac{1}{2}L\dot{q}^2 - \frac{1}{2}q^2C$
   (C) $\frac{1}{2}L\dot{q}^2 - \frac{1}{2}\frac{q^2}{C}$ (D) $\frac{1}{2}L\dot{q}^2 + \frac{1}{2}\frac{q^2}{C}$

7. Lagrangian for a charged particle in an electromagnetic field is given as—
   (A) $\frac{1}{2}mv^2 + q\phi + \frac{q}{c}\,\overline{v}\cdot\overline{A}$

(B) $\frac{1}{2}mv^2 - q\phi - \frac{q}{c}\overline{v}\cdot\overline{A}$

(C) $\frac{1}{2}mv^2 - q\phi + \frac{q}{c}\overline{v}\cdot\overline{A}$

(D) $\frac{1}{2}mv^2 + q\phi - \frac{q}{c}\overline{v}\cdot\overline{A}$

8. Lagrangian for compound pendulum is—

(A) $\frac{1}{2}I\dot{\theta}^2 - mgl\cos\theta$

(B) $\frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta$

(C) $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2 - mgl\cos\theta$

(D) $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta$

9. The path followed by a particle in sliding from one point to another in the absence of friction in the shortest time is a—

(A) Sphere
(B) Sigmoid
(C) Cycloid
(D) Catenary of revolution

10. If a co-ordinate corresponding to a rotation is cyclic, rotation of the system about given axis remains invariant then the following quantity is conserved—

(A) Linear momentum
(B) Angular momentum
(C) Kinetic energy
(D) Potential energy

11. A physical system is invariant under rotation about a fixed axis. Then the following quantity is conserved—

(A) Total linear momentum
(B) Linear momentum along the axis of rotation
(C) Total angular momentum
(D) Angular momentum along the axis of rotation

12. The period of oscillation for compound pendulum is—

(A) $2\pi\sqrt{\frac{(k^2+l^2)}{gl}}$ (B) $2\pi\sqrt{\frac{gl}{k^2+l^2}}$

(C) $2\pi\sqrt{\frac{(k^2+l^2)}{mgl}}$ (D) $2\pi\sqrt{\frac{mgl}{(k^2+l^2)}}$

13. A point mass, m, under no external forces is attached to a weightless cord fixed to a cylinder of radius R. Initially the cord is completely wound up so that mass touches the cylinder. A radially directed impulse is now given to the mass, which starts unwinding, then the angular momentum of the mass about the cylinder axis—

(A) $mR\,v_0^2\,t$ (B) $m \times (2R\,v_0^3\,t)^{1/2}$

(C) $m\,(2R\,v_0^3\,t)^{1/3}$ (D) $m \times (2R\,v_0^3\,t)$

14. A solid homogeneous cylinder of radius, r, rolls without slipping on the inside of a stationary large cylinder of radius R the period of small oscillations about the stable equilibrium position is—

(A) $2\pi\sqrt{\left(\frac{3(R-r)}{2g}\right)}$

(B) $2\pi\sqrt{\left(\frac{2(R-r)}{3g}\right)}$

(C) $2\pi\sqrt{\frac{2g}{3(R-r)}}$

(D) $2\pi\sqrt{\frac{3g}{2(R-r)}}$

15. Hamilton's canonical equations of motion are—

(A) $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = \frac{\partial H}{\partial q_i}$

(B) $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

(C) $q_i = \frac{\partial H}{\partial \dot{p}_i}$ and $p_i = \frac{\partial H}{\partial \dot{q}_i}$

(D) $q_i = \frac{\partial H}{\partial \dot{p}_i}$ and $p_i = -\frac{\partial H}{\partial \dot{q}_i}$

16. If a co-ordinate is cyclic, Hamiltonian would reduce the number of variables in new formulation by—

(A) One (B) Two
(C) Three (D) Four

17. For a charged particle in an electromagnetic field, the canonical momenta are—

(A) $mv + \frac{q}{c}A$ (B) $\frac{1}{2}mv^2 + \frac{q}{c}A$

(C) $mv - \frac{q}{c}A$ (D) $\frac{1}{2}mv^2 - \frac{q}{c}A$

18. For a charged particle in a electromagnetic field, the Hamiltonian $\overline{H}$ is represented as—

(A) $\frac{1}{2}m\left(\frac{\overline{p}}{m}+\frac{q}{mc}\overline{A}\right)^2+q\phi$

(B) $\frac{1}{2}m\left(\frac{\overline{p}}{m}+\frac{q}{mc}\overline{A}\right)^2-q\phi$

(C) $\frac{1}{2}m\left(\frac{\overline{p}}{m}-\frac{q}{mc}\overline{A}\right)^2+q\phi$

(D) $\frac{1}{2}m\left(\frac{\overline{p}}{m}-\frac{q}{mc}\overline{A}\right)^2-q\phi$

19. The Jacobi's form of the least action principle—

(A) $\Delta\int\sqrt{2[H-V(q)]}\,d\rho=0$

(B) $\Delta\int\sqrt{2[H+V(q)]}\,d\rho=0$

(C) $\Delta\int\sqrt{2[L-V(q)]}\,d\rho=0$

(D) $\Delta\int\sqrt{2[L+V(q)]}\,d\rho=0$

20. An artificial satellite revolves about the earth at height H above the surface, the orbital period so that a man in the satellite will be in the state of weightlessness is—

(A) $2\pi\sqrt{\frac{g}{R}}$ (B) $2\pi\sqrt{\frac{R}{g}}$

(C) $\frac{1}{2\pi}\sqrt{\frac{g}{R}}$ (D) $\frac{1}{2\pi}\sqrt{\frac{R}{g}}$

21. If a body is thrown vertically upwards, it strikes the ground at—

(A) $\frac{16}{3}\omega h\cos\phi\left(\frac{2h}{g}\right)^{1/2}$ to the west

(B) $\frac{16}{3}\omega h\cos\phi\left(\frac{2h}{g}\right)^{1/2}$ to the east

(C) $\frac{2}{3}\omega h\cos\phi\left(\frac{2h}{g}\right)^{1/2}$ to the west

(D) $\frac{2}{3}\omega h\cos\phi\left(\frac{2h}{g}\right)^{1/2}$ to the east

22. The α-particle scattering cross-section and hence the number of α-particle scattered must be proportional to—

(A) E (B) E^{-1}

(C) E^2 (D) E^{-2}

23. The mechanical equivalent of an LCR series circuit with a voltage source is a—

(A) Damped harmonic oscillator

(B) Forced harmonic oscillator

(C) Free linear harmonic oscillator

(D) Damped and forced harmonic oscillator

24. The value of m and n for which the transformations are

$$Q=q^m\cos np\ ;\ P=q^m\sin np$$

represents a canonical transformations are—

(A) $m=1, n=2$ (B) $m=\frac{1}{2}, n=2$

(C) $m=2, n=\frac{1}{2}$ (D) $m=2, n=1$

25. Jacobi identity for Poisson bracket—

(A) $[X,[Y,H]]+[Y,[H,X]]+[H,[X,Y]]=0$

(B) $[X,[Y,H]]-[Y,[H,X]]+[H,[X,Y]]=0$

(C) $[X,[Y,H]]+[Y,[H,X]]-[H,[X,Y]]=0$

(D) $[X,[Y,H]]-[Y,[H,X]]-[H,[X,Y]]=0$

26. The operator which represents the two variables should commute if the Poisson bracket of two variables have value—

(A) 1 (B) 0

(C) $i\hbar$ (D) $-i\hbar$

27. An inverted pendulum consists of a particle of mass m supported by a rigid massless rod of length l. The pivot 0 has a vertical motion given by $z=A\sin\omega t$, the Lagrangian of the system is—

(A) $\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta-mlA\omega^2\sin\omega t\cos\theta$

(B) $\frac{1}{2}ml^2\dot{\theta}^2-mgl\cos\theta-mlA\omega^2\sin\omega t\cos\theta$

(C) $\frac{1}{2}ml^2\dot{\theta}^2-mgl\cos\theta+mlA\omega^2\sin\omega t\cos\theta$

(D) $\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta+mlA\omega^2\sin\omega t\cos\theta$

28. Lagrange's equations of motion are second order equations, the degrees of freedom for this are—

(A) $2n$ (B) $2n-1$

(C) $2n+1$ (D) $2n+2$

29. For the transformation $Q=\log(1+q^{1/2}\cos p)$; $P=2q^{1/2}(1+q^{1/2}\cos p)$. The generating

function is—
(A) $-(e^Q - 1)^2 \tan p$
(B) $(e^Q - 1)^2 \cot p$
(C) $(e^Q - 1)^2 \tan p$
(D) $-(e^Q - 1)^2 \cot p$

30. For generating function $F_1 = \frac{1}{2} m\omega q^2 \cot Q$, an expression for the displacement of linear harmonic oscillator is given by—

(A) $\sqrt{\frac{1}{m\omega^2}} \sin(\omega t + \beta)$

(B) $\sqrt{\frac{1}{m\omega^2}} \sin(\omega t - \beta)$

(C) $\sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \beta)$

(D) $\sqrt{\frac{2E}{m\omega^2}} \sin(\omega t - \beta)$

31. The force which is always directed away or towards a fixed centre and magnitude of which is a function only of the distance from the fixed centre, known as—
(A) Coriolis force
(B) Centripetal force
(C) Centrifugal force
(D) Central force

32. Differential equation for planetary motion is given as—

(A) $\frac{d^2u}{d\theta^2} = u - \frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$

(B) $\frac{d^2u}{d\theta^2} = u + \frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$

(C) $\frac{d^2u}{d\theta^2} = -u - \frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$

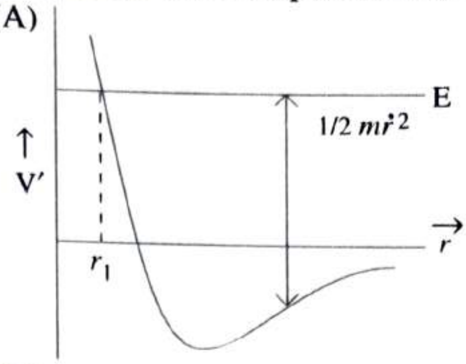
(D) $\frac{d^2u}{d\theta^2} = -u + \frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$

33. For orbits under inverse square law of force, the effective potential energy is given by—

(A) $\frac{k}{r} + \frac{l^2}{2mr^2}$ (B) $\frac{k}{r} - \frac{l^2}{2mr^2}$

(C) $\frac{k}{r^2} + \frac{l^2}{2mr^2}$ (D) $-\frac{k}{r^2} - \frac{l^2}{2mr^2}$

34. Plot for unbounded motion at positive energies for inverse square law of force—

(A)


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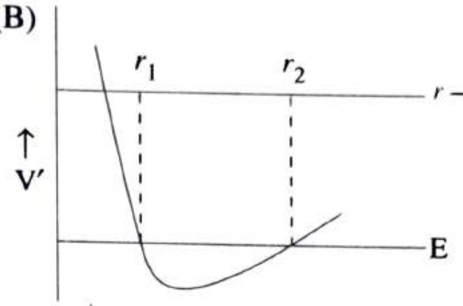

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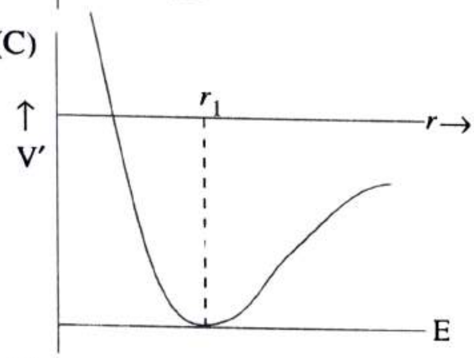

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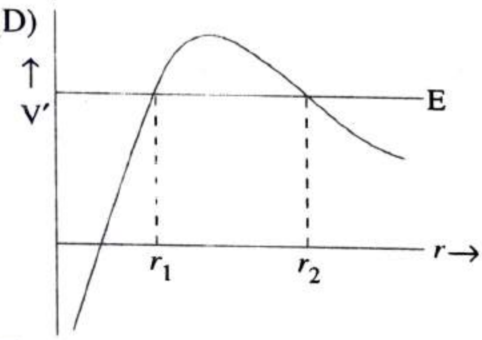
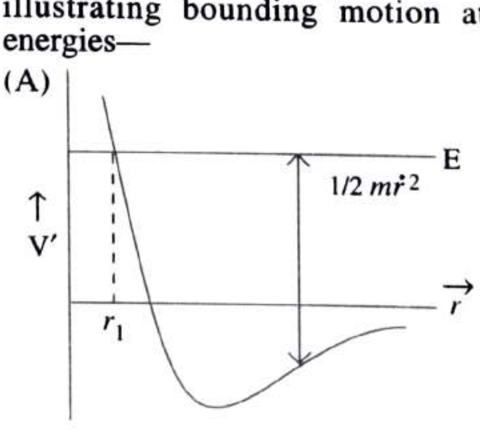
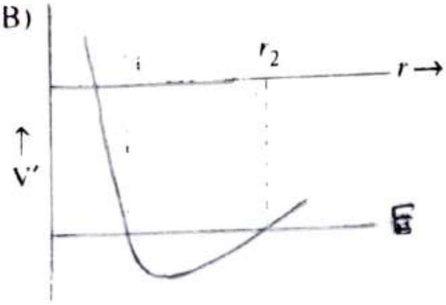


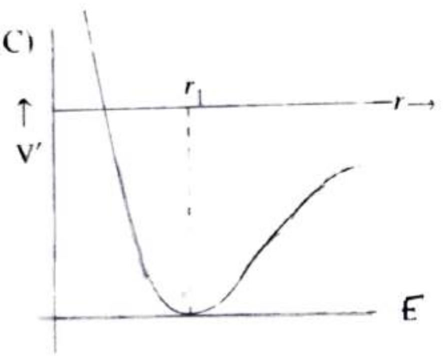
35. Plot for the equivalent one-dimensional potential for inverse square law of force, illustrating bounding motion at negative energies—

(A)


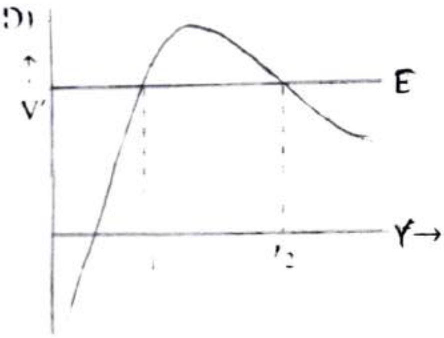
(B)



(C)

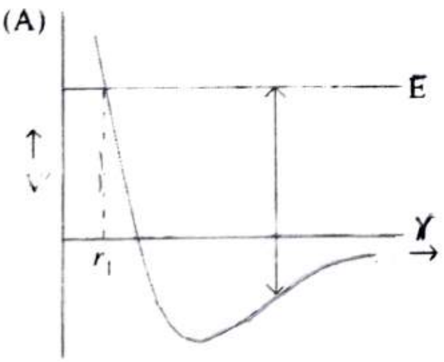


(D)

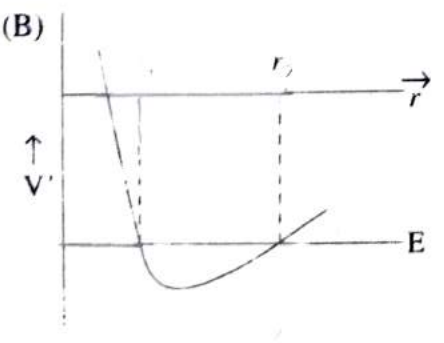


36 Plot for the equivalent one dimensional potential for inverse square law of force, illustrating the condition for circular orbits—

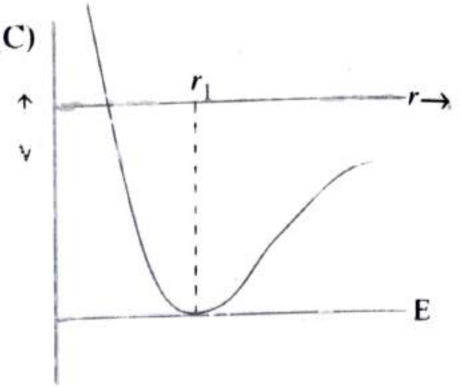
(A)



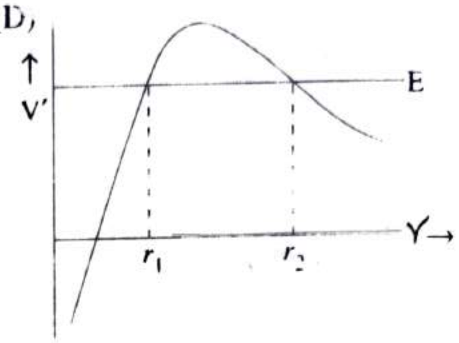
(B)



(C)

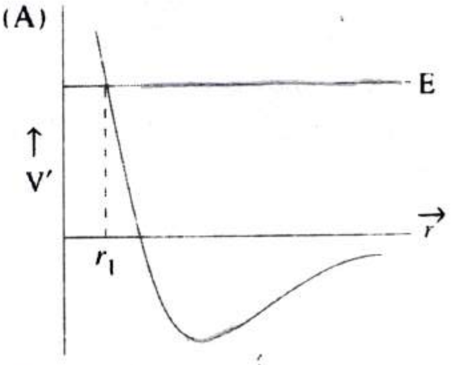


(D)

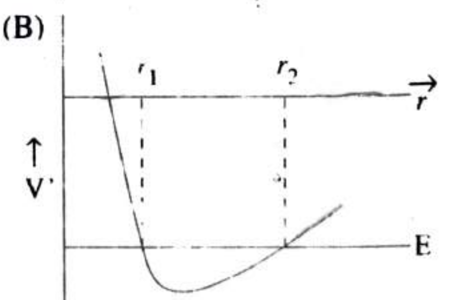


37. Plot for the equivalent one dimensional potential for an attractive inverse fourth law of force—

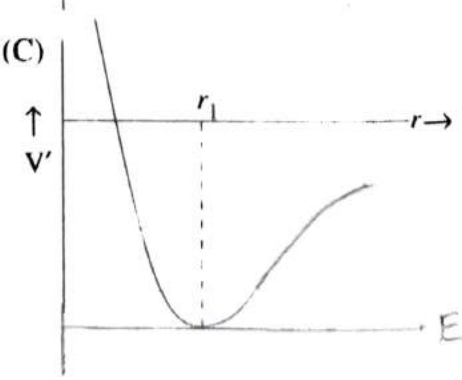
(A)

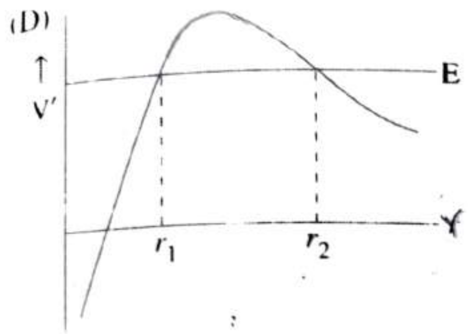


(B)

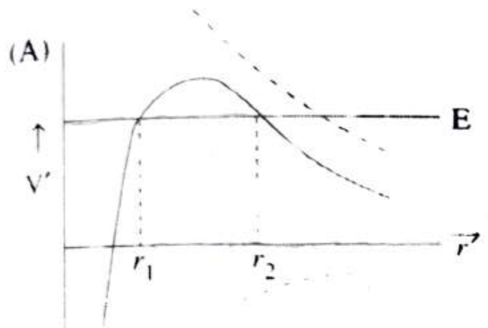


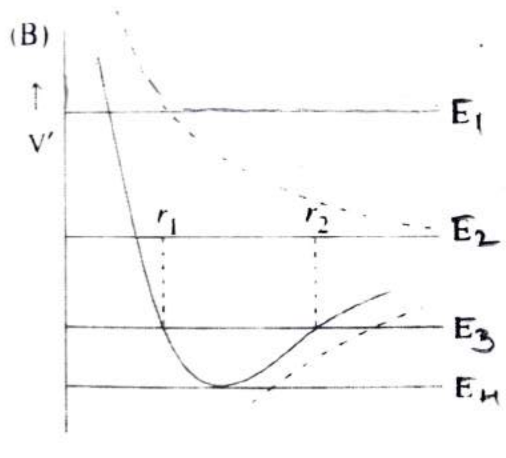
(C)

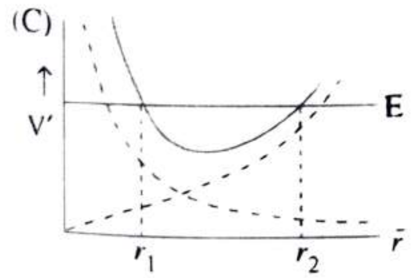


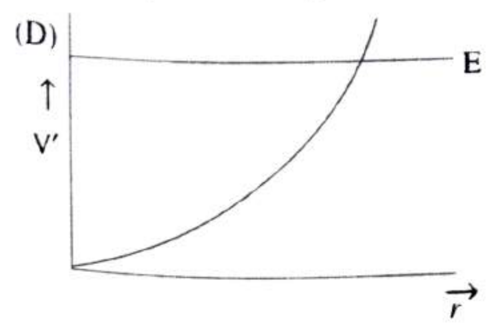


38. Plot for a linear restoring force (isotropic harmonic oscillator) for any positive energy at which motion is bounded—

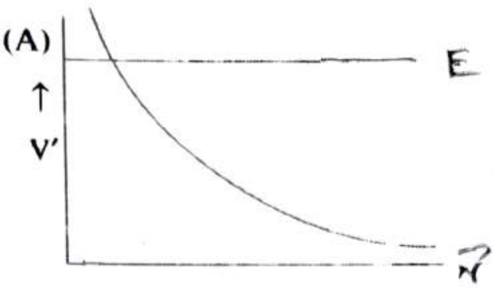


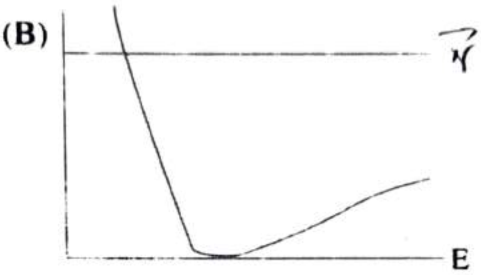


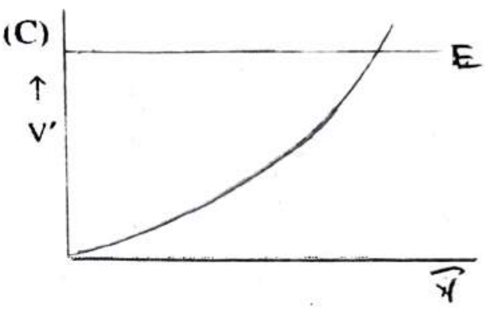


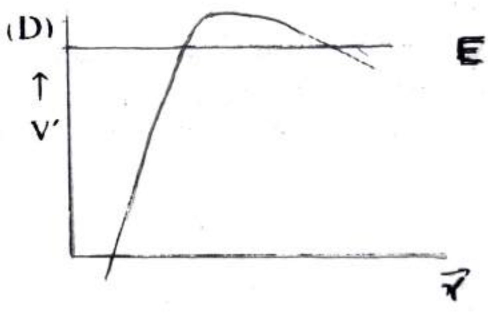


39. Plot for a linear restoring force (isotropic harmonic oscillator) for zero angular momentum—









40. For attractive inverse square forces. the shape of orbit will be—
(A) Elliptic (B) Parabolic
(C) Hyperbolic (D) All of these

41. For repulsive inverse square forces, the shape of orbit—
(A) Elliptic (B) Parabolic
(C) Hyperbolic (D) All of these

42. In the case of elliptic orbits, energy is proportional to—
(A) a (B) a^{-1}
(C) a^{-3} (D) a^3
where a is semi major axis of elliptic orbit.

43. A particle of mass, m, moves under the action of a central force whose potential is $V(r) = kmr^3$ $(k > 0)$, then energy for which the orbit will be a circle of radius a, about the origin is—

(A) $\frac{3}{2}mka^3$ (B) $\frac{3}{2}mka^2$

(C) $\frac{1}{2}mka$ (D) $\frac{1}{2}mka^2$

44. A particle of mass, m, moves under the action of a central force whose potential is $V(r) = kmr^3$ $(k > 0)$, then the period of circular motion is—

(A) $\frac{2\pi}{\sqrt{ka^2}}$ (B) $\frac{2\pi}{\sqrt{3ka}}$

(C) $\frac{2\pi}{\sqrt{mka}}$ (D) $\frac{2\pi}{\sqrt{ka}}$

45. A particle of mass, m, moves under the action of a central force whose potential is $V(r) = kmr^3$ $(k > 0)$, then angular momentum for which the orbit will be a circle of radius a, about the origin is—

(A) $ma\sqrt{3ka}$ (B) $ma^2\sqrt{ka}$

(C) $ma^2\sqrt{3ka}$ (D) $ma\sqrt{ka}$

46. A particle of mass, m, moves under the action of a central force whose potential is $V(r) = kmr^3$ $(k > 0)$, then the angular frequency is—

(A) $\sqrt{3ka}$ (B) $\sqrt{ka}$

(C) $\sqrt{5ka}$ (D) $\sqrt{15ka}$

47. The mutual potential energy V, of two particles depends on their mutual distances, r, as follows

$$V = \frac{a}{r^2} - \frac{b}{r}\ ;\ a > 0, b > 0$$

if the particles are in static equilibrium, then the separation is—

(A) $\frac{2a}{b}$ (B) $\frac{2b}{a}$

(C) $\frac{a}{b}$ (D) $\frac{b}{a}$

48. A particle of mass m moves in a central force field defined by $\vec{F} = -k\,\vec{r}/r^4$, if E is the total energy supplied to the particle, then its speed is given by—

(A) $\frac{k}{mr^2} + \frac{2E}{m}$ (B) $\frac{k}{mr^2} - \frac{2E}{m}$

(C) $\sqrt{\frac{k}{mr^2} + \frac{2E}{m}}$ (D) $\sqrt{\frac{k}{mr^2} - \frac{2E}{m}}$

49. A particle moving in a central force located at $r = 0$ describes the spiral $r = e^{-\theta}$, the magnitude of force is inversely proportional to—

(A) r (B) r^2

(C) r^3 (D) r^4

50. A particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle. The force inversely proportional to—

(A) r^2 (B) r^3

(C) r^4 (D) r^5

51. For orbits, the conic depends on the value of eccentricity given by—

(A) $\sqrt{1 - \frac{2El^2}{mk^2}}$ (B) $\sqrt{1 + \frac{2El^2}{mk^2}}$

(C) $\sqrt{1 - \frac{mk^2}{2El^2}}$ (D) $\sqrt{1 + \frac{mk^2}{2El^2}}$

52. The impact parameter, s, defind as the perpendicular distance between the centre of force and the incident velocity this parameter proportional to— [E → Energy]

(A) E (B) $E^{1/2}$

(C) E^{-1} (D) $E^{-1/2}$

53. The orbit is symmetric about the direction of periapsis, the scattering angle is given by—

(A) $\Theta = \pi - 2\Psi$ (B) $\Theta = \pi + 2\Psi$

(C) $\Theta = 2\pi - \Psi$ (D) $\Theta = 2\pi + \Psi$

54. The desired relationship between the impact parameter and the scattering angle is—

(A) $\frac{ZZ'e^2}{2E}\operatorname{cosec}\frac{\Theta}{2}$

(B) $\frac{ZZ'e^2}{2E}\sin\frac{\Theta}{2}$

(C) $\frac{ZZ'e^2}{2E}\cot\frac{\Theta}{2}$

(D) $\left(\frac{ZZ'e^2}{2E}\right)^2\operatorname{cosec}^4\frac{\Theta}{2}$

55. In the famous Rutherford scattering cross section, differential scattering cross section is proportional to— [$e \rightarrow$ Charge]

(A) e (B) e^2

(C) e^3 (D) e^4

56. In Rutherford scattering cross section, the differential scattering cross section is inversely proportional to—

(A) $\sin \Theta$ (B) $\sin^2 \Theta$

(C) $\sin^3 \Theta$ (D) $\sin^4 \Theta$

57. The angle of recoil of the target particle relative to the incident direction of the scattered particle is—

(A) $\frac{1}{2}(\pi - \Theta)$ (B) $\frac{1}{2}(\pi + \Theta)$

(C) $\pi - \Theta$ (D) $\pi + \Theta$

58. A body is freely falling on the earth's surface, the body deflects by—

(A) 3 cm towards west

(B) 3 cm towards east

(C) 3 cm towards south

(D) 3 cm towards north

59. A ball is released from rest from a great height above the ground in Delhi. It will fall on the ground—

(A) Exactly below the point of release

(B) Slightly east of the vertical

(C) Slightly west of the vertical

(D) Slightly north of the vertical

60. The Lorentz transformation matrix, where the relative velocity is along the Z-axis, given by—

(A) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \gamma & i\beta\gamma \\ 0 & 1 & -i\beta\gamma & \gamma \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma & i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i\beta\gamma & \gamma \\ 0 & 0 & -r & i\beta\gamma \end{pmatrix}$

61. The length contraction—

(A) Predicts that the length of an object approaches zero as its speed approaches the speed of light in vacuum

(B) Predicts that there is no change in the length of an object when its speed approaches the speed of light in vacuum

(C) Predicts that the length of an object reduces to half when its speed approaches the speed of light in vacuum

(D) Predicts that the length of the object is directly proportional to its velocity

62. An astronaut moves in a super spaceship travelling at a speed of $0{\cdot}8c$. The astronaut observes a photon approaching him from space. The speed of photon with respect to the astronaut is—

(A) $1{\cdot}8c$ (B) c

(C) $0{\cdot}2c$ (D) $0{\cdot}9c$

63. A spaceship is travelling with a velocity $0{\cdot}4c$, where c is the velocity of light. A person performing an experiment in these spaceship observes a particle moving with a velocity $0{\cdot}4c$ in the same direction as that of the motion of the spaceship. A stationary observer on the earth would observe the particle to have the velocity—

(A) $0{\cdot}69c$ (B) $0{\cdot}50c$

(C) $0{\cdot}80c$ (D) $0{\cdot}73c$

64. Lorentz transformations assume—

(A) Space and time are both relative

(B) Space is relative, but time is absolute

(C) Space is absolute but time is relative

(D) Space and time are both absolute

65. The difference vector X_{μ}, is space like if the two world points are separated such that—

(A) $|\bar{r}_1 - \bar{r}_2|^2 \geq c^2 (t_1 - t_2)^2$

(B) $|\bar{r}_1 - \bar{r}_2|^2 \leq c^2 (t_1 - t_2)^2$

(C) $|\bar{r}_1 - \bar{r}_2|^2 > c^2 (t_1 - t_2)^2$

(D) $|\bar{r}_1 - \bar{r}_2|^2 < c^2 (t_1 - t_2)^2$

66. The difference vector X_{μ} is time like if the two world points are separated such that—

(A) $|\bar{r}_1 - \bar{r}_2|^2 \geq c^2 (t_1 - t_2)^2$

(B) $|\bar{r}_1 - \bar{r}_2|^2 \leq c^2 (t_1 - t_2)^2$

(C) $|\bar{r}_1 - \bar{r}_2|^2 > c^2 (t_1 - t_2)^2$

(D) $|\bar{r}_1 - \bar{r}_2|^2 < c^2 (t_1 - t_2)^2$

67. The Lorentz transformations are equivalent to rotation of axes in four dimensional space through an imaginary angle—

(A) $\tan (i\beta)$ (B) $\sin \left(\frac{i\beta}{\sqrt{1-\beta^2}}\right)$

(C) $\tan^{-1} (i\beta)$ (D) $\cos^{-1} \left(\frac{i\beta}{\sqrt{1-\beta^2}}\right)$

68. The proper length of a space vehicle is l_0. According to an observer on earth, the length of the spaceship is 25% of its proper length. The speed of the spaceship according to the observer on earth is—

(A) $\frac{c\sqrt{3}}{2}$ (B) $c\sqrt{\frac{3}{2}}$

(C) $0{\cdot}968c$ (D) $0{\cdot}87c$

69. On the annihilation of a particle and its anti-particle, the energy released is E, mass of each particle is—

(A) $\frac{E}{c^2}$ (B) $\frac{E}{2c^2}$

(C) $\frac{E}{c}$ (D) $\frac{E}{2c}$

70. A cube has side l_0 when at rest. If the cube moves with velocity v parallel to its one edge then its volume becomes—

(A) l_0^3 (B) $l_0^3 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

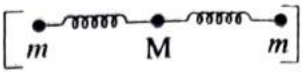
(C) $l_0^3 \left(1 - \frac{v^2}{c^2}\right)$ (D) $l_0^3 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$

71. In the normal co-ordinates of the system, each of the new co-ordinates involving ………… resonant frequencies.

(A) One (B) Two

(C) Three (D) Four

72. Normal frequencies for free vibration of a linear triatomic molecules



(A) $\sqrt{\frac{k}{m}\left(1 + \frac{2M}{m}\right)}$

(B) $\sqrt{\frac{M}{m}\left(1 + \frac{2k}{m}\right)}$

(C) $\sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$

(D) $\sqrt{\frac{M}{m}\left(1 + \frac{2m}{k}\right)}$

73. Possible longitudinal normal modes of the linear symmetric triatomic molecule is/are—

(A) One (B) Two

(C) Three (D) Four

74. Number of possible modes of vibration perpendicular to the axis in linear symmetric triatomic molecules—

(A) Two (B) Three

(C) Four (D) Five

75. Normal frequency for free vibrations of the parallel pendula is given as—

(A) $\sqrt{\frac{g}{l} - \frac{2k}{m}}$ (B) $\sqrt{\frac{g}{l} + \frac{2k}{m}}$

(C) $\sqrt{\frac{g}{l} - \frac{m}{2k}}$ (D) $\sqrt{\frac{g}{l} + \frac{m}{2k}}$

76. A massless spring of force constant k has masses m_1 and m_2 attached to its two ends. The system rests on a horizontal table. The angular vibrational frequency ω of this system is—

(A) $[k/(m_1 - m_2)]^{1/2}$

(B) $[k/(m_1 + m_2)]^{1/2}$

(C) $[k(m_1 + m_2)/m_1 m_2]^{1/2}$

(D) $\left[k\left(\frac{1}{m_1} - \frac{1}{m_2}\right)\right]^{1/2}$

77. There are six particles lying on a plane. The degrees of freedom associated with them are—
(A) 6 (B) 18
(C) 12 (D) None

78. A sphere rolling down from the top of a fixed sphere is an example of—
(A) Scleronomic, non holonomic and conservative system
(B) Only conservative system
(C) Only Scleronomic system
(D) Only non-holonomic system

79. A cylinder rolling without slipping down a rough inclined plane of angle θ is an example of—
(A) Scleronomic, conservative system only
(B) Scleronomic, holonomic, conservative system
(C) Only conservative system
(D) Only Scleronomic system

80. A particle moving on a very long frictionless wire which rotates with constant angular velocity about a horizontal axis is an example of—
(A) Rheonomic, holonomic, conservative system
(B) Only conservative system
(C) Only holonomic and conservative system
(D) Rheonomic, non-holonomic and non-conservative

81. How many degree's of freedom a rigid body possess—
(A) 3 (B) 6
(C) 9 (D) Infinite

82. When a rigid body rotates about a given axis, the degrees of freedom it will have, is—
(A) 1 (B) 2
(C) 3 (D) 4

83. When a cylinder rolls without slipping on a plane, how many degrees of freedom it has ?
(A) 1 (B) 2
(C) 3 (D) 4

84. Two particles moving on a space curve and have fixed distance between them, have degrees of freedom numbering—
(A) 1 (B) 2
(C) 3 (D) 4

85. Three particles moving in space so that the distance between any two of them always remain fixed have degrees of freedom equal to—
(A) 1 (B) 3
(C) 6 (D) 9

86. The number of degrees of freedom for a system of a rigid rod moving freely in space and a particle is constraint to move on that rod is equal to—
(A) 1 (B) 2
(C) 3 (D) 4

87. Scleronomous constraints are—
(A) Independent of time
(B) Dependent on time
(C) Both (A) and (B)
(D) None of these

88. Constraint in the case of a rigid body is—
(A) Dynamic constraint
(B) Scleronomous constraint
(C) Rheonomous constraint
(D) Static constraint

89. An example of a rheonomous constraint is—
(A) Bead rotating on a wire loop
(B) Bead on a rotating wire loop
(C) A simple pendulum
(D) A torsional pendulum

90. Name the type of constraint that may be expressed in the form of an equation relating the co-ordinates of the system and time—
(A) Holonomic (B) Non-holonomic
(C) Scleronomous (D) All of these

91. A non-holonomic constraint may be expressed in the form of—
(A) Equality (B) Inequality
(C) Vector (D) None of these

92. The constraint on the motion of a particle in a plane reduces the number of degrees of freedom by—
(A) One (B) Three
(C) Four (D) None of these

93. A particle is constrained to move along the inner surface of a hemisphere number of degrees of freedom of the particle is—
(A) One (B) Two
(C) Three (D) Four

94. The generalised co-ordinate θ for the motion of a simple pendulum oscillating in a vertical plane is—

(A) $\cos^{-1}\frac{x}{l}$

(B) $\sin^{-1}\frac{y}{l}$

(C) (A) and (B) both

(D) None of these

95. The Lagrangian method of undetermined multipliers can be used for the holonomic constraints if—

(A) The forces of constraints are required

(B) It is inconvenient to reduce all the co-ordinates of the system to independent ones

(C) Both (A) and (B)

(D) None of these

96. Let two unequal masses m_1 and m_2 $(m_1 < m_2)$ be connected by a string of length l, passes over a frictionless and massless pulley such that the distance of m_2 from the pulley be x, then the Lagrangian of the system is—

(A) $\frac{1}{2}(m_1 - m_2)\dot{x}^2 + (m_1 + m_2)x$

(B) $\frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)x$

(C) $\frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_2 - m_1)x$

(D) $\frac{1}{2}(m_2 - m_1)\dot{x}^2 - (m_2 - m_1)x$

97. A particle of mass m moves along a straight line and is attracted towards a point on this line with a force proportional to the distance x from that point. The Lagrangian of the system is—

(A) $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ (B) $\frac{1}{2}mv^2 - \frac{1}{2}kx^2$

(C) $\frac{1}{2}mv^2 + kx^2$ (D) $mv^2 + \frac{1}{2}kx^2$

98. A particle of mass m moves in a plane, its motion defined by (r, θ) under the influence of a force $F = -kr$ directed towards the origin. The Lagrangian of the system is given by—

(A) $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}kr^2$

(B) $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}kr^2$

(C) $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}kr^2$

(D) None of these

99. In Q. 98, the r-component of the motion is given by—

(A) $m\ddot{r} + kr = 0$

(B) $m\ddot{r} + mr^2\dot{\theta} + kr = 0$

(C) $m\ddot{r} - mr\dot{\theta}^2 + kr = 0$

(D) None of these

100. The equation of motion for a small particle of mass m at position x is $m\ddot{x} + \gamma\dot{x} - mg = 0$. Assuming initial speed to be v_0, the terminal speed of particle will be—

(A) $\frac{mg}{\gamma}$ (B) $\sqrt{v_0 + 2gx}$

(C) $v_0 + gt$ (D) $\frac{mg}{\gamma^2 t}$

101. Lagrangian of the sun-earth system is—

(A) $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GMm}{r}$

(B) $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r}$

(C) $\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r}$

(D) $\frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r}$

where r is the sun and earth distance M and m are mass of sun and earth respectively $\dot{\theta}$ is the angular speed and G is gravitational constant.

102. The generalised velocity co-ordinate q_k of a classical system with Lagrangian 'L' is said to be cyclic if—

(A) $\frac{\partial L}{\partial q_k} = \dot{q}_k$ (B) $\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$

(C) $\frac{\partial L}{\partial q_k} = 0$ (D) $\frac{\partial L}{\partial \dot{q}_k} = 0$

103. The Lagrangian of a particle moving in a plane under the influence of a central potential is given by $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) -$

V(r). The generalised moments corresponding to r and θ are given by—

(A) $m\dot{r}$ and $mr^2\dot{\theta}$ (B) $m\dot{r}$ and $mr\dot{\theta}$

(C) $m\dot{r}^2$ and $mr^2\dot{\theta}$ (D) $m\dot{r}^2$ and $mr^2\dot{\theta}^2$

104. The Hamiltonian corresponding to the Lagrangian $L = a\dot{x}^2 + b\dot{y}^2 - kxy$ is—

(A) $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} + kxy$ (B) $\frac{p_x^2}{4a} + \frac{p_y^2}{4b} - kxy$

(C) $\frac{p_x^2}{4a} + \frac{p_y^2}{4b} + kxy$ (D) $\frac{p_x^2 + p_y^2}{4ab} + kxy$

105. The Lagrangian of a particle of mass m moving in a plane is given by $L = \frac{1}{2} m (v_x^2 + v_y^2) + a (xv_y - yv_x)$ where v_x and v_y are velocity components and a is a constant. The canonical momenta of the particle are given by—

(A) $p_x = mv_x$ and $p_y = mv_y$

(B) $p_x = mv_x + ay$ and $p_y = mv_y + ax$

(C) $p_x = mv_x - ay$ and $p_y = mv_y + ax$

(D) $p_x = mv_x - ay$ and $p_y = mv_y - ax$

106. A system of two particles having masses m_1 and m_2 are connected by an in extensible, massless string of length l passing through a small hole in horizontal table, the Lagrangian of the system is—

(A) $(m_1 + m_2)\frac{\dot{r}^2}{2} + \frac{m_2 r\dot{\theta}^2}{2} - m_1 g (r - l)$

(B) $(m_1 + m_2)\frac{\dot{r}^2}{2} + (m_2 - m_1) g (r - l)$

(C) $(m_1 + m_2)\frac{\dot{r}^2}{2} + (m_2 - m_1) \frac{r\dot{\theta}^2}{2}$

(D) $(m_1 + m_2)\frac{\dot{r}^2}{2} + m_1 \frac{r^2 \dot{\theta}^2}{2} - m_2 g (r - l)$

107. Which of the following is incorrect for conservative systems ?

(A) $\delta L = p_i \delta q_i$ (B) $\delta L = \frac{d}{dt}(p_i \delta q_i)$

(C) $\frac{\partial T}{\partial p_i} = q_i$ (D) $\delta T + \partial V = 0$

108. Initially two co-ordinate systems are coincident. The primed system rotates with an angular velocity ω with respect to the other non-rotating frame. If i' is one of the unit vectors in the rotating co-ordinate system. Then $\frac{di'}{dt}$ in non-rotating frame is given be—

(A) i' (B) $\vec{\omega} \times \vec{i'}$

(C) $\vec{\omega} \cdot \vec{i'}$ (D) $\bar{\omega} \times (\bar{\omega} \times i')$

109. In Q. 108, $\frac{d^2 i'}{dt^2}$ in non-rotating frame—

(A) Zero (B) $\omega^2 i'$

(C) $\bar{\omega} \times i'$ (D) $\bar{\omega} \times (\bar{\omega} \times \bar{i'})$

110. In Q. 108 $\frac{di'}{dt}$ in rotating co-ordinate system is given by—

(A) Zero (B) $\vec{i'}$

(C) $\vec{\omega} \times \vec{i'}$ (D) $\bar{\omega} \times (\bar{\omega} \times \bar{i'})$

111. An xyz co-ordinate system, initially coinciding with an inertial frame xyz, rotates with an angular velocity $\bar{\omega} = 2\hat{i} + t^2\hat{j} + (2t + 4)\hat{k}$ where t = time. The position vector of a particle at time t in (xyz) system is given by—

$$\vec{r} = (t^2 + 1)\hat{i} + 6t\hat{j} + 4t^3\hat{k}$$

its apparent velocity at time t = 1 sec.

(A) $v' = 2\hat{i} + 6\hat{j} + 4\hat{k}$

(B) $v' = 2\hat{i} - 6\hat{j} + 12\hat{k}$

(C) $v' = 2\hat{i} + 6\hat{j} + 12\hat{k}$

(D) $v' = 2\hat{i} + 6\hat{j} - 4\hat{k}$

112. In Q. 111, its true velocity at t = 1 sec.

(A) $2\hat{i} + 6\hat{j} + 12\hat{k}$ (B) $34\hat{i} + 2\hat{j} + 2\hat{k}$

(C) $34\hat{i} - 2\hat{j} + 2\hat{k}$ (D) None of these

113. In Q. 111 its apparent acceleration at t = 1 sec.

(A) $2\hat{i} + 6\hat{j} + 12\hat{k}$ (B) $2\hat{i} + 24\hat{k}$

(C) $2\hat{i} - 24\hat{k}$ (D) $2\hat{i} - 6\hat{j} + 24\hat{k}$

114. In Q. 111 its true acceleration at $t = 1$ sec.

(A) $48\hat{i} - 24\hat{j} + 20\hat{k}$

(B) $-14\hat{i} + 212\hat{j} + 40\hat{k}$

(C) $4\hat{i} - 4\hat{j} - 8\hat{k}$

(D) $40\hat{i} + 18\hat{j} + 36\hat{k}$

115. The coriolis acceleration on it at $t = 1$ sec. (Q. 111)—

(A) $48\hat{i} - 24\hat{j} + 20\hat{k}$

(B) $-48\hat{i} + 24\hat{j} + 20\hat{k}$

(C) $48\hat{i} - 24\hat{j} + 20\hat{k}$

(D) $48\hat{i} + 24\hat{j} - 20\hat{k}$

116. The centripetal acceleration on the particle at $t = 1$ sec. (Q. 111)—

(A) $-14\hat{i} + 212\hat{j} + 40\hat{k}$

(B) $10\hat{i} + 200\hat{j} + 30\hat{k}$

(C) $4\hat{i} - 12\hat{j} - 10\hat{k}$

(D) $-4\hat{i} + 412\hat{j} + 70\hat{k}$

117. The magnitude of coriolis acceleration in Q. 111—

(A) $4\sqrt{205}$ (B) $8\sqrt{410}$

(C) $10\sqrt{410}$ (D) $2\sqrt{205}$

118. The magnitude of centripetal acceleration in Q. 111—

(A) $10\sqrt{410}$ (B) $8\sqrt{410}$

(C) $8\sqrt{205}$ (D) $2\sqrt{205}$

119. Which one of the following particles experiences a coriolis force ?

(A) A particle at rest w.r.t. earth at Bhopal

(B) A particle thrown vertically upward at Bhopal

(C) A particle thrown vertically upward at the north pole

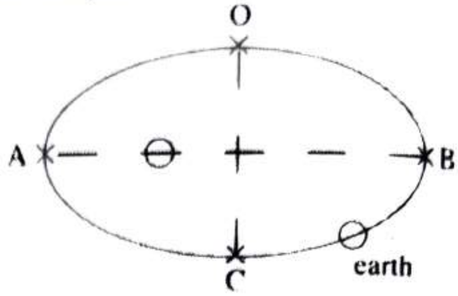
(D) A particle moving horizontally along the north-south direction at the Bhopal

120. If the escape velocity from the surface of a spherical planet of mass M is given by $\sqrt{\frac{GM}{2R}}$, the radius of the planet is—

(A) R/2 (B) R

(C) 2R (D) 4R

121. The earth revolves round the sun in elliptical orbit with sun at centre of the foci in figure. The orbital speed of the earth is maximum near the point—



(A) C (B) B

(C) A (D) D

122. A linear transformation of a generalised co-ordinates q and the corresponding momentum p to Q and P given by—

$Q = q + p$; $P = q + \alpha p$

is canonical if the value of the constant α is—

(A) -1 (B) 0

(C) $+1$ (D) $+2$

123. A particle of mass m is constrained to move on the plane curve $xy = C$ ($C > 0$) under gravity (y-axis vertical). The Lagrangian of the particle is given by—

(A) $\frac{1}{2} m\dot{x}^2 \left(1 + \frac{C^2}{x^4}\right) + \frac{mgC}{x}$

(B) $\frac{1}{2} m\dot{x}^2 \left(1 + \frac{C^2}{x^4}\right) - \frac{mgC}{x}$

(C) $\frac{1}{2} m\dot{x}^2 \left(1 + \frac{C}{x^2}\right) + \frac{mgC}{x}$

(D) $\frac{1}{2} m\dot{x}^2 \left(1 + \frac{C}{x^2}\right) - \frac{mgC}{x}$

124. A particle of mass m falls a given distance z_0 in time $t_0 = \sqrt{\frac{2z_0}{g}}$ and the distance travelled in time t is given by $z = at + bt^2$, where constant a and b are such that the time t_0 is always the same. The integral $\int_0^{t_0} L\,dt$ is an

extremum for real values of the coefficients only when—

(A) $a = 0$ and $b = \frac{g}{2}$

(B) $a = \frac{g}{2}$ and $b = 0$

(C) $a = g, b = 0$

(D) $a = \frac{g}{2}$ and $b = g$

125. The Hamiltonian corresponding to Lagrangian

$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \alpha x^3 + \beta x\dot{x}^2$

(A) $H = \frac{p^2}{2(1+2\beta x)} - \frac{1}{2}\omega^2 x^2 + \alpha x^3$

(B) $H = \frac{p^2}{2(1+2\beta x)} + \frac{1}{2}\omega^2 x^2 + \alpha x^3$

(C) $H = \frac{p^2}{2(1+2\beta x)^2} + \frac{1}{2}\omega^2 x^3 + \alpha x^2$

(D) $H = \frac{p^2}{2(1+2\beta x)^2} + \frac{1}{2}\omega^2 x + \alpha x^3$

126. q_1 and q_2 are generalised co-ordinate and p_1, p_2 are the corresponding generalised momentum. The Poisson bracket [X, Y] of $X = q_1^2 + q_2^2$ and $Y = 2p_1 + p_2$ is—

(A) $(q_1^2 + q_2^2)\, p_1$ (B) $3(q_1^2 + q_2^2)$

(C) $4q_1 + 2q_2$ (D) 0

127. The transformation $q = PQ^2$, $p = \frac{1}{Q}$ is canonical—

(A) True (B) False

128. Generating function for Q 127 is—

(A) $F = qp$ (B) $F = \frac{P}{q}$

(C) $F = Pq$ (D) $F = p^2 q$

129. A particle moves in a circular orbit about the origin under the action of a central force $\overline{F} = -\frac{k\hat{r}}{r^3}$. If the potential energy is zero at infinity, the total energy of the particle is—

(A) $-\frac{k}{r^2}$ (B) $-\frac{k}{2r^2}$

(C) Zero (D) $+\frac{k}{r^2}$

130. A particle is moving in $-\frac{1}{r}$ potential. Which of the following statements is incorrect in this case ?

(A) Angular momentum of the particle is always conserved

(B) Kinetic energy of the particle is always conserved

(C) The particle always follows a closed path

(D) Force on the particle is always radial

131. A planet is revolving around a star in an elliptic orbit. The ratio of the farthest distance to the closest distance of the planet from the star is 4. The ratio of kinetic energies of the planet at the farthest to the closest position is—

(A) 1 : 16 (B) 16 : 1

(C) 1 : 4 (D) 4 : 1

132. A particle moves in a central force field $\overline{f} = -kr^n\hat{r}$, where k is a constant, r the distance of the particle from the origin and $\hat{r}$ is the unit vector in the direction of position vector $\vec{r}$. Closed stable orbits are possible for—

(A) $n = 1$ and $n = 2$

(B) $n = 1$ and $n = -1$

(C) $n = 2$ and $n = -2$

(D) $n = 1$ and $n = -2$

133. The Lagrangian for the Kepler problem is given by

$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \frac{\mu}{r}$ $(\mu > 0)$

where r, θ denote the polar co-ordinates and the mass of the particle is unity. Then—

(A) $p_\theta = 2r^2\dot{\theta}$

(B) $p_r = 2\dot{r}$

(C) The angular momentum of the particle about the centre of attraction is a constant

(D) The total energy of the particle is time dependent

134. The mean distance of Mars from the sun being 1·524 times that of the earth, the time of revolution of Mars about sun—

(A) 1 year (B) 10·24 years

(C) 1·8814 years (D) 18·814 years

135. A particle describes the curve is luminscate under a force P towards the pole. The law of force—

(A) $P \propto \frac{1}{r^5}$ (B) $P \propto \frac{1}{r^7}$

(C) $P \propto \frac{1}{r^3}$ (D) $P \propto \frac{1}{r^9}$

136. The speed v of a particle moving in an elliptical path in an inverse square field is given by—

(A) $v^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$

(B) $v^2 = \frac{k}{m}\left(\frac{2}{r} + \frac{1}{a}\right)$

(C) $v^2 = \frac{k}{m}\left(\frac{r}{2} + \frac{1}{a}\right)$

(D) $v^2 = \frac{k}{m}\left(\frac{2}{r} + \frac{a}{2}\right)$

137. For orbits under inverse square law of force the effective potential energy is given by—

(A) $\frac{k}{r^2} + \frac{l^2}{2mr^2}$ (B) $-\frac{k}{r^2} - \frac{l^2}{2mr^2}$

(C) $\frac{k}{r} - \frac{l^2}{2mr^2}$ (D) $\frac{k}{r} + \frac{l^2}{2mr^2}$

138. The force which is always directed away or towards a fixed centre and magnitude of which is a function only of the distance from the fixed centre is known as—

(A) Central force

(B) Coriolis force

(C) Centrifugal force

(D) Centripetal force

139. The motion in which the distance between two bodies never exceeds a finite limit is—

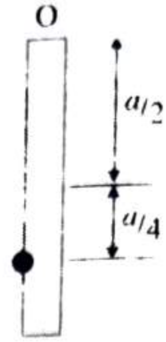
(A) Unbound motion

(B) Bound motion

(C) Fixed motion

(D) Rotational motion

140. A particle of mass m is attached to a thin uniform rod of length a and mass 4m. The distance of the particle from the centre of mass of the rod is $\frac{a}{4}$. The moment of inertia of the combination about an axis passing through O normal to the rod is—



(A) $\frac{64}{48} ma^2$ (B) $\frac{91}{48} ma^2$

(C) $\frac{27}{48} ma^2$ (D) $\frac{51}{48} ma^2$

141. Two solid spheres of radius R and mass M each are connected by a thin rigid rod of negligible mass. The distance between the centres is 4R. The moment of inertia about an axis passing through the centre of symmetry and perpendicular to the line joining the spheres is—

(A) $\frac{11}{5}$ MR^2 (B) $\frac{22}{5}$ MR^2

(C) $\frac{44}{5}$ MR^2 (D) $\frac{88}{5}$ MR^2

142. A particle of mass m moves in a potential V $(x) = \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\mu v^2$ where x is the position co-ordinate, v is the speed, and ω and μ are constants. The canonical (conjugate) momentum of the particle is—

(A) $p = m(1 + \mu)v$ (B) $p = mv$

(C) $p = m\mu v$ (D) $p = m(1 - \mu)v$

143. A circular hoop of mass M and radius a rolls without slipping with constant angular speed ω along the horizontal x-axis in the $x-y$ plane. When the centre of the hoop is at a distance $d = \sqrt{2}\,a$ from the origin, the magnitude of the total angular momentum of the hoop about the origin is—

(A) $Ma^2\omega$ (B) $\sqrt{2}\,Ma^2\omega$

(C) $2Ma^2\omega$ (D) $3Ma^2\omega$

144. If a particle moves outward in a plane along a curved trajectory described by $r = a\theta$, $\theta = \omega t$, where a and ω are constants then its—

(A) Kinetic energy is conserved

(B) Angular momentum is conserved

(C) Total momentum is conserved

(D) Radial momentum is conserved

145. For a particle moving in a central field—

(A) The kinetic energy is a constant of motion

(B) The potential energy is velocity dependent

(C) The motion is confined in a plane

(D) The total energy is not conserved

146. A bead of mass m slides along a straight frictionless rigid wire rotating in a horizontal plane with a constant angular speed ω. The axis of rotation is perpendicular to the wire and passes through one end of the wire. If r is the distance of the mass from the axis of rotation and v is its speed then the magnitude of the coriolis force is—

(A) $\frac{mv^2}{r}$ (B) $\frac{2mv^2}{r}$

(C) $mv\omega$ (D) $2mv\omega$

147. A particle of charge q, mass m and linear momentum $\bar{p}$ enters an electromagnetic field of vector potential $\bar{A}$ and scalar potential ϕ. The Hamiltonian of the particle is—

(A) $\frac{p^2}{2m} + q\phi + \frac{A^2}{2m}$

(B) $\frac{1}{2m}\left(\vec{p} - \frac{q}{C}\bar{A}\right)^2 + 2\phi$

(C) $\frac{1}{2m}\left(\vec{p} - \frac{q}{C}\bar{A}\right)^2 + \vec{p}\cdot\vec{A}$

(D) $\frac{p^2}{2m} q\phi - \vec{p}\cdot\vec{A}$

148. A particle is moving in an inverse square force field. If the total energy of the particle is positive, the trajectory of the particle is—

(A) Circular (B) Elliptical

(C) Parabolic (D) Hyperbolic

149. A particle of mass 2 kg is moving such that at time t, its position, in metre, is given by $\vec{r}(t) = 5\hat{i} - 2t^2\hat{j}$. The angular momentum of the particle at $t = 2s$ about the origin in kg $m^{-2}s^{-1}$ is—

(A) $-40\hat{k}$ (B) $-80\hat{k}$

(C) $80\hat{k}$ (D) $40\hat{k}$

150. A system of four particles in $x - y$ plane. Of these, two particles each of mass m are located at $(-1, 1)$ and $(1, -1)$. The remaining two particles each of mass $2m$ are located at $(1, 1)$ and $(-1, -1)$. The xy-component of the moment of inertia tensor of this system of particles is—

(A) 10 m (B) –10 m

(C) 2 m (D) –2 m

151. For the given transformations (i) $Q = p$, $P = -q$ and (ii) $Q = p$, $P = q$, where p and q are canonically conjugate variables, which of the following statements is true ?

(A) Both (i) and (ii) are canonical

(B) Only (i) is canonical

(C) Only (ii) is canonical

(D) Neither (i) nor (ii) is canonical

152. The mass m of a moving particle is $\frac{2m_0}{\sqrt{3}}$, where m_0 is its rest mass. The linear momentum of the particle is—

(A) $2m_0c$ (B) $\frac{2m_0c}{\sqrt{3}}$

(C) m_0c (D) $\frac{m_0c}{\sqrt{3}}$

153. A particle of mass m is constrained to move in a vertical plane along a trajectory given by $x = A\cos\theta$, $y = A\sin\theta$, where A is a constant. The Lagrangian of the particle is—

(A) $\frac{1}{2}mA^2\dot{\theta}^2 - mgA\cos\theta$

(B) $\frac{1}{2}mA^2\dot{\theta}^2 - mgA\sin\theta$

(C) $\frac{1}{2}mA^2\dot{\theta}^2$

(D) $\frac{1}{2}mA^2\dot{\theta}^2 + mgA\cos\theta$

154. In Q. 153 the equation of motion of the particle is—

(A) $\ddot{\theta} - \frac{g}{A}\cos\theta = 0$ (B) $\ddot{\theta} + \frac{g}{A}\sin\theta = 0$

(C) $\ddot{\theta} = 0$ (D) $\ddot{\theta} - \frac{g}{A}\sin\theta = 0$

155. A rigid frictionless rod rotates anticlockwise in a vertical plane with angular velocity $\vec{\omega}$. A bead of mass m move outward along the rod with constant velocity $\vec{u_0}$. The bead will experience a coriolis force—

(A) $2mu_0\omega\hat{\theta}$ (B) $-2mu_0\omega\hat{\theta}$

(C) $4mu_0\omega\hat{\theta}$ (D) $-mu_0\omega\hat{\theta}$

156. The Hamiltonian corresponding to the Lagrangian $L = \frac{1}{2}(q_1^2 + q_1q_2 + q_2^2) - V(q)$ is—

(A) $(p_1^2 - p_1p_2 + p_2^2) + V(q)$

(B) $\frac{2}{3}(p_1^2 - p_1p_2 + p_2^2) + V(q)$

(C) $\frac{2}{3}(p_1^2 - p_1p_2 + p_2^2) + V(q)$

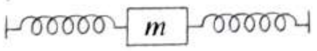
(D) None of these

157. The value of the Poisson bracket [$\vec{a} \cdot \vec{r}$, $\vec{b} \cdot \vec{p}$], where $\vec{a}$ and $\vec{b}$ are constant vectors is—

(A) $\vec{a}\,\vec{b}$ (B) $\vec{a} - \vec{b}$

(C) $\vec{a} + \vec{b}$ (D) $\vec{a} \cdot \vec{b}$

158. A mass m is connected on either side with a spring each of spring constants k_1 and k_2. The free ends of springs are tied to rigid supports. The displacement of the mass is x from equilibrium position which one of the following is TRUE ?



(A) The force acting on the mass is $-(k_1k_2)^{1/2}\,x$.

(B) The angular momentum of the mass is zero about the equilibrium point and its Lagrangian is $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k_1 + k_2)x^2$

(C) The total energy of the system is $\frac{1}{2}m\dot{x}^2$

(D) The angular momentum of the mass is $mx\dot{x}$ and the Lagrangian of the system is $\frac{m}{2}\dot{x}^2 + \frac{1}{2}(k_1 + k_2)x^2$

159. The distance between the two bodies in infinite initially and finally in—

(A) Unbounded motion

(B) Rotational motion

(C) Bounded motion

(D) Translational motion

160. If the total energy of a particle in a conservative force field is zero, then the velocity obtained in such case is—

(A) Zero

(B) Escape velocity

(C) Recoil velocity

(D) None of these

161. The force exerted by one particle on the other varies inversely as the square of the distance between them by—

(A) $F(r) = -\frac{k}{r^2}$ (B) $F(r) = -\frac{k^2}{r^2}$

(C) $F(r) = -\frac{k\hat{x}}{r^2}$ (D) $F(r) = \frac{k}{r^2}$

162. What is the nature of the orbit if energy is less than zero ?

(A) Ellipse (B) Circle

(C) Hyperbola (D) Parabola

163. What is the nature of the orbit if the value of eccentricity is equal to one ?

(A) Hyperbola (B) Ellipse

(C) Parabola (D) Circle

164. The value of eccentricity for an elliptic orbit is—

(A) $\in > 1$ (B) $\in = 1$

(C) $\in = 0$ (D) $0 < \in < 1$

165. The polar equation of the parabola is—

(A) $r = \frac{a}{1 - \sin\theta}$ (B) $r = \frac{a}{1 - \cos\theta}$

(C) $r = \frac{a}{\cos\theta - 1}$ (D) None of these

166. The equation of the orbit under central field is—

(A) $\theta - \theta_0 = -\int \frac{du}{\sqrt{\frac{2\mu E}{\lfloor 2} + \frac{2\mu k}{\lfloor 2} u - u^2}}$

(B) $\theta_0 = \int \frac{du}{\sqrt{\frac{2\mu E}{\lfloor 2} + \frac{2\mu k}{\lfloor 2} u - u^2}}$

(C) $\theta = \int \frac{du}{\sqrt{\frac{2\mu E}{\lfloor 2} - u^2}}$

(D) $\theta - \theta_0 = -\int \frac{du}{\sqrt{\frac{2\mu E}{\lfloor 2} + \frac{2\mu k}{\lfloor 2}}}$

167. The conservation of angular momentum is the central force-field motion leads us to—

(A) Kepler's Ist law

(B) Kepler's 2nd law

(C) Kepler's 3rd law

(D) None of these

168. A particle moving in a central force located at $r = 0$ describes the spiral $r = e^{-\theta}$, the magnitude of force is inversely proportional to—

(A) r^3 (B) r

(C) r^2 (D) r^4

169. In Rutherford Scattering, an α-particle of energy E is scattered through an angle θ, the differential scattering cross-section is proportional to—

(A) $E \cot \frac{\theta}{2}$ (B) $E^2 \sin^4 \frac{\theta}{2}$

(C) $E^{-2} \left(\sin \frac{\theta}{2}\right)^{-4}$ (D) $E^2 \left(\sin \frac{\theta}{2}\right)^{-4}$

170. According to special theory of relativity a particle cannot travel with the speed of light because its—

(A) Velocity will soon be infinite

(B) Mass will be infinite

(C) Mass will reduce to zero

(D) None of these

171. If the Galilean transformation were correct then the aberration angle would be given by—

(A) $\tan \theta = \frac{v}{c}$ (B) $\sin \theta = \frac{v}{c}$

(C) $\cos \theta = \frac{v}{c}$ (D) None of these

172. Photographs of rapidly moving distant objects will—

(A) Not show Lorentz contraction

(B) Show Lorentz contraction

(C) Not show any change

(D) None of these

173. A small sphere of radius R in its proper frame is moving with half the velocity of light, when viewed by an observer in a laboratory frame it looks like—

(A) A sphere (B) An ellipsoid

(C) A paraboloid (D) A hyperboloid

174. In Q. 173 instead of being viewed it is photographed then the space would be—

(A) Spherical (B) Ellipsoidal

(C) Paraboloidal (D) Hyperboloidal

175. The perpendicular component of acceleration as compared to the parallel component is—

(A) Less

(B) More

(C) Less than by a factor $\frac{v}{c}$

(D) More than by a factor $\frac{v}{c}$

176. The total kinetic energy of a system of particles about an arbitrary origin is equal to the kinetic energy of its centre of mass plus the kinetic energy of—

(A) Nothing extra

(B) Any particle system in an inertial frame of reference

(C) The system w.r.t. its centre of mass

(D) The inertial frame of reference

177. A passenger in a supersonic jet liner tosses a coin vertically upwards the coin will fall—

(A) Behind him

(B) Infront of him

(C) Right in his hand
(D) He is unable to toss the coin

178. If the speed of light were $\frac{2}{3}$ of its present value the energy released in a given atomic explosion will be decreased to a factor—

(A) $\frac{2}{3}$ (B) $\frac{4}{9}$
(C) $\frac{5}{9}$ (D) $\sqrt{\frac{5}{9}}$

179. A body of mass m_0 is placed in a rocket. The rocket is moving with a velocity $v = 0{\cdot}6c$. Then the mass of the rocket as observed by a person sitting in the rocket is—

(A) m_0 (B) $\frac{5}{4} m_0$
(C) $\frac{4}{5} m_0$ (D) $2m_0$

180. A body with a charge q starts from rest and acquire a velocity $v = 0{\cdot}5c$. Then the new charge on it is—

(A) q (B) $q\sqrt{1-(0{\cdot}5)^2}$
(C) $q\sqrt{1-0{\cdot}5}$ (D) $\frac{q}{\sqrt{1-(0{\cdot}5)^2}}$

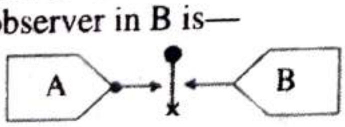
181. A slowly moving electron collides with a position at rest and annihilates it producing two photons. If the rest mass of the electron and position be m_0, then the frequency of each photon is—

(A) $2m_0c^2$ (B) m_0c^2
(C) $\frac{m_0c^2}{h}$ (D) $\frac{2m_0c^2}{h}$

182. A gamma ray of energy 2·2 MeV produces an electron positron pair. Then the energy imparted to each of the charge particles is nearly—

(A) 1·1 MeV (B) 0·51 MeV
(C) 0·59 MeV (D) 1·18 MeV

183. An observer 'X' sees two rockets A and B approaching him from opposite directions, each with a velocity 0·8 c, then the speed of the observer 'X' w.r.t. rocket B, as observed by an observer in B is—



(A) 0·64 c (B) 0·8 c
(C) 0·975 c (D) 1·60 c

184. In Q. 183 the speed of the rocket A w.r.t. the rocket B as observed by an observer 'X' is—

(A) 0·64 c (B) 0·8 c
(C) 0·975 c (D) 1·60 c

185. In Q. 183 the speed of the rocket A as observed by an observer on B is—

(A) 0·64 c (B) 0·8 c
(C) 0·975 c (D) 1·60 c

186. The postulate of co-variance of physical states that—

(A) The physical laws should have the same meaning in all inertial frames
(B) Physical laws should have the same meaning in only linearly moving frames
(C) Physical laws should have the same meaning in rotating frames
(D) None of these

187. An inertial frame is one in which—

(A) Newton's 2nd law of motion is valid
(B) Newton's 1st law of motion is valid
(C) Newton's 3rd law of motion is valid
(D) None of these

188. A particle with a mean proper life time of 1 μsec. moves through the laboratory at a velocity of $2{\cdot}7 \times 10^{10}$ cm/sec. What is its life time, as measured by an observer in the laboratory ?

(A) More than one microsecond
(B) Same as above
(C) Less than one microsecond
(D) Data appears to insufficient

189. When an observer moves so fast that the lengths that he measures are reduced to half, his time interval measurements—

(A) Be invariant (B) Reduced to half
(C) Becomes twice (D) Reduced to $\frac{1}{4}$th

190. Different physical theories must yield convergent description of the same phenomenon in their overlapping realms of validity this is known as—

(A) Principle of invariance of Lorentz
(B) Principle of correspondence of Bohr
(C) Principle of general rule of Einstein
(D) None of these

191. The rest mass of an electron is m_0 when it moves with a velocity $v = 0{\cdot}6$ c, then its rest mass is—

(A) m_0 (B) $\frac{5}{4} m_0$

(C) $\frac{4}{5} m_0$ (D) $2m_0$

192. The mass of an electron is double its rest mass then the velocity of the electron is—

(A) $\frac{c}{2}$ (B) $2c$

(C) $\frac{\sqrt{3}}{2} c$ (D) $\sqrt{\frac{3}{2}} c$

193. Rest mass energy of an electron is 0·51 MeV. A moving electron has a kinetic energy of 9·69 MeV. The ratio of the mass of the moving electron to its rest mass is—

(A) 19 : 1 (B) 20 : 1

(C) 1 : 19 (D) 1 : 20

194. Rest mass of an electron is $9{\cdot}1 \times 10^{-31}$ kg. The mass equivalent energy of its electron is—

(A) 0·511 ergs (B) 0·511 J

(C) 0·511 eV (D) 0·511 MeV

195. A Galilean transformation applies between two frames of reference P and Q if—

(A) Q is rotating with uniform angular velocity relative to P

(B) Q is moving with uniform acceleration relative to P

(C) Q is moving with uniform velocity relative to P

(D) Q goes round P at a constant distance with a constant speed

196. Out of the following quantities, pick out one that is invariant under a Galilean transformation—

(A) Displacement

(B) Velocity

(C) Force

(D) Momentum

197. With respect to the reference frame S_2, a particle and a reference frame S_1 are moving along the x-direction, the particle is moving with the velocity $\frac{c}{4}$ w.r.t S_1 and S_1 is moving with velocity $\frac{c}{4}$ w.r.t. S_2. The velocity of the particle w.r.t. S_2 is—

(A) $\frac{8}{17}$ c (B) $\frac{c}{2}$

(C) Zero (D) $\frac{c}{4}$

198. K. E. of a relativistic particle of rest mass is moving with speed v is—

(A) $\frac{1}{2} mv^2$ (B) $\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$

(C) $\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2$ (D) $\frac{1}{2} m (v^2 - c^2)$

199. The momentum of an electron (mass m) which has the same kinetic energy as its rest mass energy is—

(A) $\sqrt{3}\, mc$ (B) $\sqrt{2}\, mc$

(C) mc (D) $\frac{mc}{\sqrt{2}}$

200. Two events are separated by a distance of 6×10^5 km and the first event occurs 1s before the second event. The interval between the two events—

(A) Is time like

(B) Is light like (null)

(C) Is space like

(D) Cannot be determined from the information given

201. Which of the following equation is relatively invariant (α, β, γ and δ are constant of suitable dimension) ?

(A) $\frac{\partial \phi\,(x, t)}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} (x, t)$

(B) $\frac{\partial^2 \phi}{\partial t^2} (x, t) = \beta^2 \frac{\partial^2 \phi}{\partial x^2} (x, t)$

(C) $\frac{\partial^2 \phi}{\partial t^2} (x, t) = \gamma \frac{\partial \phi}{\partial x} (x, t)$

(D) $\frac{\partial \phi}{\partial t} = \delta \frac{\partial^3 \phi}{\partial x^3} (x, t)$

202. Although mass-energy equivalence of special relativity allows conversion of a photon to an electron-positron pair such a process cannot occur in free space because—

(A) The mass is not conserved
(B) The energy is not conserved
(C) The momentum is not conserved
(D) The charge is not conserved

203. In an electromagnetic field which one of the following remains invariant under Lorentz transformation ?

(A) $\bar{E} \times \bar{B}$
(B) $E^2 - C^2B^2$
(C) B^2
(D) E^2

204. An electron gains energy so that its mass because $2m_0$. Its speed is—

(A) $\frac{\sqrt{3}}{2}C$
(B) $\frac{3}{4}C$
(C) $\frac{3}{2}C$
(D) $\sqrt{\frac{3}{2}}C$

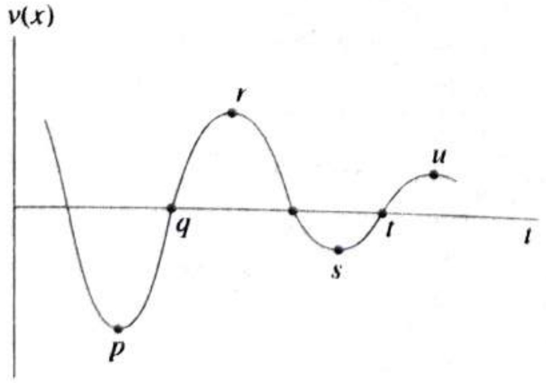
205. The Lagrangian for a three particle system is given by

$$L = \frac{1}{2}(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2) - a^2(\eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_1\eta_3)$$

where a is real, then one of the normal co-ordinates has frequency ω—

(A) $\omega^2 = a^2$
(B) $\omega^2 = \frac{a^2}{2}$
(C) $\omega^2 = 2a^2$
(D) $\omega^2 = \sqrt{2}\,a^2$

206. Identify the points of unstable equilibrium for the potential shown in the figure—



(A) p and s
(B) q and t
(C) r and u
(D) r and s

Classical dynamics

Objective type question

Answer key

ANSWERS WITH HINTS

1. (C) 2. (B) 3. (C)

4. (A) Constraints is time dependent in this case and given by

$$\dot{\theta} = \omega;$$

ω = Angular velocity of rotation

Transformation equations

$$x = r\cos\theta$$

$$y = r\sin\theta \qquad \text{where } \theta = \omega t$$

Therefore, the kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2]$$

$$= \frac{1}{2}m[\dot{r}^2 + r^2\omega^2] \qquad \text{...(i)}$$

Equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

Therefore, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

Since wire is rotating uniformly in a force free space,

$$\Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = 0$$

From equation (i) we get

$$\frac{d}{dt}(m\dot{r}) - mr\,\dot{\theta}^2 = 0$$

$$m\ddot{r} - mr\,\dot{\theta}^2 = 0$$

$$\boxed{\ddot{r} = r\omega^2}$$

5. (A) Equation of constraints on the co-ordinates x and θ is

$$r\,d\theta = dx \qquad \text{...(1)}$$

or

$$r\,d\theta - dx = 0$$

Compare it with

$$\Sigma a_{ik}\,dq_k - a_{it}\,dt = 0$$

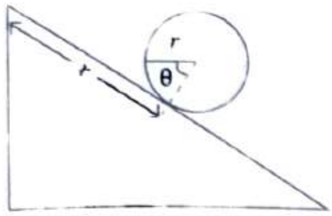
We get

$$a_\theta = r$$

and

$$a_x = -1$$

According to Question.



Kinetic energy. $(T) = \frac{1}{2}M\dot{x}^2 + \frac{1}{4}Mr^2\dot{\theta}^2$

Potential energy. $(V) = Mg(l - x)\sin\phi$

Lagrangian, $L = T - V$

$= \frac{1}{2}M\dot{x}^2 + \frac{1}{4}Mr^2\dot{\theta}^2 - Mg(l - x)\sin\phi$ ...(2)

Using equation of motion

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= a_x\lambda \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= a_\theta\lambda \end{aligned}\right\} \quad ...(3)$$

Gives $M\ddot{x} - Mg\sin\phi + \lambda = 0$ ...(4)

and $\frac{1}{2}Mr\ddot{\theta} = \lambda$ ...(5)

respectively.

From equation (1),

$$rd\theta = dx$$

we get $r\ddot{\theta} = \ddot{x}$

Substituting it in equation (5), we get

$$\lambda = \frac{1}{2}M\ddot{x}$$

Using this value of λ in equation (4), we get

$$\ddot{x} = \frac{2g\sin\phi}{3}$$

Hence, $\ddot{\theta} = \frac{2g\sin\phi}{3r}$ ...(6)

$$\Rightarrow \quad \lambda = \frac{M\ddot{x}}{2} = \frac{Mg\sin\phi}{3}$$

$$\ddot{x} = v\frac{dv}{dx} = \frac{2g\sin\phi}{3}$$

$$\int_0^v v\,dv = \int_0^l \frac{2g\sin\phi}{3}dx$$

Gives $\frac{v^2}{2} = \frac{2gl\sin\phi}{3}$

or $$\boxed{v = \left(\frac{4gl\sin\phi}{3}\right)^{\frac{1}{2}}}$$

6. (C) For such circuit,

Magnetic energy,

$$T_M = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2$$

$$\left[\text{because } i = \frac{dq}{dt} = \dot{q}\right]$$

Electrical energy. $V_E = \frac{1}{2}\frac{q^2}{C}$

∴ Lagrangian, $L_E = T_M - V_E$

$T_M \to$ Magnetic energy of the electrical circuit (analogous to the kinetic energy of the mechanical system)

$V_E \to$ Electrical energy of the electrical circuit (analogous to the potential energy of the mechanical system)

$$\therefore \boxed{L_E = \frac{1}{2}L\dot{q}^2 - \frac{1}{2}\frac{q^2}{C}}$$

L → Inductance
C → Capacitance

7. (C) Maxwell's equations in Gaussian units are

$$\bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial\bar{B}}{\partial t} = 0$$
$$\bar{\nabla}\cdot\bar{D} = 4\pi\rho \quad ...(1)$$
$$\bar{\nabla}\times\bar{H} - \frac{1}{c}\frac{\partial\bar{D}}{\partial t} = \frac{4\pi\bar{j}}{c}$$
$$\bar{\nabla}\cdot\bar{B} = 0$$

The complete force is

$$\bar{F} = q\left\{E + \frac{1}{c}(\bar{v}\times\bar{B})\right\} \quad ...(2)$$

Also $\bar{B}$ can be written as

$$\bar{B} = \text{curl}\,\bar{A}$$
$$\vec{A} = \text{Magnetic vector potential}$$

Curl E equation becomes

$$\bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{A}) = \nabla\times\left(\bar{E} + \frac{1}{c}\frac{\partial\bar{A}}{\partial t}\right)$$
$$= 0$$

We can set

$$\bar{E} + \frac{1}{c}\frac{\partial\bar{A}}{\partial t} = -\nabla\phi$$

or $$E = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t} \qquad \ldots(3)$$

So, $F = q\left\{-\nabla\phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t} + \frac{1}{c}\left[\bar{v}\times(\bar{\nabla}\times\bar{A})\right]\right\}$

Now we have,

$$\left[\bar{v}\times\left(\bar{\nabla}\times\bar{A}\right)\right]_x = \frac{\partial}{\partial x}\left(\bar{v}.\bar{A}\right) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

So, $F_x = q\left\{-\frac{\partial}{\partial x}\left(\phi - \frac{1}{c}\bar{v}.\bar{A}\right) - \frac{1}{c}\frac{d}{dt}\left(\frac{\partial}{\partial v_x}(A.\bar{v})\right)\right\}$

Which is equivalent to,

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial U_x}$$

With $U = q\phi - \frac{q}{c}\bar{A}.\bar{v}$ (4)

Now Lagrangian

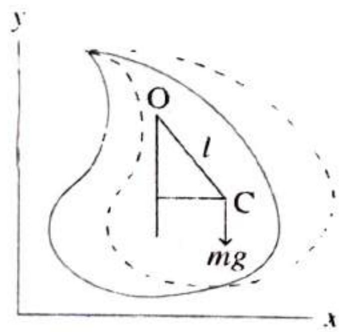
$$L = T - U$$

With $T = \frac{1}{2}mv^2$

So, $$\boxed{L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\bar{A}.\bar{v}}$$

8. (B) Let M → Mass of pendulum.

I → Moment of inertia about axis of rotation.



A rigid body capable of oscillating in a vertical plane above a fixed horizontal axis is called a compound pendulum.

So, kinetic energy. $T = \frac{1}{2}I\dot{\theta}^2$

Potential energy, $V = -mgl\cos\theta$

Lagrangian, $L = T - V$

$$\boxed{L = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta}$$

9. (C) 10. (B) 11. (D)

12. (A) For compound pendulum

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$$

where $\omega = \sqrt{\frac{mgl}{I}}$

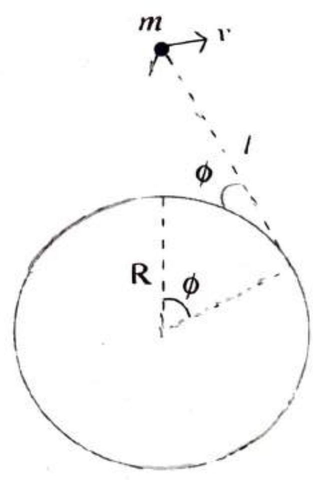
If $k \rightarrow$ Radius of gyration, then

$$I = mk^2 + ml^2$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{mk^2 + ml^2}{mgl}}$$

or $$\boxed{T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}}$$

13. (B) As there is no force there is no potential energy



$$\Rightarrow \quad K.E = \frac{1}{2}mv^2$$

or $$L = \frac{1}{2}ml^2\frac{\dot{l}^2}{R^2} \qquad \ldots(1)$$

as $$\phi = \frac{l}{R}$$

Equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) - \frac{\partial L}{\partial l} = 0$$

Using equation (1), we get

$$l\ddot{l} + \dot{l}^2 = 0$$

or $$\frac{d}{dt}(l\dot{l}) = 0$$

By integrating, we have

$$\frac{1}{2}\frac{d}{dt}(l^2) = \text{Constant} = c, \text{ say}$$

Initial conditions are

$$\left.\begin{array}{l} l=0 \\ v=v_0 \end{array}\right\} \text{at } t = 0$$

$$d(l^2) = 2c\,dt$$

or

$$l^2 = 2ct + c_1$$

by applying Initial conditions we get

$$c_1 = 0$$

and

$$l^2 = 2ct$$

Differentiating w.r.t. t, we get

$$2l\dot{l} = 2c$$

Further

$$v_0 = l\dot{\phi} = l\frac{\dot{l}}{R}$$

$$l\dot{l} = Rv_0$$

$\Rightarrow$

$$c = Rv_0$$

or

$$l^2 = 2Rv_0\,t$$

Angular momentum about the cylinder axis

$$= mvl$$

$$= mv\sqrt{2Rv_0 t}$$

$$= mv_0\sqrt{2Rv_0}\,t$$

$$\boxed{\text{Angular momentum} = m \times (2Rv_0^3 t)^{1/2}}$$

$$v = v_0$$

[as there is no external force]

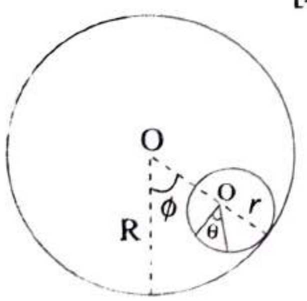
14. (A) Since the cylinder rolls without slipping. equation of constraints

$$r\theta = (R-r)\phi$$

or

$$r\dot{\theta} = (R-r)\dot{\phi} \quad \ldots(1)$$

[From figure]



K.E = Due to translation motion + Due to rotational motion

$$= \frac{1}{2}m[(R-r)\dot{\phi}]^2 + \frac{1}{2}I_0\dot{\theta}^2$$

$$= \frac{3}{4}m(R-r)^2\dot{\phi}^2 \left[\text{since } I_0 = \frac{1}{2}mr^2\right]$$

[Using equation (1)]

Potential energy.

$$V = -mg(R-r)\cos\phi$$

and

$$V = 0 \text{ for } \phi = 90°$$

Lagrangian,

$$L = T - V$$

$$= \frac{3}{4}m(R-r)^2\dot{\phi}^2 + mg(R-r)\cos\phi \quad \ldots(2)$$

Equation of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$

Using equation (2), we get

$$\frac{3}{2}(R-r)\ddot{\phi} + g\phi = 0$$

[for small oscillations $\sin\phi = \phi$]

or

$$\ddot{\phi} + \frac{2g}{3(R-r)}\phi = 0 \quad ..(3)$$

Compare with equation

$$\ddot{\phi} + \omega^2\phi = 0$$

So ω, the frequency of small oscillations

$$\omega = \sqrt{\frac{2g}{3(R-r)}}$$

The period of oscillation

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3(R-r)}{2g}}}$$

15. (B) 16. (B)

17. (A) We have, Lagrangian for a charged particle in an e.m. field

$$L = \frac{1}{2}mv^2 - q\phi + q\frac{\bar{v}\cdot\bar{A}}{c}$$

The canonical momenta are

$$\boxed{p = \frac{\partial L}{\partial v} = m\bar{v} + \frac{q}{c}\bar{A}}$$

18. (C) Lagrangian for a charged particle in an e.m. field

$$L = \frac{1}{2}mv^2 - q\phi + q\frac{\bar{v}\cdot\bar{A}}{c}$$

$$\bar{p} = \frac{\partial L}{\partial v} = m\bar{v} + \frac{q}{c}\bar{A}$$

or

$$v = \frac{1}{m}\left[p - \frac{q}{c}A\right]$$

$$= \text{K.E.} + \text{P.E.}$$

Now, $\overline{H} = \frac{1}{2}mv^2 + q\phi$

or $H = \frac{1}{2m}\left[\overline{p} - \frac{q}{c}\overline{A}(\overline{r}, t)\right]^2 + q\phi(\overline{r}, t)$

or $$\boxed{H = \frac{1}{2}m\left[\frac{\overline{p}}{m} - \frac{q}{mc}\overline{A}(\overline{r}, t)\right]^2 + q\phi(\overline{r}, t)}$$

19. (A)

20. (B) If orbit is circular then

Attractive force = Centrifugal force

$$\Rightarrow \frac{GMm}{(R+H)^2} = \frac{gR^2m}{(R+H)^2} = \frac{mv_o^2}{R+H}$$

$v_o \to$ Orbital velocity

$R \to$ Radius of earth

$$\Rightarrow v_o = \frac{R}{R+H}\sqrt{[(R+H)g]} \quad ...(1)$$

If H << R

$\therefore v_o = \sqrt{Rg}$

Also orbital speed

$$v_0 = \frac{2\pi(R+H)}{\tau}$$

or $$\tau = \frac{2\pi(R+H)}{v_o}$$

From equation (1),

$$\tau = 2\pi\left(\frac{R+H}{R}\right)\sqrt{\frac{R+H}{g}}$$

if H << R then $\boxed{\tau = 2\pi\sqrt{\frac{R}{g}}}$

21. (B) The eastward derivation of the falling bodies is $y' = \frac{1}{3}\omega g t^3 \cos\phi$

Time taken by the body in going up to the point it will come at rest is $\left(\frac{2h}{g}\right)^{1/2}$.

Similar time will be taken by this body in coming down to the surface of the earth.

Total time of flight $= 2\left(\frac{2h}{g}\right)^{1/2}$

$$\therefore y' = \frac{1}{3}\omega g\left[2\left(\frac{2h}{g}\right)^{1/2}\right]^3 \cos\phi\left(\frac{2h}{g}\right)^{1/2}$$

$$\boxed{y' = \frac{16}{3}\omega h\cos\phi\left(\frac{2h}{g}\right)^{1/2}}$$

22. (D) 23. (A)

24. (B) Condition for canonical transformation

$$[Q, P] = 1 \quad ...(1)$$

where $$[Q, P] = \frac{\partial Q}{\partial q_i}\frac{\partial P}{\partial p_i} - \frac{\partial Q}{\partial p_i}\frac{\partial P}{\partial q_i} \quad ...(2)$$

$$Q = q^m \cos np;$$
$$P = q^m \sin np$$

$$\left.\begin{aligned}\frac{\partial Q}{\partial q_i} &= mq^{m-1}\cos np; \quad \frac{\partial P}{\partial q_i} = mq^{m-1}\sin np\\ \frac{\partial Q}{\partial p_i} &= -q^m. n\sin np; \quad \frac{\partial P}{\partial p_i} = nq^m\cos np\end{aligned}\right\} \quad ...(3)$$

Using equation (3) in (1) and (2)

$$m. q^{m-1}\cos np. nq^m\cos np + q^m. n\sin np.$$
$$mq^{m-1}\sin np = 1$$
$$\Rightarrow mnq^{2m-1}(\cos^2 np + \sin^2 np) = 1$$
$$mnq^{2m-1} = 1$$

Thereby giving

$$mn = 1$$

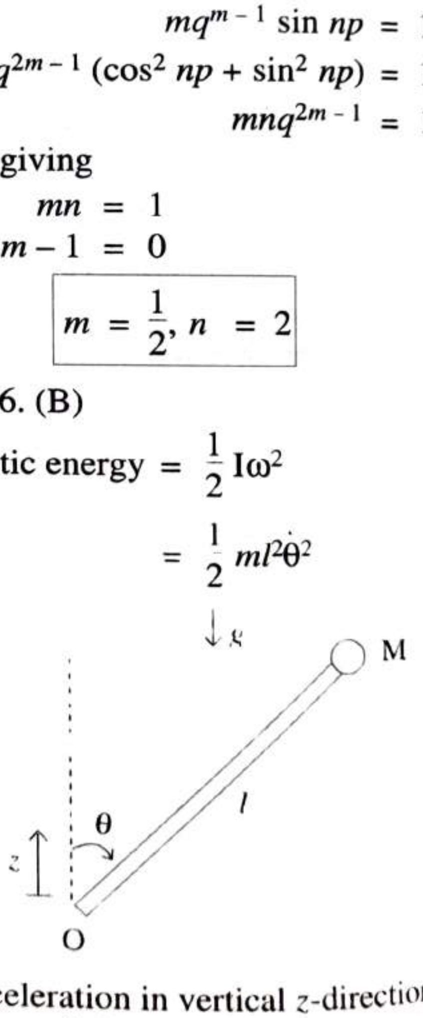
and $$2m - 1 = 0$$

gives $\boxed{m = \frac{1}{2}, n = 2}$

25. (A) 26. (B)

27. (C) Kinetic energy $= \frac{1}{2}I\omega^2$

$$= \frac{1}{2}ml^2\dot{\theta}^2$$



Now acceleration in vertical z-direction due to the motion of pivot

$$\ddot{z} = A\omega^2 \sin\omega t$$

Net acceleration

$$a = g - \ddot{z} = g - A\omega^2\sin\omega t$$

Then potential energy

$$V = mal\cos\theta$$
$$= m(g - A\omega^2\sin\omega t)\, l\cos\theta$$

$$\therefore \quad L = T - V$$

$$L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl \cos\theta + mlA\omega^2 \sin\omega t \cos\theta$$

28. (C)

29. (A) $\quad Q = \log(1 + q^{1/2} \cos p)$

We get,

$$e^Q = 1 + q^{1/2} \cos p$$

or $\quad q = (e^Q - 1)^2 \sec^2 p$

Now, we know.

$$\frac{\partial F_3}{\partial p} = -q = -(e^Q - 1)^2 \sec^2 p$$

$$F_3 = \int -(e^Q - 1)^2 \sec^2 p\,dp$$

Choosing constant of integration as zero

$$F_3 = -(e^Q - 1)^2 \tan p$$

30. (C) Given generating function

$$F(q, Q) = \frac{1}{2} m\omega q^2 \cot Q \quad \ldots(1)$$

Canonical transformation are

$$\left.\begin{aligned} p &= \frac{\partial F}{\partial q} = m\omega q \cot Q \quad \ldots(a) \\ \rho &= -\frac{\partial F}{\partial Q} = \frac{1}{2} m\omega q^2 \operatorname{cosec}^2 Q \quad \ldots(b) \end{aligned}\right\} \ldots(2)$$

From equation (2b).

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \quad \ldots(3)$$

From equation (2) and (3)

$$p = \sqrt{2m\omega P} \cos Q \quad \ldots(4)$$

$k \rightarrow$ Force constant of linear harmonic oscillator potential energy

$$V = -\int_0^q F dq$$

$$= -\int_0^q (-kq)\, dq = \frac{1}{2} kq \quad \ldots(5)$$

$$\therefore \quad H = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} kq^2$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

$$\left(\omega^2 = \frac{k}{m}\right) \quad \ldots(6)$$

Now, $H(p, q) = H(P, Q)$

Substituting p and q from (3) and (4) in (6)

$$H = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P \quad \ldots(7)$$

or $\quad P = \frac{H}{\omega} = \frac{E}{\omega} \quad \ldots(8)$

where $H(P, Q) = E =$ Constant.

Other's equation of motion

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega \quad \text{[From equation (7)]}$$

or $\quad Q = \omega t + \beta$

$\therefore$ Displacement of linear harmonic oscillator is

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \beta)$$

31. (D)

32. (C) Equation of motion of particle is

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{dV}{dr} = \frac{l^2}{mr^3} + f(r) \quad \ldots(1)$$

Introduce new variables

$$u = \frac{1}{r} \text{ or, } r = \frac{1}{u}$$

$$\dot{r} = -r^2\dot{\theta} \frac{du}{d\theta} = -\frac{l}{m} \frac{du}{d\theta}$$

$$\left(\text{since } \dot{\theta} = \frac{l}{mr^2}\right)$$

$$\ddot{r} = -\frac{l^2u^2}{m^2} \frac{d^2u}{d\theta^2}$$

Substitute in equation (1), we get

$$m \times \frac{(-l^2u^2)}{m^2} \frac{d^2u}{d\theta^2} = mr \frac{l^2}{m^2r^4} + f(r)$$

$$\Rightarrow \quad \frac{d^2u}{d\theta^2} = -u - \frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$$

33. (A) 34. (A) 35. (B) 36. (C) 37. (D)

38. (C) 39. (C) 40. (D) 41. (C)

42. (B) In case of elliptic orbits, semi major axis is

$$a = \frac{p}{1 - \epsilon^2}$$

Substitute, $\quad p = \frac{l^2}{mk}$

$$a = \frac{l^2}{mk(1 - \epsilon^2)}$$

$$\Rightarrow \quad (1 - \epsilon^2) = \frac{l^2}{mka} \quad \ldots(1)$$

Now, we have

$$E = \frac{1}{2} m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}$$

E → Constant so

$$\dot{r} = 0$$

$$\Rightarrow \quad E = 0 + \frac{l^2}{2mr^2_{min}} - \frac{k}{r_{min}} \quad ..(2)$$

We have $r_{min} = \frac{p}{1+6}$,

$$\Rightarrow \quad r_{min} = \frac{l^2}{mk(1+\epsilon)}$$

So equation (2) gives

$$E = \frac{mk^2}{2l^2}(\epsilon^2 - 1) \quad ..(3)$$

From equation (3) and (1), we get

$$E = -\frac{mk^2}{2l^2} \cdot \frac{l^2}{mka}$$

$$E = -\frac{k}{2a}$$

or

$$\boxed{E \propto \frac{1}{a}}$$

43. (A) If the orbit is circular then,

Attractive force = Centrifugal force

$$\Rightarrow \quad f = \left(\frac{dV}{dr}\right) = \frac{mv^2}{r} \quad [V = kmr^3]$$

$$\Rightarrow \quad \frac{d}{dr}(kmr^3) = \frac{mv^2}{r}$$

$$3kr^2 = \frac{v^2}{r}$$

or

$$v = r\sqrt{3kr}$$

at

$$r = a$$

$$v|_{r=a} = a\sqrt{3ka}$$

∴ Kinetic energy $= \frac{1}{2}mv^2$

$$E = \frac{1}{2}m.\,a^2.\,3ka$$

$$\boxed{E = \frac{3}{2}mka^3}$$

44. (B) From previous question

$$v|_{r=a} = a = \sqrt{3ka}$$

Angular velocity with $r = a$

$$\dot{\theta} = \frac{v}{a}\sqrt{3ka}$$

Time period of revolution is

$$T = \frac{2\pi}{\dot{\theta}}$$

$$\Rightarrow \quad \boxed{T = \frac{2\pi}{\sqrt{3ka}}}$$

45. (C) Angular momentum

$$l = mva$$

$$= m.\,a\sqrt{3ka}.\,a$$

$$\left[v|_{r=a} = a\sqrt{3ka}\right]$$

$$\boxed{l = ma^2\sqrt{3ka}}$$

46. (D) For angular frequency, we have to find out equation of motion.

We use Taylor's series expansion

$$V(a+x) = V(a) + V'(a)\,x + \frac{1}{2}V''(a)x^2 + $$

$$V(a) = kma^3$$

$$V(r) = kmr^3$$

$$\Rightarrow \quad V(a+x) = kma^3 + 3.\,kma^2.\,x + \frac{1}{2}\cdot 6\,kma.\,x^2 + .$$

So, equation of motion

$$m\ddot{x} + \left[\frac{3V'(a)}{a} + V''(a)\right]x = 0$$

$$\Rightarrow m\ddot{x} + \left[\frac{3.\,3kma^2}{a} + 6kma\right]x = 0$$

$$\Rightarrow \quad \ddot{x} + 15kax = 0$$

Compare with $\ddot{x} + \omega^2 x = 0$

We get angular frequency $\boxed{\omega = \sqrt{15ka}}$

47. (A) In static equilibrium $\frac{dV}{dr} = 0$

$$V = \frac{a}{r^2} - \frac{b}{r}$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{a}{r^2} - \frac{b}{r}\right) = 0$$

$$\Rightarrow \quad -2\cdot\frac{a}{r^3} + \frac{b}{r^2} = 0$$

Separation $\boxed{r = \frac{2a}{b}}$

48. (C) Given $F = \frac{k|r|}{r^4} = -\frac{k}{r^3}$

So, $V(r) = -\int F dr = -\frac{k}{2r^2}$

Now energy, $E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$

$$= \frac{1}{2}m\dot{r}^2 + \frac{(mr^2\dot{\theta})^2}{2mr^2} - \frac{k}{2r^2}$$

$$\Rightarrow E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{2r^2}$$

$$= \frac{1}{2}mv^2 - \frac{k}{2r^2}$$

$$\Rightarrow \boxed{v = \sqrt{\frac{2E}{m} + \frac{k}{mr^2}}}$$

49. (C) $r = e^{-\theta}$

$$\Rightarrow u = \frac{1}{r} = e^{\theta}; \frac{d^2u}{d\theta^2} = e^{\theta}$$

Equation of orbit

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$$

$$e^{\theta} + e^{\theta} = -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$$

$$\Rightarrow f\left(\frac{1}{u}\right) = -2.e^{\theta}. \frac{l^2u^2}{m}$$

$$f\left(\frac{1}{u}\right) = -\frac{2l^2}{m} u \cdot u^2$$

or $\boxed{f(r) \propto \frac{1}{r^3}}$

50. (D) For attractive central force

$$r = \frac{1}{u} = 2a\cos\theta$$

so $u = \frac{\sec\theta}{2a}$ ...(1)

$$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{\sec^3\theta + \sec\theta\tan^2\theta}{2a} \quad ...(2)$$

Now substitute equation (1) and (2) in equation of orbit

$$f\left(\frac{1}{u}\right) = -\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= -\frac{l^2u^2}{2am}[\sec^3\theta + \sec^3\theta]$$

$$= -\frac{l^2\sec^3\theta}{am} \cdot u^2$$

and from equation (1),

$$\sec\theta = 2au$$

$$\Rightarrow f\left(\frac{1}{u}\right) = \frac{l^2}{am} 8a^3 u^3 u^2$$

$$= -\frac{8a^2l^2}{m} \cdot u^5$$

$$\propto u^5$$

or $\boxed{f(r) \propto \frac{1}{r^5}}$

51. (B) We have. $E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$

for inverse square law

$$V(r) = -\frac{k}{r}$$

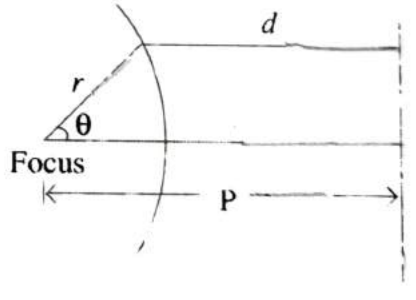
$$= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}$$

Since E is constant and at turning point for r_{min}

$$\dot{r} = 0$$

$$\Rightarrow E = 0 + \frac{l^2}{2mr^2_{min}} - \frac{k}{r_{min}} \quad (1)$$

For conic section



From fig. $\frac{r}{d}$ = Constant = $\in$

$\in \rightarrow$ Eccentricity

Now, $P = d + r\cos\theta$

$$= \frac{r}{\in} + r\cos\theta$$

$$p = \in P$$

$$\frac{p}{\in} = \frac{r}{\in} + r\cos\theta$$

or $r = \frac{p}{1 + \in\cos\theta}$

or $r_{min} = \frac{p}{1+\in}$ with $p = \frac{l^2}{mk}$

$$r_{min} = \frac{l^2}{mk(1+\in)} \quad ..(2)$$

∴ Using equation (2) in (1), we get

$$E = \frac{mk^2}{2l^2}(\epsilon^2 - 1)$$

$$\Rightarrow \quad \boxed{\epsilon = \sqrt{\frac{2El^2}{mk^2} + 1}}$$

52. (C) 53. (A) 54. (D) 55. (D)
56. (D) 57. (A) 58. (B) 59. (B)
60. (C) 61. (A)

62. (B) Speed of photon = c

The speed of astronaut = $0{\cdot}8c$

Let the photon and astronauts super space-ship moving along positive and negative directions of x-axis respectively.

Let the electron moving with velocity $-0{\cdot}8c$

So, $$u_x = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{c + 0{\cdot}8c}{1+\frac{0{\cdot}8c\,(c)}{c^2}}$$

$$= \frac{1{\cdot}8c}{1{\cdot}8} = c$$

$$\boxed{u_x = c}$$

63. (A) Speed of spaceship $u = 0{\cdot}4\,c$
Speed of particle $v = 0{\cdot}4\,c$

Speed of particle observing by observer

$$u' = \frac{u+v}{1+\frac{uv}{c^2}}$$

$$u' = \frac{0{\cdot}4c + 0{\cdot}4c}{1+\frac{0{\cdot}4c \times 0{\cdot}4c}{c^2}}$$

$$= \frac{0{\cdot}8c}{1+\frac{0{\cdot}16c^2}{c^2}}$$

or $$u' = \frac{0{\cdot}8c}{1{\cdot}16}$$

$$\boxed{u' = 0{\cdot}69c}$$

64. (A) 65. (A) 66. (D) 67. (C)
68. (C) 69. (B)

70. (D) Let two system s and s'.

s' moving with velocity v relative to s along +ve direction of x-axis.

Volume of the cube in system $s = l_0^3$.

One edge of the cube = l_0.

Along x-axis as observed by the observer in s',

$$l_x = l_0\sqrt{1-\frac{v^2}{c^2}}$$

∴ Volume of the cube as observed from s'

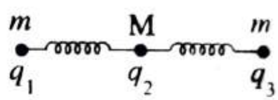
$$l_x\,l_y\,l_z = l_0\sqrt{\left(1-\frac{v^2}{c^2}\right)} \times (l_0) \times (l_0)$$

$$= l_0^3\sqrt{1-\frac{v^2}{c^2}}$$

$$\boxed{\text{Volume} = l_0^3\sqrt{1-\frac{v^2}{c^2}}}$$

71. (A)

72. (C)



All three atoms are on one straight line. Two masses m are symmetrically located on each side of an atom of mass M.

There exists an elastic bond between central atom and the end atoms, of force constants k.

Kinetic energy

$$T = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_3^2) + \frac{1}{2}M\dot{q}_2^2$$

So, that the T matrix is diagonal :

$$T = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \quad \ldots(1)$$

Potential energy

$$V = \frac{1}{2}k(q_2 - q_1)^2 + \frac{1}{2}k(q_3 - q_2)^2$$

$$= \frac{1}{2}k(q_1^2 + 2q_2^2 + q_3^2 - 2q_1q_2 - 2q_2q_3)$$

Hence, the V matrix has the form

$$V = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \quad \ldots(2)$$

Combining these two matrices, the secular equation appears as

$$|V - \omega^2 T| = \begin{vmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{vmatrix} = 0$$

Direct evaluation of the determinant leads to the cubic equation in ω^2

$$\omega^2 (k - \omega^2 m)(k(M + 2m) - \omega^2 Mm) = 0 \quad \ldots(3)$$

With solution

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k}{m}}$$

$$\omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$$

$\omega_1 = 0 \rightarrow$ Refers no oscillation.

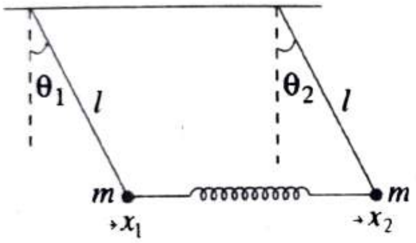
$\omega_2 \rightarrow$ Shows S.H.M.

$\Rightarrow$ so normal freq. for triatomic is

$$\boxed{\omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}}$$

73. (B) 74. (A)

75. (B) In case of parallel pendula



The kinetic energy,

$$T = \frac{1}{2} ml^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

So that the T matrix is diagonal,

$$T = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix} \quad \ldots(1)$$

Potential energy.

$$V = mgl\,[(1 - \cos\theta_1) + (1 - \cos\theta_2)] + \frac{1}{2} kl^2 (\sin\theta_2 - \sin\theta_1)$$

$$= mgl\left[\left(1 - 1 + \frac{\theta_1^2}{2}\right) + \left(1 - 1 + \frac{\theta_2^2}{2}\right)\right] + \frac{1}{2} kl^2 (\theta_2 - \theta_1)^2$$

$$= \frac{1}{2} mgl\,(\theta_1^2 + \theta_2^2) + \frac{1}{2} kl^2 (\theta_1^2 + \theta_2^2) - kl^2 \theta_1\theta_2$$

$$= \frac{1}{2} [(mgl + kl^2)(\theta_1^2 + \theta_2^2) - 2kl^2\theta_1\theta_2]$$

Hence, the V matrix has the form

$$V = \begin{pmatrix} mgl + kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 \end{pmatrix} \quad \ldots(2)$$

Combining these two matrices, the secular equation appears as

$$|V - \omega^2 T| = \begin{vmatrix} mgl + kl^2 - \omega^2 ml^2 & -kl^2 \\ -kl^2 & mgl + kl^2 - \omega^2 ml^2 \end{vmatrix} = 0$$

The development of which gives

$$(mgl + kl^2 - \omega^2 ml^2)^2 = k^2 l^4$$

$$mgl + kl^2 - \omega^2 ml^2 = \pm kl^2$$

gives

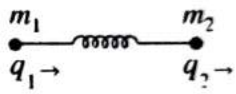
$$\omega_1 = \sqrt{\frac{g}{l}};$$

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

So,

$$\boxed{\omega = \sqrt{\left(\frac{g}{l} + \frac{2k}{m}\right)}}$$

76. (C) According to question,



The kinetic energy,

$$T = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2$$

So that the T matrix is diagonal

$$T = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad \ldots(1)$$

The potential energy

$$V = \frac{1}{2} k (q_2 - q_1)^2$$

$$= \frac{1}{2} k (q_2^2 + q_1^2 - 2q_2 q_1)$$

Hence, the V matrix has the form

$$V = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \quad \ldots(2)$$

Combining these two matrices, the secular equation appears as

$$|V - \omega^2 T| = \begin{vmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{vmatrix} = 0$$

With solution,

$$\Rightarrow \quad (k - \omega^2 m_1)(k - \omega^2 m_2) - k^2 = 0$$

$$k^2 - \omega^2 m_1 k - \omega^2 m_2 k + \omega^4 m_1 m_2 - k^2 = 0$$

$$\Rightarrow \quad m_1 k + m_2 k - \omega^2 m_1 m_2 = 0$$

or

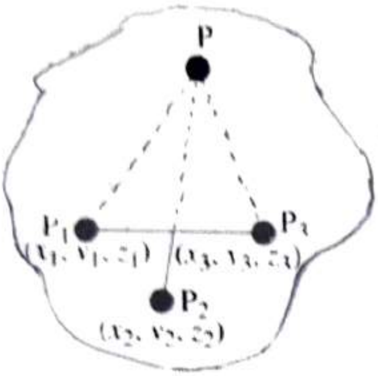
$$\omega^2 = \frac{k(m_1 + m_2)}{m_1 m_2}$$

or

$$\boxed{\omega = \sqrt{\left(\frac{k(m_1 + m_2)}{m_1 m_2}\right)}}$$

77. (C) 78. (A) 79 (B) 80. (A)

81. (B) A rigid body is a system of particles in which the distance between any two particles remain fixed throughout the motion. Let us consider three non-collinear particles P_1, P_2, P_3 of a rigid body as shown in fig.



As each particle has 3 degrees of freedom and there are three constraints of the form

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = r_{12}^2$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = r_{23}^2$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = r_{13}^2$$

Hence, the degrees of freedom for these particles are $3 \times 3 - 3 = 6$

82. (A) A rigid body rotating about a fixed axis, has one degrees of freedom, because relative to the origin, say on the fixed axis, taken as z-axis, z = constant and $x^2 + y^2 = r_0^2$ for a particle, where r_0 is the radius of the circle about the fixed axis.

83. (B)

84. (A) Two particle moving on a space curve have $2 \times 1 = 2$ degrees of freedom. Since they fixed distance between them so a constraint is imposed so total degrees of freedom is $2 - 1 = 1$.

85. (C)

86. (D) The number of degrees of freedom for a system of a rigid rod moving freely in space and a particle is constraint to move on that rod is $3 + 1 = 4$ because 3 degrees of freedom for rigid rod and one for that particle which is constraint to move on rod

87. (A) 88. (B) 89. (B) 90. (A) 91. (B)

92. (A)

93. (B) For a particle which is constraint to move along the inner surface of a hemisphere there are two variables to describe their motion (θ and ϕ). So, the degree of freedom is 2

94. (C) For the motion of a simple pendulum oscillating in a vertical plane we have

$$x = l \sin\theta$$
$$\Rightarrow \quad \theta = \sin^{-1}\frac{x}{l}$$
$$y = l\cos\theta$$
$$\Rightarrow \quad \theta = \cos^{-1}\frac{y}{l}$$

95. (C) 96. (B) 97. (B)

98. (A) Kinetic Energy for the system

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

For potential energy

$$V = -\int F dr.$$

Now
$$F = -kr$$
$$V = -\int -kr\, dr$$
$$= \frac{kr^2}{2}$$

Lagrangian

$$L = T - V$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{kr^2}{2} \quad \text{...(i)}$$

99. (C) For r-component of motion, Lagrangian eqn. is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{...(ii)}$$

from eqn. (i)

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$
$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - kr$$

Substitute in eqn. (ii), we get

$$\frac{d}{dt}(m\dot{r}) - [mr\dot{\theta}^2 - kr] = 0$$

$$\Rightarrow \quad m\ddot{r} - mr\dot{\theta}^2 + kr = 0$$

100. (A) For terminal speed

$$\ddot{x} = 0 \quad \text{...(i)}$$

eqn is

$$m\ddot{x} + \gamma\dot{x} - mg = 0 \quad \text{...(ii)}$$

Substitute eqn. (i) in (ii), we get

$$\gamma\dot{x} - mg = 0$$

or
$$\dot{x} = \frac{mg}{\gamma}$$

101. (B) For sun-earth system

$$T = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2\dot{\theta}^2$$

and potential energy

$$V = -\frac{GMm}{r}$$

So, Lagrangian

$$L = T - V$$

$$L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2\dot{\theta}^2 + \frac{GMm}{r}$$

102. (C) 103. (A)

104. (C) Here $L = a\dot{x}^2 + b\dot{y}^2 - kxy$

Now $p_x = \frac{\partial L}{\partial \dot{x}^2} = 2a\dot{x}$ ...(i)

$p_y = \frac{\partial L}{\partial \dot{y}} = 2by$ ...(ii)

$\Rightarrow \quad H = \sum p_i q_i - L$

$= p_x\dot{x} + p_y y - (a\dot{x}^2 + b\dot{y}^2 - kxy)$

$= 2a\dot{x}^2 + 2b\dot{y}^2 - a\dot{x}^2 - b\dot{y}^2 + kxy$

$= a\dot{x}^2 + b\dot{y}^2 + kxy$

$x = \frac{p_x}{2a}$ from (i)

$y = \frac{p_y}{2b}$ from (ii)

So $H = a\frac{p_x^2}{4a^2} + 2b\frac{p_y^2}{4b^2} + kxy$

$= \frac{p_x^2}{4a} + \frac{p_y^2}{4b} + kxy$

105. (C) $L = \frac{1}{2} m(v_x^2 + v_y^2) + a(xv_y - yv_x)$

So

$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial v_x} = mv_x - ay$

$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial v_y} = mv_y + ax$

106. (D) 107. (A) 108. (B) 109. (B)

110. (D)

111. (C) Let S′ be the co-ordinate system, rotates with an angular velocity $\bar{\omega}$ about a point fixed in Newtonian frame. The apparent velocity is

$$\bar{v}' = \frac{d\bar{r}}{dt} = \frac{d}{dt}[(t^2+1)\hat{i} + 6t\hat{j} + 4t^3]\hat{k}$$

$$= 2t\hat{i} + 6\hat{j} + 12t^2\hat{k}$$

$$\bar{v}'\Big|_{\text{at } t = 1 \text{ sec}} = 2\hat{i} + 6\hat{j} + 12\hat{k}$$

112. (B) In Q. 111.

$$\bar{v}' = \frac{d\bar{r}}{dt} = \frac{\delta\bar{r}}{\partial t} + \bar{\omega}\times r$$

$$\bar{v}' = \bar{v} + \bar{\omega}\times\bar{r}$$

So true velocity

$$\bar{v} = \bar{v}' - \bar{\omega}\times\bar{r}$$

$$= \bar{v}' + \bar{r}\times\bar{\omega}$$

$$\Rightarrow \quad \bar{r}\times\bar{\omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2+1 & 6t & 4t^3 \\ 2 & t^2 & 2t+4 \end{vmatrix}$$

$= i\,[6t(2t+4) - 4t^5] + \hat{j}\,[8t^3 - (2t+4)(t^2+1)] + \hat{k}\,[(t^2+1)t^2 - 12t]$

$$\bar{r}\times\bar{\omega}\Big|_{\text{at } t=1 \text{ sec}} = 32\hat{i} - 4\hat{j} - 10\hat{k}$$

$$\Rightarrow \quad \bar{v} = (2\hat{i} + 6\hat{j} + 12\hat{k}) + (32\hat{i} - 4\hat{j} - 10\hat{k})$$

$$= 34\hat{i} + 2\hat{j} + 2\hat{k}$$

113. (B) Apparent acceleration at $t = 1$ sec

$$\frac{dv'}{dt} = 2\hat{i} + 24t\hat{k}$$

$$\vec{a} = \frac{dv'}{dt}\Big|_{\text{at } t=1 \text{ sec}} = 2\hat{i} + 24\hat{k}$$

114. (D) For true acceleration

$$\frac{d\bar{v}}{dt} = \frac{\delta\bar{v}}{dt} + \bar{\omega}\times\bar{v}$$

Since for rotating frame

$$\frac{d}{dt} = \frac{\delta}{\delta t} + \bar{\omega}\times$$

$$\vec{a} = \frac{d\bar{v}}{dt} = \frac{\delta}{\delta t}\left(\frac{\delta\bar{r}}{\delta t} + \vec{\omega}\times\vec{r}\right) + \vec{\omega}\times\left[\frac{\delta\bar{r}}{\delta t} + \bar{\omega}\times\bar{r}\right]$$

$$= \frac{\bar{\delta}^2 r}{\delta t^2} + \frac{\delta}{\delta t} \times \bar{r} + 2\bar{\omega} \times \frac{\delta \vec{r}}{\delta t} + \vec{\omega} \times (\vec{\omega} \times \bar{r})$$

$$\vec{a} = \vec{a}' + \vec{a}_t + a_c$$

$$\Rightarrow \quad \vec{a}' = \vec{a} - \bar{a}_t - \vec{a}_c$$

where

$$a_t = \frac{\delta \bar{\omega}}{\delta t} \times \bar{r} + 2\bar{\omega} \times \frac{\delta \bar{r}}{\delta t}$$

$$a_c = \bar{\omega} \times (\bar{\omega} \times \bar{r})$$

Using these relations true acceleration

$$= 40\hat{i} + 184\hat{j} + 36\hat{k}$$

115. (A) 116. (B) 117. (A) 118. (A)

119. (B)

120. (D) Escape velocity $= \sqrt{\frac{2GM}{r}}$

Given escape velocity for spherical planet of mass M

$$= \sqrt{\frac{GM}{2R}}$$

Compare these two

$$\sqrt{\frac{2GM}{r}} = \sqrt{\frac{GM}{2R}}$$

or $$\frac{2}{r} = \frac{1}{2R}$$

or $$\boxed{r = 4R}$$

121. (C) Gravitational force is a central force. Angular momentum is conserved

$$m\vec{v} \times \vec{r} = \text{constant}$$

$$v \propto \frac{1}{r}$$

At A, B, C, D $\vec{v}$ and $\bar{r}$ are at 90°

where r is mininum, v is max mum

122. (D) For linear transformation

$$[Q, P] = 1$$

or $$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} - 1 \quad \ldots(i)$$

Now $$Q = q + p$$

$$P = q + \alpha$$

$$\frac{\partial Q}{\partial q} = 1; \ \frac{\partial Q}{\partial p} = 1; \ \frac{\partial P}{\partial q} = 1; \ \frac{\partial P}{\partial p} = \alpha$$

Substitute all these values in (i)

$$1.\alpha - 1.1 = 1$$

$$\boxed{\alpha = 1 + 1 = 2}$$

123. (B) The Lagrangian of the particle is given by

$$L = T - V$$

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2$$

Also $$xy = c$$

$$y = \frac{c}{x}$$

$$\dot{y} = -\frac{c}{x^2}\dot{x}$$

$$\dot{y}^2 = \frac{c^2}{x^4}\dot{x}^2$$

$$\therefore \ T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m \cdot \frac{c^2}{x^4} \dot{x}^2$$

$$V = mgy$$

$$= mg\frac{c}{x}$$

Hence $$L = \frac{1}{2} m\dot{x}^2 \left(1 + \frac{c^2}{x^4}\right) - \frac{mgc}{x}$$

124. (A)

125. (B) $$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \alpha x^3 + \beta x\dot{x}^2$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} + 2\beta x\dot{x}$$

or $$\dot{x} = \frac{p_x}{1 + 2\beta x}$$

The Hamiltonian

$$H = p_x\dot{x} - L$$

$$= \dot{x}^2 + 2\beta x\dot{x}^2 - \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \alpha x^3 - \beta x\dot{x}^2$$

$$= \frac{1}{2}\dot{x}^2 + \beta x\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \alpha x^3$$

$$= \frac{\dot{x}^2(1 + 2\beta x)}{2} + \frac{1}{2}\omega^2 x^2 + \alpha x^3$$

126. (C) $$X = q_1^2 + q_2^2$$

$$Y = 2p_1 + p_2$$

Poisson Bracket condition

$$\left[\frac{\partial X}{\partial q_1}\frac{\partial Y}{\partial p_1}-\frac{\partial Y}{\partial q_1}\frac{\partial X}{\partial p_1}\right]+\left[\frac{\partial X}{\partial q_2}\frac{\partial Y}{\partial p_2}-\frac{\partial X}{\partial q_2}\frac{\partial X}{\partial p_2}\right]$$
$$= 2q_1 . 2 + 2q_2$$
$$= 4q_1 + 2q_2$$

127. (A) Check the condition

$$[q, p] = 1$$
$$[q, p] = \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P}-\frac{\partial p}{\partial Q}\frac{\partial q}{\partial P}$$
$$q = PQ^2 ;\ p = \frac{1}{Q}$$
$$\frac{\partial q}{\partial Q} = 2PQ ;\ \frac{\partial q}{\partial P} = Q^2$$
$$\frac{\partial p}{\partial Q} = -\frac{1}{Q^2},\ \frac{\partial p}{\partial P} = 0$$

Substitute in condition

$$[q, p] = 2PQ.0 - \left(-\frac{1}{Q^2}\right) Q^2$$

$$\boxed{[q, p] = 1}$$

128. (A) Since $P = \frac{q}{Q^2}$

Functions of q, Q^2

$$\therefore \quad P = \frac{\partial F}{\partial Q}$$

or $$\frac{q}{Q^2} = -\frac{\partial F}{\partial Q}$$

or $$F = \frac{q}{Q}$$
$$F = qp$$

129. (C) Since particle moves in a circular orbit, therefore

$$\frac{mv^2}{r} = \frac{k}{r^3}$$

or $$E = \frac{1}{2}mv^2 = \frac{k}{2r^2}$$

$$P.E. = -k\int_r^\infty \frac{1}{r^3}\,dr$$
$$= -\frac{k}{2r^2}$$
$$K.E. + P.E. = \frac{k}{2r^2}-\frac{k}{2r^2} = 0$$

130. (B) K. E. of the particle is not conserved infact total energy of the particle is always conserved.

131. (A) Angular **momentum is conserved during** revolution of planet, because gravitational force is a central force.

$$K.E. = \frac{1}{2}mv^2$$

or $$= \frac{m^2v^2r^2}{2mr^2}$$
$$= \frac{L}{2mr^2}$$

or since L is constant hence

$$K.E. \propto \frac{1}{r^2}$$
$$\frac{E_2}{E_1} = \frac{r_1^2}{r_2^2} = \frac{1}{16}$$
$$1 : 16$$

132. (D) Since $V_e = V(r)$

$$= -kr^n\hat{r} + br^{-2}$$

$$\Rightarrow \quad -nkr^{n-1}\hat{r} - 2br^{-3} = 0 \quad \ldots(i)$$

or $$nkr^{n-1}\hat{r} = -2br^{-3}$$

or $$\frac{n}{2}kr^{n+2}\hat{r} = b$$

Further diff. eqn. (i)

$$-n(n-1)\,kr^{n-2}\hat{r} + 6br^{-4} > 0$$

or $$2\,(n-1\,r^{-4}\,b + 6br^{-4} > 0$$

or $$(2n-4) > 0$$

or $$n > 2$$

Further an orbit is said to be **closed** if the particle eventually retraces its path. The stable and closed orbits (circular and non-circular) for $n = 1$ and $n = -2$ have the force law as follows :

For $n = 1$, $f(r) = -kr$ Hook's law

For $n = -2$, $f(r) = -\frac{k}{r^2}$ Inverse square law

133. (C) $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu}{r}\right]$$
$$= r^2\dot{\theta}$$

and $$p_r = \frac{\partial L}{\partial \dot{x}} = 2\dot{r} - \frac{\mu}{r^2}$$

and total energy is not time dependent hence only option is (C).

134. (C) For Kepler's third law

$$T_1 \propto a_1^{3/2} \; . \; T_2 \propto a_2^{3/2}$$

$$\therefore \quad \frac{T_1}{T_2} = \left(\frac{a_1}{a_2}\right)^{3/2}$$

$$T_1 = T_2 (1{\cdot}524)^{3/2}$$
$$= 1{\cdot}8814\, T_2$$
$$= 1{\cdot}8814 \text{ years}$$

135 (B) For luminscate curve ($r^n = a^n \cos n\,\theta$) the law of force

$$P \;\alpha\; \frac{1}{r^{2n+3}}$$

Here curve $r^2 = a^2 \cos 2\theta$

$$n = 2$$
$$P = \frac{1}{r^7}$$

136 (A) $K.E. = \frac{1}{2} mv^2 = E - V$

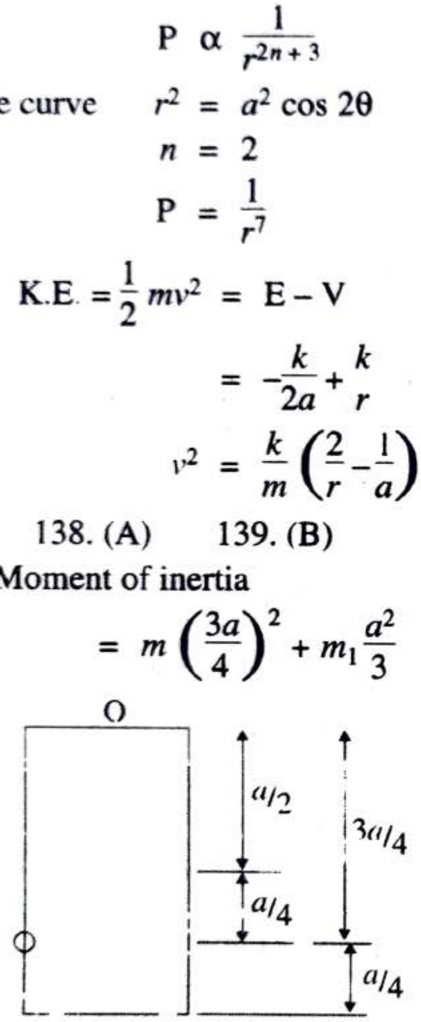
$$= -\frac{k}{2a} + \frac{k}{r}$$

or $$v^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$$

137. (D) 138. (A) 139. (B)

140 (B) Moment of inertia

$$= m\left(\frac{3a}{4}\right)^2 + m_1 \frac{a^2}{3}$$



For the centre of rod

$$\left(\frac{m_1 a^2}{12} + \frac{m_1 a^2}{4}\right) = \frac{m_1 a^2}{3}$$

$$\therefore \quad m_1 = 4m$$

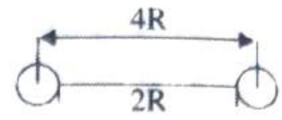
Total $$I = m\left(\frac{3a}{4}\right)^2 + \frac{4ma^2}{3}$$

$$= \frac{9\,ma^2}{16} + \frac{4ma^2}{3}$$

$$= \frac{(27 + [illegible])}{48} ma^2$$

$$= \frac{91}{48} ma^2$$

141. (C) Total moment of inertia



$$= I_1 + I_2$$

$$= \left[\frac{2}{5} MR^2 + M(2R)^2\right] + \left[\frac{2}{5} MR^2 + M \times (2R)^2\right]$$

$$= \left[\frac{22}{5} MR^2\right] + \left[\frac{22}{5} MR^2\right]$$

$$= \frac{44}{5} MR^2$$

Total M. I. $\Big|_{R=a} = \frac{44}{5} Ma^2$

142. (D) $$L = T - V$$

$$= \frac{1}{2} mv^2 - \frac{1}{2} m\omega^2 x^2 - \frac{1}{2} m\mu v^2$$

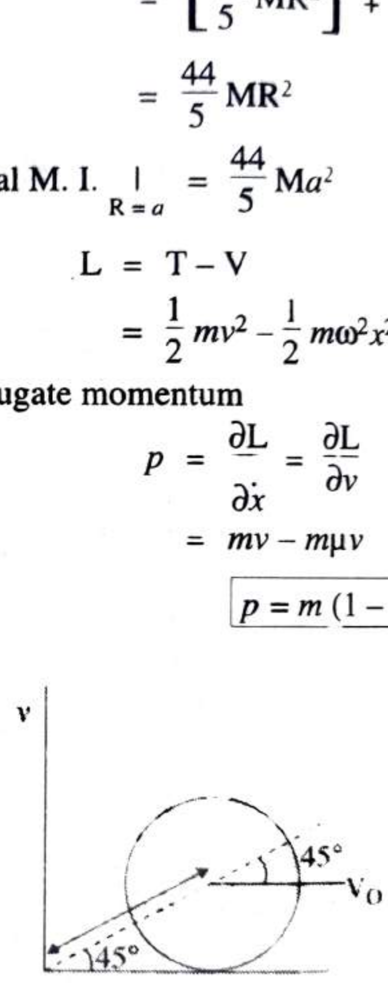
Conjugate momentum

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial v}$$

$$= mv - m\mu v$$

or $$\boxed{p = m(1 - \mu)\, v}$$

143. (C)



$$\vec{L} = \vec{L}CM + M\, \bar{r}_0 \times \bar{v}_0$$

$$= I\omega + M r_0 v_0 \sin 45^\circ$$

$$= Ma^2\omega + M\sqrt{2}\;.\;\omega a \frac{1}{\sqrt{2}}$$

$$= 2Ma^2\omega$$

144. (B) 145. (C)

146 (D) Coriolis force $2m\vec{\omega} \times \vec{v}$

$$2m\omega v \sin\theta$$

Since the axis o rotate perpendicular to the w

$$\theta = 90^\circ$$

∴ Coriolis force = $2m\omega v$

Since $\sin 90° = 1$

147. (B) 148. (D)

149. (B) Angular momentum

$$= \bar{r} \times \bar{p}$$
$$= r \times mv$$
$$= (5\hat{i} - 2t^2\hat{j}) \times m\,(-4t\,\hat{j})$$
$$= m\,(-20\,t)\,[\hat{i} \times \hat{j}] + 2t^2 \times 4t\,(\hat{j} \times \hat{j})$$
$$= -20\,mt\,\hat{k}$$

$t = 2s$ and $m = 2\text{kg}$

Angular momentum

$$= -20 \times 2 \times 2\hat{k}\ (\text{kg}\ m^2\ s^{-1})$$
$$= -80\,\hat{k}$$

150. (D) The xy component of the moment of inertia tensor

$$I_{xy} = -\sum mxy = I_{yx}$$

Two particles of mass m located at $(-1, 1)$ and $(1, -1)$ other two particles of mass $2m$ are located at $(1, 1)$ and $(-1, -1)$

$$I_{xy} = -[m_1x_1y_1 + m_2x_2y_2 + m_3x_3y_3 + m_4x_4y_4]$$
$$= -[m \times -1 \times 1 + m \times 1 \times -1 + 2m \times 1 \times 1 + 2m \times 1 \times -1]$$
$$= -[-m - m + 2m + 2m]$$
$$= -[2m]$$

151. (B) (i) $Q = p, \quad P = -q$

Condition for canonically conjugate pair

$$[Q, P] = 1$$

$$\Rightarrow \quad \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 0 - 1 \times -1 = 1$$

Hence this is canonically conjugate

(ii) $Q = p, \quad P = q$

Since condition is not satisfied it is not canonically conjugate

152. (B)

153. (B)

$$x = A\cos\theta$$
$$y = A\sin\theta$$

$$(\text{K.E.})\ T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}m\,[A^2\cos^2\theta\dot{\theta}^2 + A^2\sin^2\theta\,\dot{\theta}^2]$$
$$= \frac{1}{2}mA^2\dot{\theta}^2$$
$$V = mgy$$
$$= mgA\sin\theta$$

The Lagrangian

$$L = T - V = \frac{1}{2}mA^2\dot{\theta}^2 - mgA\sin\theta$$

154. (A) Equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$mA^2\ddot{\theta}^2 - mgA\cos\theta = 0$$
$$\ddot{\theta} - \frac{g}{A}\cos\theta = 0$$

155. (B) 156. (B)

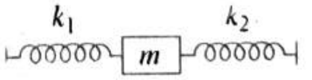
157. (A) $[Q, P] = \sum\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right)$

Here $Q = \vec{a}\ \vec{r}$

$P = \bar{b}\ .\ \vec{b}$

$$[\bar{a}\ \ r,\ \bar{b}\ \ \bar{p}] = \left(a.\frac{\partial r}{\partial r}\ .\ b\frac{\partial p}{\partial p} - a\frac{\partial r}{\partial p}\,b\frac{\partial p}{\partial r}\right)$$
$$= (a \times 1\ .\ b \times 1 - a \times 0\ .\ b \times 0)$$
$$= ab$$

158. (B)



$$L = T - V$$

where $T = \frac{1}{2}m\dot{x}^2$

$$V = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2x^2$$
$$= \frac{1}{2}(k_1 + k_2)\,x^2$$
$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k_1 + k_2)\,x^2$$

159. (A) 160. (B) 161. (D) 162. (A) 163. (C)

164. (D) 165. (B) 166. (A) 167. (B) 168. (A)

169. (C) 170. (B) 171. (A) 172. (A) 173. (B)

174. (A) 175. (B) 176. (C) 177. (C)

178. (B) $E = mc^2$

$$E \propto c^2$$

Decreased to factor $\frac{4}{9}$

179. (A) 180. (A) 181. (C)

182. (C) 1.02 MeV is used in pair production. So total energy imparted

$$= 2.2 - 1.02$$

$$= \frac{1\cdot18}{2} = 0\cdot59 \text{ MeV}$$

183. (B)

184. (C) The speed of rocket A

$$= \frac{0\cdot8\,c + 0\cdot8\,c}{1 + \frac{0\cdot64\,c^2}{c^2}}$$

$$= \frac{1\cdot6\,c}{1\cdot64} = 0\cdot975\,c$$

185. (D) Speed of rocket A as observed by an observer on B is

$$= 0\cdot8\,c + 0\cdot8\,c$$

$$= 1\cdot60\,c$$

186. (A) 187. (B) 188. (A)

189. (C) $l = l_0\sqrt{1-\frac{v^2}{c^2}}$

Now $l = \frac{1}{2}l_0$

$$\frac{1}{2}l_0 = l_0\sqrt{1-\frac{v^2}{c^2}}$$

$$\frac{1}{4} = 1 - \frac{v^2}{c^2}$$

$$\Rightarrow \quad \frac{v^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

Now time period

$$\tau = \frac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{\tau_0}{\sqrt{1-\frac{3}{4}}}$$

$$= \frac{\tau_0}{\sqrt{\frac{1}{4}}}$$

$$= 2\tau_0$$

190. (B)

191. (B) $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1-\frac{0\cdot36\,c^2}{c^2}}}$

$$= \frac{m_0}{\sqrt{0\cdot64}} = \frac{m_0}{0\cdot8}$$

$$= \frac{5}{4}m_0$$

192. (C) $2m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

or $\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2}$

$$\frac{v^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

or $\frac{v}{c} = \frac{\sqrt{3}}{2}$

or $v = \frac{\sqrt{3}}{2}c$

193. (B) $E = mc^2 - m_0c^2$

$$= \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

$$9\cdot69 = \frac{0\cdot51}{\sqrt{1-\frac{v^2}{c^2}}} - 0\cdot51$$

$$\Rightarrow \quad 10\cdot2 = \frac{0\cdot51}{\sqrt{1-\frac{v^2}{c^2}}}$$

or $\sqrt{1-\frac{v^2}{c^2}} = \frac{0\cdot51}{10\cdot2} = 0\cdot05$

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \quad \frac{m}{m_0} = \frac{1}{0\cdot05} = \frac{100}{5}$$

or $\frac{m}{m_0} = \frac{20}{1}$

194. (D) 195. (C) 196. (C) 197. (A)

198. (C)

199. (A) $E^2 = p^2c^2 + m^2c^4$

$$(2mc^2)^2 = p^2c^2 + m^2c^4$$

$$4m^2c^4 - m^2c^4 = p^2c^2$$

or $3m^2c^4 = p^2c^2$

or $p = \sqrt{3}\,mc$

200. (C)
$$\frac{r}{c} = \frac{6 \times 10^5 \text{ km/s}}{3 \times 10^8 \text{ m/s}} = \frac{6 \times 10^8 \text{ m/s}}{3 \times 10^8 \text{ m}} = 2\text{s}$$

event $t_2 - t_1 = 1$ s

$$\frac{r}{c} > 1$$

Hence it is a space like

201. (B)

202. (B) For conservation

$$h\upsilon = 2mc^2$$

But here

$$h\upsilon < 2mc^2$$

so, energy is not conserved

203. (B) 204. (A)

205. (C)
$$L = T - V = \frac{1}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2\right) - a^2\left(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2 - \eta_1\eta_3\right) \quad \ldots(i)$$

The normal frequencies are determined by

$$|V - \omega^2 T|$$

Here

$$VT = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$$T = \frac{1}{2}\left[T_{11}\dot{\eta}_1^2 + T_{12}\dot{\eta}_1\dot{\eta}_2 + T_{22}\dot{\eta}_2^2 + T_{33}\dot{\eta}_3^2 + \ldots\right]$$

Comparing the T's from eqn. (i)

$T_{11} = 1 = T_{22} = T_{33}$

$$V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}$$

Here
$$V_{11} = \left(\frac{\partial^2 V}{\partial \eta_1^2}\right)_0 = 2a^2$$

$$V_{22} = \left(\frac{\partial^2 V}{\partial \eta_2^2}\right) = 2a^2$$

$$V_{33} = \left(\frac{\partial^2 V}{\partial \eta_2^3}\right) = 2a^2$$

$$V_{13} = \left(\frac{\partial^2 V}{\partial \eta_1 \partial \eta_3}\right)_0 = -a^2$$

$$V_{31} = \left(\frac{\partial^2 V}{\partial \eta_3 \partial \eta_1}\right) = -a^2$$

$$V = \begin{bmatrix} 2a^2 & 0 & -a^2 \\ 0 & 2a^2 & 0 \\ -a^2 & 0 & 2a^2 \end{bmatrix}$$

Condition

$$|V - \omega^2 T| = \begin{vmatrix} 2a^2 - \omega^2 & 0 & -a^2 \\ 0 & 2a^2 - \omega^2 & 0 \\ -a^2 & 0 & 2a^2 - \omega^2 \end{vmatrix}$$

$$(2a^2 - \omega^2)\,[(2a^2 - \omega^2)(2a^2 - \omega^2)] + 0\,(0) + a^2\,(a^2\,(2a^2 - \omega^2)] = 0$$

$$(2a^2 - \omega^2)\,[(2a^2 - \omega^2)(2a^2 - \omega^2) + a^4] = 0$$

One co-ordinate will be

$$2a^2 - \omega^2 = 0$$

or
$$\omega^2 = 2a^2$$

Other co-ordinates are

$$(2a^2 - \omega^2)(2a^2 - \omega^2) + a^4 = 0$$

206. (C)

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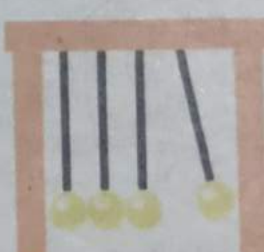

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PHYSICAL SCIENCES

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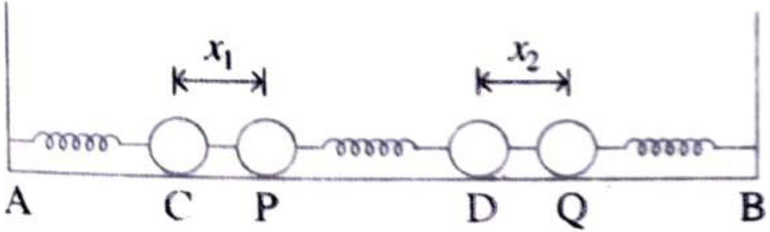
Classical dynamics

Descriptive type question

DESCRIPTIVE QUESTIONS

Q. 1. Two equal masses m are connected by springs having equal spring constant c, so that the masses are free to slide on a frictionless table. The ends of the springs are attached with the fixed walls. Using Lagrangian equation, set up the differential equation of vibrating masses.



Sol. x_1 and x_2 $\rightarrow$ Displacements of two masses from equilibrium positions C and D.

Kinetic energy :

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$$

The stretches of springs AP, PQ and QB are numerically equal to x_1, $x_2 - x_1$ and x_2.

Potential energy :

$$V = \frac{1}{2} Cx_1^2 + \frac{1}{2} C (x_2 - x_1)^2 + \frac{1}{2} Cx_2^2$$

So. $L = T - V = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$

$$- \frac{1}{2} Cx_1^2 - \frac{1}{2} C (x_2 - x_1)^2 - \frac{1}{2} Cx_2^2 \quad ...(1)$$

The Lagrangian equation of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

Using equation (1)

$$\frac{d}{dt}\left(m\dot{x}_1\right) - [- Cx_1 + C (x_2 - x_1)] = 0$$

$$m\ddot{x}_1 - C (x_2 - 2x_1) = 0$$

or $$\boxed{m\ddot{x}_1 = C (x_2 - 2x_1)} \quad ..(A)$$

$$\frac{d}{dt} (m\dot{x}_2) - [- C (x_2 - x_1) - Cx_2] = 0$$

$$m\ddot{x}_2 - [- 2Cx_2 + Cx_1] = 0$$

$$m\ddot{x}_2 + C (2x_2 - x_1)$$

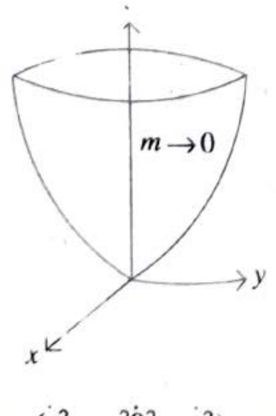
$$m\ddot{x}_2 - [Cx_1 - 2Cx_2] = 0$$

or $$\boxed{m\ddot{x}_2 = C (x_1 - 2x_2)} \quad ..(B)$$

(A) and (B) are desired equation of motion.

Q. 2. A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$. Obtain the equations of motion.

Sol. The Lagrangian in cylindrical co-ordinates



$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz \quad ...(I)$$

Since $x^2 + y^2 = r^2$, the given constraints are

$$r^2 - az = 0,$$

So that $$2rdr - adz = 0$$

If $$q_1 = r,\ q_2 = \theta \text{ and } q_3 = z$$

then $$2q_1dq_1 - adq_3 = 0 \quad ...(1)$$

Compare with equation of constraints

$$\sum_k a_k dq_k + a_t dt = 0$$

We get, $a_1 = 2r$ and $a_2 = 0$; $a_3 = -a$

Only one constraints is given

Lagrangian equation then can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \lambda_1 a_j \ (j = 1, 2, 3) \quad ...(2)$$

We have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 2\lambda_1 r$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{2\theta} = 0$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = - \lambda_1 a$$

Using equation (2) and (1) we get

$$\left.\begin{aligned} m (\ddot{r} - \dot{r}\theta^2) &= 2\lambda_1 r \\ \text{or} \quad m\frac{d}{dt}\dot{\theta} &= 0 \\ \text{and} \quad m\ddot{z} &= - mg - \lambda 1a \end{aligned}\right\} \quad ..(3)$$

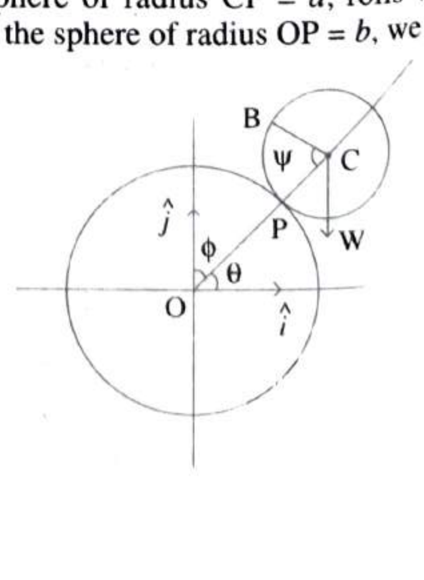
With constraints condition

$$2r\dot{r} - a\dot{z} = 0 \quad ...(4)$$

and equation (3), (4) we can find the four unknowns r, θ, z and λ.

Q. 3. A sphere of radius a and mass m rests on the top of a fixed rough sphere of radius b. The first sphere is slightly displaced so that it rolls without slipping down the second sphere.

Sol. Sphere of radius CP = a, rolls without slipping on the sphere of radius OP = b, we have,



$$b\frac{d\phi}{dt} = a\frac{d\psi}{dt}$$

or $$b\dot{\phi} = a\psi \quad ...(1)$$

ϕ and $\psi \rightarrow$ Generalised co-ordinate.

If $\phi = 0$, where $\psi = 0$

then $$b\phi = a\psi \quad ...(1a)$$

Compare equation (1) with

$$\sum_k a_{lk}\, ^d q_k - a_{lt}\, dt = 0$$

We have. $$a_1 = b.$$
$$a_2 = -a \quad .(2)$$

The kinetic energy of rolling sphere

$$T = \frac{1}{2} m (a+b)^2 \dot{\phi}^2 + \frac{1}{2} I\omega^2$$

Now $$I = \frac{2}{5} ma^2$$

Since moment of inertia of a sphere about a horizontal axis through the centre.

$$T = \frac{1}{2} m (a+b)^2 \dot{\phi}^2 + \frac{1}{2}\left(\frac{2}{5} ma^2\right)(\phi + \dot{\psi})^2 \quad (3)$$

and $$V = mg (a+b)\cos\phi \quad ..(4)$$

Hence $$L = T - V$$
$$= \frac{1}{2} m (a+b)^2 \dot{\phi}^2 + \frac{1}{5} ma^2 (\dot{\phi} + \dot{\psi})^2 - mg (a+b)\cos\phi \quad .(5)$$

Equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} - \lambda_1 b$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} - -\lambda_2 a$$

using equation (5)

$$m (a+b)^2 \ddot{\phi} + \frac{2}{5} ma^2 (\ddot{\phi} + \ddot{\psi}) - mg (a+b) \sin\phi = \lambda_1 b$$

and $$\frac{2}{5} ma^2 (\ddot{\phi} + \ddot{\psi}) = -\lambda_1 a$$

Now from equation (1a) we further get

$$\psi = \frac{b}{a}\phi$$

$$m (a+b)^2 \ddot{\phi} + \frac{2}{5} ma^2 \left(\ddot{\phi} + \frac{b}{a}\ddot{\phi}\right) - mg (a+b)\sin\phi = \lambda_1 b \quad ...(6)$$

and $$\frac{2}{5} ma^2 \left(1 + \frac{b}{a}\right)\ddot{\phi} \qquad \lambda_1 a$$

i.e $$\lambda_1 = -\frac{2}{5} ma\left(1 + \frac{b}{a}\right)\phi$$

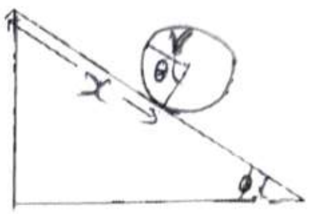
Substituting this value of λ_1 in (6), we get

$$\boxed{\dot{\phi} = \frac{5g}{7(a+b)}\sin\phi}$$

Required equation of motion.

Q. 4. A cylinder rolling down an inclined plane, calculate its velocity at the bottom of the inclined plane.

Sol. Hoop is rolling down on inclined plane without slipping.



Equation of constraints

$$r d\theta = dx$$
$$r d\theta - dx = 0 \quad .(1)$$

Compare with

$$\sum_k a_{lk}\, ^d q_k + a_{lt}\, dt = 0$$
$$l = 1, 2, .. m$$

We get $$a_\theta = r \text{ and } a_x = -1$$

Now K.E. = Translation + Rotational

$$= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} I\dot{\theta}^2$$
$$= \frac{1}{2} M\dot{x}^2 + \frac{1}{4} Mr^2\dot{\theta}^2 \quad \left[I = \frac{1}{2} Mr^2\right]$$

Potential energy is

$$V = Mgh = Mg (l - x)\sin\phi$$

Lagrangian

$$L = T - V = \frac{1}{2} M\dot{x}^2 + \frac{1}{4} Mr^2\dot{\theta}^2 - Mg (l - x)\sin\phi \quad ...(2)$$

Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = a_x\lambda$$

or $$M\ddot{x} - Mg\sin\phi + \lambda = 0 \quad ...(3)$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = a_\theta\lambda$$

$$\frac{1}{2}\text{M}r^2\ddot{\theta} - r\lambda = 0$$

$$\frac{1}{2}\text{M}r\ddot{\theta} = \lambda \quad ...(4)$$

Further from equation (1)

$$rd\theta = dx$$

or

$$r\dot{\theta} = \dot{x}$$

$$r\ddot{\theta} = \ddot{x}$$

From equation (4)

$$\frac{1}{2}\text{M}\ddot{x} = \lambda \quad ...(5)$$

and from equation (3) using equation (5)

$$\text{M}\ddot{x} - \text{M}g \sin\phi + \frac{1}{2}\text{M}\ddot{x} = 0$$

$$\frac{3}{2}\text{M}\ddot{x} - \text{M}g\sin\phi = 0$$

$$\Rightarrow \quad \ddot{x} = \frac{2g\sin\phi}{3}$$

Hence

$$\ddot{\theta} = \frac{2g\sin\phi}{3r}$$

The frictional force of constraints, λ, is given by

$$\lambda = \frac{\text{M}\ddot{x}}{2} = \frac{\text{M}g\sin\phi}{3}$$

Now using,

$$\ddot{x} = v\frac{dv}{dx} = 2g\frac{\sin\phi}{3}$$

or

$$\int_0^v v\,dv = \int_0^l \frac{2g\sin\phi}{3}dx$$

$$\frac{v^2}{2} = \frac{2gl\sin\phi}{3}$$

$$\boxed{v = \left(\frac{4gl\sin\phi}{3}\right)^{1/2}}$$

Velocity at the bottom of inclined plane.

Q. 5. Show that for a particle moving on a sphere, $\cos^{-1}\frac{2}{3}$ is the angle at which the particle flies off the surface ?

Sol. Lagrangian for this system

$$\text{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \quad ...(1)$$

Particle moves on the surface of sphere, the constraints condition is,

$$r = l\text{, radius of the sphere}$$

$$dr = 0$$

Coefficient of constraints are

$$a_r = 1,\ a_\theta = 0 \quad ...(2)$$

Also $\dot{r} = 0$, so that

$$\text{L} = \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cos\theta \quad ...(3)$$

The equation of motion in θ will be

$$\frac{d}{dt}\left(\frac{\partial \text{L}}{\partial\dot{\theta}}\right) - \frac{\partial \text{L}}{\partial\theta} = a_\theta\lambda \quad ...(4)$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mgr\sin\theta = 0$$

or

$$mr^2\ddot{\theta} - mgr\sin\theta = 0 \quad ...(5)$$

The equation of motion in r will be

$$\frac{d}{dt}\left(\frac{\partial \text{L}}{\partial\dot{r}}\right) - \frac{\partial \text{L}}{\partial r} = a_r\lambda$$

$$-mr\dot{\theta}^2 + mg\cos\theta = \lambda \quad ...(6)$$

Consider, the undetermind multiplier, λ, is dependent on θ,

So,

$$-mr\dot{\theta}^2 + mg\cos\theta = \lambda(\theta)$$

Differentiating with respect to t, we get

$$-2mr\dot{\theta}\ddot{\theta} - mg\sin\theta\,\dot{\theta} = \frac{d\lambda(\theta)}{dt} = \frac{d\lambda}{d\theta}\frac{d\theta}{dt} = \frac{d\lambda}{d\theta}\dot{\theta}$$

$$\Rightarrow \quad -2mr\ddot{\theta} - mg\sin\theta = \frac{d\lambda}{d\theta}$$

Putting $\ddot{\theta}$ from equation (5)

$$-2mr\left(\frac{g\sin\theta}{r}\right) - mg\sin\theta = \frac{d\lambda}{d\theta}$$

$$\Rightarrow \quad -3mg\sin\theta = \frac{d\lambda}{d\theta}$$

Integrating

$$\lambda(\theta) = 3mg\cos\theta + \text{C} \quad ...(7)$$

At the top of sphere,

$$\theta = 0 \text{ and } \lambda = mg$$

So,

$$mg = 3mg\cos\theta + \text{C}$$

$$\text{C} = -2mg$$

From equation (7)

$$\lambda(\theta) = 3mg\cos\theta - 2mg$$

For the particle to move on the surface, the force of constraints should remain positive.

$$\therefore \quad \lambda(\theta) \geq \theta$$

$$\Rightarrow \quad 3mg\cos\theta \quad 2mg \geq 0$$

$$\Rightarrow \quad \cos\theta_c = \frac{2}{3}$$

or $$\theta_c = \cos^{-1}\left(\frac{2}{3}\right)$$

where $\theta_c \rightarrow$ Angle at which the particle flies off the surface.

Q. 6. A particle of mass m falls a given distance z_0 in time $t_0 = \left(\frac{2z_0}{g}\right)^{1/2}$ and the distance travelled in time t is given by $z = at + bt^2$ where constants a and b are such that the time t_0 is always the same. Show that the integral $\int_0^{t_0} Ldt$ is extremum for real values of the coefficients only when $a = 0$ and $b = \frac{g}{2}$.

Sol. Lagrangian for this system

$$L = T - V = \frac{1}{2}m\dot{z}^2 + mgz$$

Hamilton's principle

$$\delta\int\left(\frac{1}{2}m\dot{z}^2 + mgz\right)dt = 0 \qquad (1)$$

$\left[\int f(z, \dot{z}, t)\, dt\right.$ is extremum

$$\left. f(z, \dot{z}, t) = \frac{1}{2}m\dot{z}^2 + mgz \right]$$

For path to be extremum, function $f(z, \dot{z}, t) = \frac{1}{2}m\dot{z}^2 + mgz$ must satisfy the Euler-Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right) - \frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(m\dot{z}) - (mg) = 0$$

$$\Rightarrow \quad m\ddot{z} - mg = 0 \qquad \ldots(2)$$

But $$z = at + bt^2 \qquad \text{(given)}$$

and $$z_0 = \frac{gt_0^2}{2} \qquad \text{(given)}$$

$$\therefore \quad \ddot{z} = 2b \qquad \ldots(3)$$

Compare equation (2) and (3)

$$b = \frac{g}{2}$$

Now, when $t = t_0, z = z_0$

So that $$z_0 = at_0 + bt_0^2$$

Putting $$z_0 = \frac{gt_0^2}{2}$$

and $$b = \frac{g}{2}$$

We get $$\frac{gt_0^2}{2} = at_0 + \frac{gt_0^2}{2}$$

$$\Rightarrow \quad a = 0$$

$\therefore$ The path will be extremum or the integral $\int_0^{t_0} Ldt$ is extremum when $a = 0$ and $b = \frac{g}{2}$.

Q. 7. A particle of mass M moves on a plane in field of force given by $\overline{F} = -\hat{r}kr\cos\theta$, where k is constant and $\hat{r}$ is the radial unit vector.

(a) Will the angular momentum of the particle about the origin be conserved ? Justify your statement.

(b) Obtain the differential equation of the orbit of the particle.

Sol. Kinetic energy of the particle will be

$$T = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) \qquad \ldots(1)$$

(a) Lagrange's equation in term of θ

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta \qquad \ldots(2)$$

From equation (1)

$$\frac{d}{dt}(Mr^2\dot{\theta}) \quad 0 = Q_\theta$$

Since force is only radial

$$Q_\theta = 0$$

$$\therefore \quad \frac{d}{dt}(Mr^2\dot{\theta}) = 0$$

or $$Mr^2\dot{\theta} = \text{Constant}$$

$Mr^2\dot{\theta} \rightarrow$ Angular momentum of the particle about the origin.

$\therefore$ Conserved.

(b) Lagrange's equation in terms of r

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

From equation (1)

$$\frac{d}{dt}(M\dot{r}) - Mr\dot{\theta}^2 = Q_r$$

$$Q_r = F_r = -kr\cos\theta$$

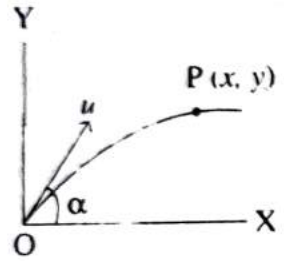
$$M\ddot{r} - Mr\dot{\theta}^2 = -kr\cos\theta$$

$$M\ddot{r} - Mr\dot{\theta}^2 + kr\cos\theta = 0$$

Differential equation of the orbit of the particle.

Q. 8. A particle of mass *m* is projected with initial velocity *u* at an angle α with the horizontal. Use Lagrange's equations to describe the motion of the projectile. The resistance of the air may be neglected ?

Sol. P (x, y) → Position of particle in (x, y) plane at time t. X-axis horizontal.



Kinetic energy.

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Potential energy,

$$V = mgy$$

Lagrangian,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

Lagrangian's equation of motions are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \qquad m\ddot{x} = 0$$

$$\text{or} \qquad \ddot{x} = 0 \qquad \ldots(1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$m\ddot{y} + mg = 0$$

$$\text{or} \qquad \ddot{y} + g = 0 \qquad \ldots(2)$$

Integrating equation (1) and (2) twice, we get

$$x = At + B \qquad \ldots(3)$$

$$y = -\frac{1}{2}gt^2 + Ct + D \qquad \ldots(4)$$

In equation (3) and (4) A, B, C and D are constant of integration.

Now we applying boundary conditions at $t = 0$.

$$x = 0 \qquad y = 0$$

$$\dot{x} = \frac{dx}{dt} \qquad \dot{y} = u\sin\alpha$$

$$= v = u\cos\alpha$$

Now at $x = 0$,

$\Rightarrow$ $B = 0$

at $t = 0$

and $\dot{x} = u\cos\alpha$

$A = u\cos\alpha$

Equation (3) becomes

$$x = u\cos\alpha\, t \qquad \ldots(5)$$

Similarly by applying boundary conditions equation (4) becomes

$$y = u\sin\alpha \cdot t - \frac{1}{2}gt^2 \qquad \ldots(6)$$

Eliminating t from equation (6) with the help of equation (5), we get

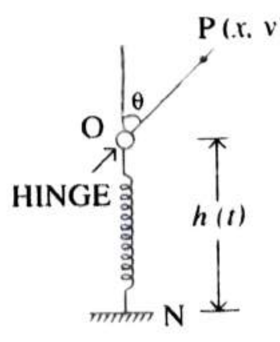
$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

Equation of the path followed by the projectile and is parabola.

Q. 9. A massless rod of length *l* is hinged at the extremity of a vertical spring that is fixed to the ground. A mass point, *m*, rests on the rod. Assuming harmonic motion of the spring, find the equation of motion of the system.

Sol. Let harmonic motion of the spring be described by

$$h(t) = h_0\cos\omega t.$$



$h(t)$ → Height of hinge at time t.

h_0 → Height at $t = 0$

$m \rightarrow$ Mass at the point P (x, y)

Co-ordinate of mass point P be (x, y) so,

$$x = OP \sin\theta = l\sin\theta$$

$$y = h(t) + l\cos\theta = h_0\cos\omega t + l\cos\theta \quad ...(1)$$

Kinetic energy

$$T = \frac{1}{2}m(x^2 + y^2)$$

$$= \frac{1}{2}m[(\theta l\cos\theta)^2 + (\theta l\sin\theta + \omega h_0\sin\omega t)^2]$$

$$= \frac{1}{2}m[\dot{\theta}^2 l^2 + \omega^2 h_0^2\sin^2\omega t + 2\omega h_0\dot{\theta}l\sin\theta\sin\omega t]$$

Potential energy

$$V = mgy$$

$$= mg(h_0\cos\omega t + l\cos\theta)$$

Lagrangian of the system is

$$L = T - V$$

$$= \frac{1}{2}m[\theta^2 l^2 + \omega^2 h_0^2\sin^2\omega t + 2\omega h_0\dot{\theta}l\sin\theta\sin\omega t] - mg(h_0\cos\omega t + l\cos\theta) \quad (2)$$

The Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \quad ...(3)$$

Using equation (2) and (3), we get

$$l^2\theta + \omega^2 h_0 l\sin\theta\cos\omega t + \omega h_0 l\dot{\theta}\cos\theta\sin\omega t - \omega h_0 l\theta\cos\theta\sin\omega t - gl\sin\theta = 0$$

$$l^2\ddot{\theta} + \omega^2 h_0 l\sin\theta\cos\omega t - gl\sin\theta = 0$$

In case the hinge is fixed.

$$h_0 = 0$$

So that, $\quad l\ddot{\theta} - g\theta = 0$

which is equation of motion of simple pendulum executing small oscillations.

Q. 10. Show that the transformation,

$$\mathbf{Q = \sqrt{2q}\, e^{a}\cos p}$$

$$\mathbf{P = \sqrt{2q}\, e^{-a}\sin p}$$

is a canonical transformation.

Sol. We can find

$$dQ = (2q)^{-1/2}e^{a}\cos p\, dq - (2q)^{1/2}e^{a}\sin p\, dp$$

So that.

$$pdQ - pdq = \sin p\sqrt{2q}\, e^{-a}[(2q)^{-1/2}e^{a}\cos p dq - (2q)^{1/2}e^{a}\sin p dp] - pdq$$

$$= \sin p[\cos p dq - 2q\sin p dp] - pdq$$

$$= (\sin p\cos p - p)\,dq - 2q\sin^2 p dp$$

$$= \left(\frac{1}{2}\sin 2p - p\right)dq - 2q\sin^2 p dp$$

$$= \frac{\partial}{\partial q}\left(\frac{1}{2}q\sin 2p - pq\right)dq + \frac{\partial}{\partial p}\left(\frac{1}{2}q\sin 2p - pq\right)dp$$

$$= \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp$$

$$= dF$$

$$= \text{Exact differential}$$

$$\text{where } F = \left(\frac{1}{2}q\sin 2p - pq\right)$$

Hence the transformation is canonica

Q. 11. Show that the transformation

$$\mathbf{Q = \log\left(1 + \sqrt{q}\cos p\right)}$$

$$\mathbf{P = 2\sqrt{q}\left(1 + \sqrt{q}\cos p\right)\sin p}$$

is canonical. Find the generating function F (p, Q)

Sol. We can find

$$dQ = \frac{\cos p dq - 2q\sin p dp}{2q^{1/2}(1 + q^{1/2}\cos p)}$$

Here,

$$pdq - PdQ = pdq - 2q^{1/2}\left(1 + \sqrt{q}\cos p\right)\sin p\cdot\frac{\cos p dq - 2q\sin p dp}{2q^{1/2}(1 + q^{1/2}\cos p)}$$

$$= pdq - \sin p\cos p dq + 2q\sin^2 p dp$$

$$= \left(p - \frac{1}{2}\sin 2p\right)dq + 2q\sin^2 p dp$$

$$= \left(p - \frac{1}{2}\sin 2p\right)dq + q(1 - \cos 2p)\,dp$$

$$= d\left\{q\left(p - \frac{1}{2}\sin 2p\right)\right\}$$

$$= \text{An exact differential}$$

Hence given transformation is canonical.

Now we have $F(p, Q) = F_3(p, Q)$ which satisfies the canonical equation.

$$\frac{\partial F_3}{\partial p} = -q\ ;\ \frac{\partial F_3}{\partial Q} = -p$$

Also from

$$Q = \log\left(1+\sqrt{q}\cos p\right), \text{ we get}$$

$$q = (e^Q - 1)^2 \sec^2 p$$

Hence $\frac{\partial F_3}{\partial p} = -q = -(e^Q-1)^2 \sec^2 p$

Integrating

$$F_3 = \int -(e^Q-1)^2 \sec^2 p\,dp$$

Choosing constant of integration to be zero for a particular choice of reference frame

$$= -\ (e^Q-1)^2 \tan p$$

Hence, the generating function for the transformation is

$$\boxed{F(p, Q) = -(e^Q-1)^2 \tan p}$$

Q. 12. Show that the transformation

$$Q = \frac{1}{p}$$

is canonical

$$P = qp^2$$

Sol. (Using bilinear form)

$$dQ = \frac{\partial}{\partial p}\left(\frac{1}{p}\right) dp + \frac{\partial}{\partial q}\left(\frac{1}{p}\right) dq$$

$$= -\frac{1}{p^2} dp + 0$$

and $\delta Q = \frac{\partial}{\partial p}\left(\frac{1}{p}\right) \delta p + \frac{\partial}{\partial q}\left(\frac{1}{q}\right) \delta q$

$$= -\frac{1}{p^2} \delta p + 0$$

Similarly, $dP = p^2 dq + 2qp dp$

$$\delta P = p^2 \delta q + 2qp\,\delta p$$

So that

$$\delta P dQ - \delta Q dP = (p^2\,\delta q + 2qp\,\delta p)\left(-\frac{1}{p^2}dp\right) - \left(-\frac{1}{p^2}\delta p\right) \times (p^2 dq + 2qp dp)$$

$$= -\delta q dp + \delta p dq$$

$$= (\delta p dq - \delta q dp)$$

This is bilinear form and hence transformation is canonical.

Q. 13. For what values of α and β do the transformation equations

$$Q = q^{\alpha} \cos \beta p$$

$$P = q^{\alpha} \sin \beta p$$

represents a canonical transformation ? Obtain the generating function.

Sol. Transformation is canonical if

$$[Q, P] = 1$$

or $\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1$ ...(1)

From given equations

$$\frac{\partial Q}{\partial q} = \alpha q^{\alpha-1}\cos\beta p\ ;$$

$$\frac{\partial Q}{\partial p} = -\beta q^{\alpha}\sin\beta p$$

$$\frac{\partial P}{\partial p} = \beta q^{\alpha}\cos\beta p\ ;$$

$$\frac{\partial P}{\partial q} = \alpha q^{\alpha-1}\sin\beta p$$

Substituting in equation (1)

$$\alpha q^{\alpha-1}\cos\beta p.\ \beta q^{\alpha}\cos\beta p - \alpha q^{\alpha-1}\sin\beta p.\ -\beta q^{\alpha}\sin\beta p = 1$$

or $\alpha\beta\, q^{2\alpha-1}(\cos^2\beta p + \sin^2\beta p) = 1$

or $\alpha\beta\, q^{2\alpha-1} = 1$

which gives

$$\alpha\beta = 1$$

and $2\alpha - 1 = 0$

Hence $\alpha = \frac{1}{2}$

and $\beta = 2$

Thus, we have

$$Q = q^{1/2}\cos 2p\ ;$$

$$P = q^{1/2}\sin 2p$$

or $q = Q^2 \sec^2 2p\ ;$

$$P = Q\sec 2p \sin 2p$$

$$= Q\tan 2p$$

If F_3 (p, Q) generating function for the transformation

$$\frac{\partial F_3}{\partial p} = -q\ ;\ \frac{\partial F_3}{\partial Q} = -P$$

i.e., $\frac{\partial F_3}{\partial p} = -Q^2\sec^2 2p$

Integrating with respect to p, we get

$$F_3 = -\int Q^2 \sec^2 2p\,dp$$

$$= -\frac{1}{2}Q^2\tan 2p$$

This yield $\frac{\partial F_3}{\partial Q} = -Q \tan 2p$

Hence generating function of the given canonical transformation.

$$F_3(p, Q) = -\frac{1}{2} Q^2 \tan 2p$$

Q. 14. Use Hamilton's equations to find the differential equation for planetary motion and prove that the areal velocity is constant. Assume force $f(r) = -\frac{k}{r^2}$

Sol. The Lagrangian

$$L = T - V$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} \quad \left(V = -\frac{k}{r}\right)$$

Giving $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$;

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

The Hamiltonian

$$H = \sum_i p_i \dot{q}_i - L = p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$= p_r \frac{p_r}{m} + p_\theta \frac{p_\theta}{mr^2} - \frac{1}{2} m \left(\frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2}\right) - \frac{k}{r}$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r}$$

So that,

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = + \frac{p_\theta^2}{mr^3} - \frac{k}{r^2}$$

$$\left.\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \end{aligned}\right\} \quad \ldots(1)$$

These are Hamilton's equation of motion from equation (1), we find that

$$m\ddot{r} = \dot{p}_r$$

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \quad \ldots(2)$$

In motion under central force, angular momentum is conserved.

$\therefore$ We can put $mr^2\dot{\theta} = l = p_\theta$

$\therefore$ From equation (2), we get

$$m\ddot{r} = \frac{l^2}{mr^3} + f(r) \quad \ldots(3)$$

Now let $r = \frac{1}{u}$;

$$\ddot{r} = -\frac{l^2 u^2}{m^2} \frac{d^2 u}{d\theta^2} \quad \ldots(4)$$

Substitute in equation (3)

$$-\frac{l^2u^2}{m} \frac{d^2u}{d\theta^2} = \frac{l^2u^3}{m} + f\left(\frac{1}{u}\right)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right) \quad \ldots(5)$$

Equation (5) is the differential equation for planetary motion.

Further from equation (1), we have

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\Rightarrow \quad p_\theta = \text{Constant}$$

$$\text{or} \quad mr^2\dot{\theta} = p_\theta = \text{Constant}$$

$$\text{or} \quad \frac{1}{2} r^2\dot{\theta} = \text{Constant}$$

which shows that the areal velocity is constant.

Q. 15. Show that the velocity of a planet at any point of its orbit is the same as it would have been if it had fallen that point from rest at a distance from the sun equal to the length of the major axis.

Sol. The path of the planet being an ellipse, its velocity at any time t is given by

$$v^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right) \quad \ldots(1)$$

$a \rightarrow$ Semi major axis of elliptic path.

Equation of motion of particle falling to the sun from a distance $2a$ to the distance r from the sun is

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\frac{1}{x^2}$$

Integrating and multiply both sides by $2\frac{dx}{dt}$.

We have,

$$\left(\frac{dx}{dt}\right)^2 = -2\frac{k}{m}\int_{2a}^{r} \frac{dx}{x^2}$$

$$= -2\frac{k}{m}\left[-\frac{1}{x}\right]_{2a}^{r}$$

$$= \frac{2\frac{k}{m}}{r} - \frac{\frac{k}{m}}{a}$$

or $$V^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right) \quad ...(2)$$

where V is the velocity at $x = r$, and from (1) and (2) it is obvious that

$$V^2 = v^2$$

i.e., $$\boxed{V = v}$$

Q. 16. A particle describes the curve $r^n = a^n \cos n\theta$ under a force P towards the pole. Find the law of force.

Sol. $$r^n = a^n \cos n\theta \quad ...(1)$$

Substitute $r = \frac{1}{u}$

Equation (1), gives

$$1 = a^n u^n \cos n\theta$$

Differentiating with respect to θ

$$0 = a^n \left\{nu^{n-1}\cos n\theta \frac{du}{d\theta} - nu^n \sin n\theta\right\}$$

or $$\frac{du}{d\theta} = u \tan n\theta$$

Again differentiating it

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta + nu\sec^2 n\theta$$

or $$= u\tan^2 n\theta + nu\sec^2 n\theta$$

$$\frac{d^2u}{d\theta^2} + u = u\tan^2 n\theta + nu\sec^2 n\theta + u$$

$$= u(1 + \tan^2 n\theta) + nu\sec^2 n\theta$$

$$= (1 + n)\,u\sec^2 n\theta$$

Equation of motion

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$$

$$\Rightarrow -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right) = (1 + n)\,u\sec^2 n\theta$$

$$f\left(\frac{1}{u}\right) = -\frac{(1 + n)}{m} l^2u^3 \sec^2 n\theta \quad ...(2)$$

From equation (1)

$$u^n = \frac{1}{a^n}\sec n\theta$$

$$\sec^2 n\theta = a^{2n} u^{2n}$$

Equation (2), gives

$$f\left(\frac{1}{u}\right) = -\frac{(1 + n)}{m} l^2 a^{2n} . u^{2n+3}$$

or $$f\left(\frac{1}{u}\right) \propto u^{2n+3}$$

or $$\boxed{f(r) \propto \frac{1}{r^{2n+3}}}$$

i.e., the law of force is inverse $(2n + 3)$rd power of this distance.

Q. 17. If a planet were suddenly stopped in its orbit, supposed circular, show that it would fall into the sun in a time which is $\frac{\sqrt{2}}{8}$ times of its period.

Sol. The period of revolution is

$$\tau = 2\pi\sqrt{\frac{m}{k}}\,a^{3/2} \quad ...(1)$$

$a \rightarrow$ Radius of circular orbit, in this case.

$V = 0$, because the planet is stopped in its orbit.

Therefore, the planet will begin to move towards the centre of sun in the straight line under a law of inverse square law.

Therefore equation of motion becomes

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\frac{1}{x^2}$$

Also from the conservation of energy

$$\frac{1}{2}mv^2 = E - V = -\frac{k}{2a} + \frac{k}{r}$$

since $$V = -\frac{k}{r}$$

$$v^2 = \frac{2k}{m}\left(\frac{1}{x} - \frac{1}{a}\right)$$

or $$\left(\frac{dx}{dt}\right)^2 = \frac{2k}{m}\left(\frac{1}{x} - \frac{1}{a}\right)$$

$$\frac{dx}{dt} = \sqrt{\frac{2k}{m}}\left(\frac{a - x}{ax}\right)^{1/2}$$

or $$\int\left(\frac{ax}{a - x}\right)^{1/2} dx = \sqrt{\frac{2k}{m}}\int dt$$

Substitute $$x = a\sin^2\theta$$

$$dx = 2a\sin\theta\cos\theta$$

$$\int \frac{a\sin\theta}{a^{1/2}\cos\theta} 2a\sin\theta\cos\theta\, d\theta = \sqrt{\frac{2k}{m}}\,t$$

$$\sqrt{\frac{2k}{m}}\,t = 2a^{3/2}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= 2a^{3/2}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= 2a^{3/2}\cdot\frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{2}a^{3/2}$$

or
$$t = \sqrt{\frac{m}{2k}}\cdot\frac{\pi}{2}a^{3/2}$$

$$\boxed{t = \frac{\sqrt{2}}{8}T}$$ using equation (1)

Q. 18. A mass m, moves in a circular orbit of radius r_0 under the influence of a central force whose potential is $-\frac{k}{r^n}$. Show that the circular orbit is stable under small oscillation if $n < 2$.

Sol. Under central force, effective potential is

$$V' = V(r) + \frac{l^2}{2mr^2}$$

A circular orbit is possible at the value of r for which

$$\frac{dV'}{dr} = 0$$

and is stable if

$$\frac{d^2V'}{dr^2} = +\text{ve}$$

Here
$$V' = -\frac{k}{r^n} + \frac{l^2}{2mr^2}$$

$$\frac{dV'}{dr} = \frac{kn}{r^{n+1}} - \frac{l^2}{mr^3} = 0$$

$$\Rightarrow \quad \frac{kn}{r^{n+1}} = \frac{l^2}{mr^3} \qquad \ldots(1)$$

and
$$\frac{d^2V'}{dr^2} = -\frac{kn(n+1)}{r^{n+2}} + \frac{3l^2}{mr^4}$$

$$= -\frac{l^2}{mr^3}\frac{(n+1)}{r} + \frac{3l^2}{mr^4}$$

[using eqn. (1)]

$$= -\frac{(n+1)l^2}{mr^4} + \frac{3l^2}{mr^4}$$

which is positive only if $(n + 1) < 3$ or $n < 2$.

Q. 19. A particle describes a circular orbit given by $r = 2a\cos\theta$ under the influence of an attractive central force directed towards a point on the circle. Show that the force varies as the inverse fifth power of the distance.

Sol. We have $r = 2a\cos\theta$

Putting $r = \frac{1}{u}$

$$u = \frac{\sec\theta}{2a} \qquad \ldots(1)$$

$$\frac{du}{d\theta} = \frac{\sec\theta\tan\theta}{2a};$$

$$\frac{d^2u}{d\theta^2} = \frac{\sec\theta.\sec^2\theta + \sec\theta\tan\theta\tan\theta}{2a}$$

$$= \frac{\sec^3\theta + \sec\theta\tan^2\theta}{2a} \qquad \ldots(2)$$

Differential equation of orbit

$$f\left(\frac{1}{u}\right) = -\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= -\frac{l^2u^2}{m}\left[\frac{\sec^3\theta + \sec\theta\tan^2\theta}{2a} + \frac{\sec\theta}{2a}\right]$$

$$= -\frac{l^2u^2}{m}\left[\frac{\sec^3\theta + \sec\theta(\tan^2\theta + 1)}{2a}\right]$$

$$= -\frac{l^2u^2}{m}\left[\frac{2\sec^3\theta}{2a}\right]$$

From equation (1), we have

$$\sec\theta = 2au$$

$$= -\frac{l^2u^2}{am}[8a^3u^3]$$

$$f\left(\frac{1}{u}\right) = -\frac{8a^2l^2}{m}\cdot u^5$$

$$f(r) = -\frac{8a^2l^2}{m}\frac{1}{r^5}$$

or
$$\boxed{f(r) \propto \frac{1}{r^5}}$$

Q. 20. A charged particle is moving under the influence of point nucleus. Find the orbit of the particle and the periodic time in the case of an elliptical orbit.

Sol. Let $z \to$ Atomic number of the nucleus.

$e \to$ Charge on the electron

Central force between the nucleus and any electron in orbit is

$$F(r) = -\frac{ze^2}{r^2},\ k = ze^2$$

Now we have eccentricity

$$\in = \sqrt{1 + \frac{2EI^2}{mk^2}} = \sqrt{1 + \frac{2EI^2}{mz^2e^4}}$$

But $\quad E = -\frac{k}{2a} = -\frac{ze^2}{2a}$

$$\therefore \quad \in = \sqrt{1 + \frac{2\left(\frac{-ze^2}{2a}\right)I^2}{mz^2e^4}}$$

$$= \sqrt{1 - \frac{I^2}{maze^2}}$$

or $\quad \in < 1$

$\therefore$ The path of an electron moving under the influence of point nucleus is an ellipse.

Now, the period T

$$T^2 = 4\pi^2a^3\frac{m}{k}$$

$$k = ze^2$$

$$T^2 = 4\pi^2a^3\frac{m}{ze^2}$$

$$\therefore \quad \boxed{T = \frac{2\pi}{e}\sqrt{\frac{ma^3}{z}}}$$

Q. 21. Find the horizontal component of the Coriolis force actin; on a body of mass 1·5 kg. moving northward with a horizontal velocity of 100 m/sec, at 30° N latitude on earth.

Sol. $\quad v' = 100 \times 100$ cm/sec. j'

$$= 10^4\, j' \text{ cm/sec}$$

The angular velocity vector in terms of latitude λ is

$$\vec{\omega} = \omega\cos\lambda\, j' + \omega\sin\lambda\, k' \qquad (\lambda = 30°)$$

$$= \omega\frac{\sqrt{3}}{2}j' + \frac{1}{2}\omega k'$$

The Coriolis force is given by

$$\bar{F}_e = -2m(\bar{\omega} \times \bar{v}')$$

$$= -2m\left[\left(\sqrt{\frac{3}{2}}\omega j' + \frac{1}{2}\omega k'\right) \times 10^4 j\right]$$

$$= 2m\omega \times \frac{1}{2} \times 10^4\, i'$$

$$= 2 \times 1{\cdot}5 \times 10^3 \times 7{\cdot}29 \times 10^{-5} \times \frac{1}{2} \times 10^4\, i'$$

$$[\omega = 7{\cdot}29 \times 10^{-5}]$$

$$= 1094 \text{ dynes } i'$$

The horizontal component of Coriolis force is 1094 dynes acting along the east.

Q. 22. In a certain process, a particle of mass m_A changes into two particles of masses m_B and m_C. If $m_A = 2341\, m_e$, $m_B = 273{\cdot}2\, m_e$ and $m_C = 1839\, m_e$, where m_e is the rest mass of the electron, calculate the amount of energy which is generated in the above process.

($m_e = 9{\cdot}11 \times 10^{-31}$ kg and $c = 3 \times 10^8$ m/s)

Sol. The loss of mass when a particle m_A breaks into two particles m_B and m_C is given by

$$\Delta m = m_A - (m_B + m_C)$$

$$= 2341\, m_e - 273{\cdot}2\, m_e - 1839\, m_e$$

$$= 228{\cdot}8\, m_e$$

$$= 228{\cdot}8 \times 9{\cdot}11 \times 10^{-31}$$

$$= 2{\cdot}08 \times 10^{-28} \text{ kg}$$

Energy equivalent to this mass

$$\Delta E = \Delta m{\cdot}C^2 = 2{\cdot}08 \times 10^{-28} \times (3{\cdot}0 \times 10^8)^2$$

$$= 1{\cdot}872 \times 10^{-11} \text{ joule}$$

Q. 23. A hundred μ mesons, each of rest mass 206 electrons and energy 4·75 BeV are produced at an altitude of 30 km. If the mean life of μ-mesons at rest is $2{\cdot}2 \times 10^{-6}$ second, calculate their number expected to reach the sea level 1) allowing for time dilation and (b) neglecting time dilation. Take the electrons to travel vertically downwards without loss of energy. Given the electron rest mass = 0·5 MeV. What conclusion can you draw from your result ?

Sol. The rest mass of a μ-meson is

$$m_0c^2 = 206 \times 0{\cdot}5 \text{ MeV} = 103 \text{ MeV}$$

$$= 0{\cdot}103 \text{ BeV}$$

The kinetic energy of a μ-meson is

$$T = (m - m_0)c^2 = 4{\cdot}75 \text{ BeV}$$

or $$\left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right] m_0c^2 = 4{\cdot}75 \text{ BeV}$$

$$\left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right] \times 0{\cdot}103 \text{ BeV} = 4{\cdot}75 \text{ BeV}$$

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \times 0{\cdot}103 \text{ BeV} = (4{\cdot}75 + 0{\cdot}103) \text{ BeV}$$

$$= 4{\cdot}853$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{4{\cdot}853}{0{\cdot}103} = 47{\cdot}12$$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{47{\cdot}12}$$

$$\frac{v}{c} = \sqrt{1-\left(\frac{1}{47{\cdot}12}\right)^2} \approx 1$$

$$v \approx c$$

(a) Allowing time dilation, the average life of moving μ-mesons is

$$\tau = \frac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= 47{\cdot}12 \times 2{\cdot}2 \times 10^{-6}$$

$$= 1{\cdot}04 \times 10^{-4}\text{ s}$$

If N → Number of μ-mesons reaching the sea level undecayed.

We have

$$N = N_0 e^{-t/\tau}$$

Here $t = \frac{30 \times 10^3\text{ m}}{3 \times 10^8\text{ m/s}}$

$$= 10^{-4}\text{ s}$$

$$N = 100 \times e^{-10^{-4}/1{\cdot}04 \times 10^{-4}}$$

$$= 100e^{-1/1{\cdot}04} = 38$$

(b) Neglecting time dilation, the mean life of moving μ-mesons is to be taken as $2{\cdot}2 \times 10^{-6}$ second.

Thus the number of μ-mesons reaching the sea-level undecayed is

$$N = N_0 e^{-t/\tau_0}$$

$$= 100e^{-100/2{\cdot}2 \times 10^{-6}}$$

$$= 1{\cdot}7 \times 10^{-18}$$

Conclusion

Availability of cosmic rays μ-mesons at sea level can be explained only on the basis of relativistic time dilation.

Q. 24. Discuss the harmonic oscillator problem using Hamilton Jacobi method.

Sol. Let us consider one dimensional harmonic oscillator.

Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

or $H = \frac{p^2}{2m} + \frac{k}{2} q^2$;

where $\frac{k}{m} = \omega^2$

Taking $p = \frac{\partial S}{\partial q}$, the new Hamiltonian and obtained as

$$k = H + \frac{\partial S}{\partial t} = 0 \qquad \ldots(i)$$

or $$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{kq^2}{2} + \frac{\partial S}{\partial t} = 0 \qquad \ldots(ii)$$

Its solution can be written as

$$S(q, \theta, t) = T(q, \theta) - \theta t \qquad \ldots(iii)$$

$$\therefore \frac{1}{2m}\left(\frac{\partial T}{\partial q}\right)^2 + \frac{kq^2}{2} = \theta \qquad \text{[from eqn. (ii)]}$$

Integrate

$$T = \sqrt{mk}\int\sqrt{\left(\frac{2\theta}{k} - q^2\right)}\,dq$$

so, $$S = \sqrt{mk}\int\sqrt{\left(\frac{2\theta}{k} - q^2\right)}\,dq - \theta t$$

$$\therefore \quad \phi = \frac{\partial S}{\partial \theta} = \sqrt{\frac{m}{k}}\int\frac{dq}{\sqrt{\left(\frac{2\theta}{k} - q^2\right)}} - t$$

or $$\phi + t = -\sqrt{\left(\frac{m}{k}\right)}\cos^{-1}\left[q\sqrt{\left(\frac{k}{2\theta}\right)}\right]$$

i.e., $$q = -\sqrt{\frac{2\theta}{k}}\cos\left(\sqrt{\frac{k}{m}}(\phi + t)\right)$$

$$\boxed{q = -\sqrt{\frac{2\theta}{k}}\cos\omega(t + \phi)}$$

Constants : Suppose at $t = 0$.

Particle is stationary and it is displaced from the equilibrium position by the amount q_0 then

$$\left(\frac{\partial S}{\partial q}\right)_0 = p_0 = 0 = \sqrt{2m}\sqrt{\left(\theta - \frac{kq_0^2}{2}\right)}$$

$$\theta + \frac{kq_0^2}{2} = \frac{m\omega^2 q_0}{2}$$

So that, $\boxed{q = q_0 \cos\omega(t + \phi)}$

But at $q = q_0$ at $t = 0 \Rightarrow \phi = 0$

Hence, $\boxed{q = q_0 \cos \omega t}$

Q. 25. A particle of mass m moves in a plane in the field of force given by $\vec{F} = -\hat{e}_r kr \cos\theta$, where k is a constant and $\hat{e}_r$ is the radial unit vector.

(i) Will the angular momentum of the particle about the origin be conserved ? Justify your statement.

(ii) Obtain the differential equation of the orbit of the particle.

Sol. Since radial velocity $= \dot{r}$

transverse velocity $= r\dot{\theta}$

$\therefore$ velocity at any time $= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$

and
$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

(i) Since Lagrangian equation (in θ)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_0$$

or
$$\frac{d}{dt}(mr^2\dot{\theta}) - 0 = 0$$

or
$$mr^2\dot{\theta} = \text{Constant}$$

which implies that the angular momentum about the origin is conserved.

(ii) For differential equation consider Lagrange's r equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

$$\Rightarrow \quad \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = -kr\cos\theta$$

or
$$\boxed{\ddot{r} - r\dot{\theta}^2 = -\left(\frac{k}{m}\right) r\cos\theta}$$

Required differential equation of the orbit of the particle.

Q. 26. Prove that if a rectangular parallelopiped (edges $2a$ and $2b$) rotates about its centre of gravity, its angular velocity about one principal axis is constant and about the other principal axis is periodic, the period being to the period about the first mentioned principal axis as $(b^2 + a^2) : (b^2 - a^2)$.

Sol. We have

$$A = \frac{m}{3}(a^2 + b^2) = B$$
$$C = \frac{2}{3}ma^2$$

Enter dynamical equations are :

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{constant} = x, \text{ say} \quad \ldots(i)$$

$$(a^2 + b^2)\dot{\omega}_1 - (b^2 - a^2)\omega_2 n = 0$$

$$\Rightarrow \quad (a^2 + b^2)\dot{\omega}_1 = (b^2 - a^2)\omega_2 n \quad \ldots(ii)$$

and
$$(a^2 + b^2)\dot{\omega}_2 - (a^2 - b^2)n\omega_1 = 0$$

$$\Rightarrow \quad (a^2 + b^2)\dot{\omega}_2 = -(b^2 - a^2)n\omega_1 \quad \ldots(iii)$$

From equation (ii) [Differentiating]

$$(a^2 + b^2)^2\ddot{\omega}_1 = (b^2 - a^2)\dot{\omega}_2 n$$

or
$$\dot{\omega}_2 = \frac{(a^2 + b^2)}{(b^2 - a^2)n}\ddot{\omega}_1$$

Now substitute this value of $\dot{\omega}_2$ in equation (iii) we get,

$$\frac{(a^2 + b^2)}{(b^2 - a^2)n}\ddot{\omega}_1 = -(b^2 - a^2)n\omega_1$$

or
$$\ddot{\omega}_1 + \frac{(b^2 - a^2)^2 n^2}{(a^2 + b^2)^2}\omega_1 = 0$$

$\therefore$ Periodic time
$$T_2 = \frac{2\pi(a^2 + b^2)}{n(b^2 - a^2)}$$

But
$$T_1 = \frac{2\pi}{n}$$

$$\therefore \quad \boxed{\frac{T_2}{T_1} = \frac{a^2 + b^2}{b^2 - a^2}}$$

Q. 27. An artificial satellite revolves around the earth in a circular orbit at a height H above earth's surface. Calculate the period of revolution of the satellite, so that the astronaut in it may be in a state of weightlessness.

Sol. The state of weightlessness will result when the centrifugal force just balance the earth's pull. Let the radius of the earth be R and let v_0 be the speed of satellite for weightlessness; then

Centrifugal force = Earth's pull

i.e.
$$\frac{mv_0^2}{R + H} = \frac{GMm}{(R + H)^2}$$

where m, M are respectively mass of the particle and that of earth.

Then $$v_0^2 = \frac{GM}{(R+H)} = \frac{gR^2}{(R+H)}$$

Since $$g = \frac{GM}{R^2}$$

$$\Rightarrow \quad v_0 = \left[\frac{gR^2}{R+H}\right]^{1/2}$$

Period of revalution for this orbit is given by

$$T_0 = \frac{2\pi(R+H)}{v_0}$$

$$= \frac{2\pi}{R}\sqrt{\frac{(R+H)^3}{g}}$$

Q. 28. In an orbit described under a force to a centre the velocity at any point is inversely proportional to the distance of the point from the centre of force. Show that the path is an equiangular spiral.

Sol. Let v be the velocity of the particle at any point at a distance r from the centre of force, then we have

$$v = \left(\frac{\mu}{r}\right) \quad \text{...(i)}$$

Since $$v = \frac{h}{p} \quad \text{...(ii)}$$

where p is the length of the perpendicular from pole to the tangent at any point of the path.

(i) and (ii) gives

$$\frac{\mu}{r} = \frac{h}{p}$$

$$\Rightarrow \quad p = \left(\frac{h}{\mu}\right)r$$

or $$\boxed{p = ar}$$

where $$a = \frac{h}{\mu}$$

So $p = ar$ is the pedal equation of equiangular spiral.

Q. 29. If the Hamiltonian H is independent of t explicitly prove that it is (a) a constant and (b) is equal to the total energy of the system.

Sol. (a) $$\frac{dH}{dt} = \sum_{\alpha=1}^{n}\frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha + \sum_{\alpha=1}^{n}\frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha$$

$$= \sum_{\alpha=1}^{n}\dot{q}_\alpha\dot{p}_\alpha + \sum_{\alpha=1}^{n}(-\dot{p}_\alpha)\dot{q}_\alpha$$

$$\left[\text{since } \frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha, \ \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha\right]$$

$$\Rightarrow \quad H = \text{Constant} = E, \text{ say} \quad \text{...(i)}$$

(b) By Euler's theorem on homogeneous functions, we have

$$\sum_{\alpha=1}^{n}\dot{q}_\alpha\frac{\partial T}{\partial \dot{q}_\alpha} = 2T \quad \text{...(ii)}$$

But $$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial(T-V)}{\partial \dot{q}_\alpha}$$

$$= \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial V}{\partial \dot{q}_\alpha} = \frac{\partial T}{\partial \dot{q}_\alpha}$$

$$\left[\text{Since } \frac{\partial V}{\partial \dot{q}_\alpha} = 0 \text{ as V is independent of } \dot{q}_\alpha\right]$$

equation (ii) gives

$$= \sum_{\alpha=1}^{n}\dot{q}_\alpha p_\alpha = 2T$$

Hence

$$H = \sum_{\alpha=1}^{n}p_\alpha\dot{q}_\alpha - L$$

$$= 2T - L = 2T - (T - V)$$

$$= T + V = E$$

Q. 30. Derive an expression for the relativistic Hamiltonian for an electron of mass m, momentum $\vec{p}$ moving in an external electromagnetic field ($\vec{A}$, ϕ).

Sol. In the absence of the electromagnetic field, the relativistic invariance relation is

$$p_\mu^2 = -m^2c^2$$

In the presence of an external electromagnetic field,

$$p_\mu \rightarrow p_\mu - \frac{e}{c}A_\mu$$

$$\therefore \quad \left(p_\mu - \frac{e}{c}A_\mu\right)^2 = -m^2c^2$$

Now $$A_0 = \phi, \quad E = cp_0$$

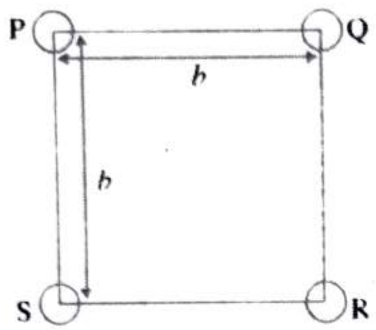
$$\therefore \quad \left(\frac{E}{c} - \frac{e}{c}\phi\right)^2 = \left(m^2c^2 - \frac{e}{c}\vec{A}\right)^2$$

By identifying E with the Hamiltonian H, we have

$$H = e\phi + \sqrt{m^2c^4 + (\vec{p}c - eA)^2}$$

Q. 31. Four spheres each of diameter $2a$ and mass m are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about any side of the square.

Sol. The four spheres of same mass m and radius are placed at the corners of the square PQRS.



MI of P and Q spheres about axis PQ

$$= \frac{2}{5} ma^2 + \frac{2}{5} ma^2$$

$$= \frac{4}{5} ma^2$$

$\Rightarrow$ For R and S sphere about axis RS

$$= \frac{4}{5} ma^2$$

Since the axis PQ is parallel to SR are at a distance b, therefore M.I. of S and R spheres about the axis PQ

$$= \frac{4}{5} ma^2 + 2mb^2$$

Therefore total M.I. of the system about the axis PQ is

$$= \frac{4}{5} ma^2 + \frac{4}{5} ma^2 + 2mb^2$$

$$= \frac{8}{5} ma^2 + 2mb^2$$

Q. 32. Calculate the inertia tensor for a rigid body consists of three particles of masses 2, 1, 4 gram located at (1, –1, 1), (2, 0, 2), (–1, 1, 0) cm respectively.

Sol. $I_{xx} = \sum_{i=1}^{2} m_i (y_i^2 + z_i^2)$

$$= 2[(-1)^2 + (1)^2] + 1[0^2 + 2^2] + 4[1^2 + 0^2]$$

$$= 4 + 4 + 4 = 12 \text{ gm-cm}^2$$

Similarly

$$I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$$

$$= 2\{1^2 + 1^2] + 1[2^2 + 2^2] + 4[(-1)^2 + 0^2]$$

$$= 4 + 8 + 4$$

$$= 16 \text{ gm-cm}^2$$

and

$$I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$$

$$= 2[1^2 + (-1)^2] + 1[2^2 + 0^2] + 4[(-1)^2 + 1^2]$$

$$= 4 + 4 + 8$$

$$= 16 \text{ gm-cm}^2$$

Also

$$I_{xy} = -\sum_{i=1}^{3} m_i x_i y_i = I_{yx}$$

$$= -[2 \times 1 \times -1 + 1 \times 2 \times 0 + 4 \times -1 \times 1]$$

$$= -[-2 + 0 - 4] = 6 \text{ gm–cm}^2$$

Similarly

$$I_{xz} = I_{zx} = -\sum_{i=1}^{3} m_i x_i z_i$$

$$= -[2 \times 1 \times 1 + 1 \times 2 \times 2 + 4 \times -1 \times 0]$$

$$= -[2 + 4] = -6 \text{ gm–cm}^2$$

$$I_{yz} = I_{zy} = -\sum m_i y_i z_i$$

$$= -[2 \times -1 \times 1 + 1 \times 0 \times 2 + 4 \times 1 \times 0]$$

$$= 2 \text{ gm–cm}^2$$

Hence the inertia tensor I is

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix}$$

Q. 33. Find the canonical transformation defined by the generating function

$$\mathbf{F_1 (q, Q) = qQ - \frac{1}{2} m\omega q^2 - Q^2/4m\omega}$$

what happens to the Hamiltonian of a simple harmonic oscillator when transformed from p, q co-ordinate to P, Q co-ordinates.

Sol. For F transformation equation are

$$p = \frac{\partial F_1}{\partial q}$$

and $$P = -\frac{\partial F_1}{\partial Q}$$

$\therefore$ $$p = \frac{\partial}{\partial q}\left[qQ - \frac{1}{2} m\omega q^2 - \frac{Q^2}{4m\omega}\right]$$

$$= Q - m\omega q$$

or $$\boxed{Q = p + m\omega q}$$

Now $$P = -\frac{\partial}{\partial Q}\left[qQ - \frac{1}{2}m\omega q^2 - \frac{Q^2}{4m\omega}\right]$$

$$= -q + \frac{2Q}{4m\omega}$$

$$= -q + \frac{1}{2m\omega}[p + m\omega q]$$

$$= -q + \frac{p}{2m\omega} + \frac{q}{2}$$

$$\boxed{P = \frac{p - m\omega q}{2m\omega}}$$

To find H'

Now find the transformed equation

$$Q = p + m\omega q$$

$$2m\omega P = p - m\omega q$$

Adding

$$Q + 2m\omega P = 2p$$

or $$p = \frac{Q + 2m\omega P}{2}$$

and $$q = \frac{Q - 2m\omega P}{2m\omega}$$

Hamiltonian for simple harmonic oscillator

$$H' = \frac{1}{2}m\,[p^2 + m^2\omega^2 q^2]$$

$$= \frac{1}{2m}\left[\frac{(Q + 2m\omega P)^2}{4} + \frac{(Q - 2m\omega P)^2}{4}\right]$$

$$= \frac{1}{2m}\left[\frac{2Q^2 + 8\,m^2\omega^2P^2}{4}\right]$$

$$\boxed{H' = \frac{Q^2 + 4m^2\omega^2P^2}{4m}}$$

Q. 34. Find the canonical transformations defined by the generating functions

$$F_3\,(Q, p) = -\,(e^Q - 1)^2 \tan p$$

what happens to the Hamiltonian of a simple harmonic oscillator when transformed from p, q co-ordinates to P, Q co-ordinates for each of these transformations.

Sol. Transformation equation for F_3

$$P = -\frac{\partial F_3}{\partial Q}$$

and $$q = -\frac{\partial F_3}{\partial p}$$

$\Rightarrow$ $$P = -\frac{\partial}{\partial Q}[-(e^Q - 1)^2 \tan p]$$

$$= 2\,(e^Q - 1)\tan p$$

and $$q = -\frac{\partial}{\partial p}[-(e^Q - 1)^2 \tan p]$$

$$q = (e^Q - 1)^2\,\frac{1}{\cos^2 p}$$

or $$\sqrt{q}\cos p = (e^Q - 1)^2$$

or $$e^Q = 1 + \sqrt{2}\cos p$$

or $$\boxed{Q = \ln\,(1 + \sqrt{q}\cos p)}$$

and $$P = 2\,(e^Q - 1)\tan p$$

$$= 2\,(\sqrt{q}\cos p)\,\frac{\sin p}{\cos p}$$

$$\boxed{P = 2\sqrt{q}\sin p}$$

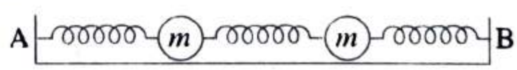
Now find out the inverse transformation in q and p and substitute in Simple harmonic oscillator Hamiltonian.

$$H' = \frac{1}{2m}[p^2 + m\omega^2 q^2]$$

We find

$$H' = \frac{1}{2m}\left\{\tan^{-1}\left[\frac{P}{2e^Q\,(e^Q - 1)}\right]\right\}^2 + \frac{1}{2}m\omega^2\left[(e^Q - 1)^2 + \frac{P^2}{4e^{2Q}}\right]^2$$

Q. 35. Find out the Lagrangian for two equal masses connected by springs having each force constant C (fig.) The masses are free to slide on a frictionless table AB. The walls are at A and B to which the ends of the springs are fixed. Find the normal frequencies of the system.



Sol.

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$V = \frac{1}{2}Cx_1^2 + \frac{1}{2}Cx_2^2 + \frac{1}{2}C\,(x_1 - x_2)^2$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}Cx_1^2 - \frac{1}{2}Cx_2^2 - \frac{1}{2}C\,(x_1 - x_2)^2$$

Now for normal frequencies, we use

$$|V - \omega^2 T| = 0$$

where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

and $$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

Compare $$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$$

With $$T = \frac{1}{2} [T_{11}\dot{x}_1^2 + T_{22}\dot{x}_2^2]$$

$$T_{11} = m,$$
$$T_{22} = m$$

Now $$V_{11} = \left(\frac{\partial^2 V}{\partial x_1^2}\right) = 2C$$

$$V_{22} = \left(\frac{\partial^2 V}{\partial x_2^2}\right) = 2C$$

$$V_{12} = V_{21} = \left(\frac{\partial^2 V}{\partial x_1 \partial x_2}\right) = -C$$

Condition becomes

$$\begin{bmatrix} 2C & -C \\ C & 2C \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = 0$$

or $$\begin{vmatrix} 2C - m\omega^2 & -C \\ -C & 2C - m\omega^2 \end{vmatrix} = 0$$

or $$(2C - m\omega^2)^2 - C^2 = 0$$

or $$C^2 - 2m\omega^2 C + m^2\omega^4 - C^2 = 0$$

$$m\omega^2 (-C + m\omega^2) = 0$$

$\Rightarrow$ $$\omega = 0, 0$$

Either $$\omega = \pm\sqrt{\frac{C}{m}}$$

Q. 36. Lagrangian of two coupled oscillators of mass m each is

$$\mathbf{L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} m\omega_0^2 (x_1^2 + x_2^2) + m\omega_0^2 \mu x_1 x_2}$$

Find out the equations of motions and the normal modes of the system.

Sol. The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

for x_1

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\left[\frac{1}{2} 2m\dot{x}_1\right] - \left[- m\omega_0^2 x_1 + m\omega_0^2 \mu x_2\right] =$$

or $$m\ddot{x}_1 + m\omega_0^2 x_1 - m\omega_0^2 \mu x_2 = 0$$

or $$\boxed{\ddot{x}_1 + \omega_0^2 x_1 = \omega_0^2 \mu x_2}$$

Similarly for x_2

$$\boxed{\ddot{x}_2 + \omega_0^2 x_2 = \omega_0^2 \mu x_1}$$

For Normal Modes of System

We use condition

$$|V - \omega^2 T| = 0$$

where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

and $$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where $$V_{11} = \left(\frac{\partial^2 V}{\partial x_1^2}\right)_0,$$

$$V_{12} = \left(\frac{\partial^2 V}{\partial x_1 \partial x_2}\right)_0 = V_{21}$$

$$V_{21} = \left(\frac{\partial^2 V}{\partial x_2^2}\right)_0$$

In the given equation

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$$

and

$$V = \frac{1}{2} m\omega_0^2 (x_1^2 + x_2^2) - m\omega_0^2 \mu x_1 x_2$$

$\therefore$ $$V_{11} = m\omega_0^2$$

$$V_{12} = -m\omega_0^2 \mu = V_{21}$$

$$V_{22} = m\omega_0^2$$

and for T_{11}, T_{22} etc., we compare our T with

$$T = \frac{1}{2} (T_{11}\dot{x}_1^2 + T_{22}\dot{x}_2^2 + T_{12}\dot{x}_1\dot{x}_2 + T_{21}\dot{x}_2\dot{x}_1)$$

We get

$$T_{11} = m \; ; \quad T_{22} = m$$

$$T_{12} = 0 = T_{21}$$

Hence

$$\begin{pmatrix} m\omega_0^2 & -m\omega_0^2\mu \\ -m\omega_0^2\mu & m\omega_0^2 \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = 0$$

or $$\begin{vmatrix} m\omega_0^2 - m\omega^2 & -m\omega_0^2\mu \\ -m\omega_0^2\,\mu & m\omega_0^2 - m\omega^2 \end{vmatrix} = 0$$

or $$\begin{vmatrix} m\,(\omega_0^2 - \omega^2) & -m\omega_0^2\,\mu \\ -m\omega_0^2\,\mu & m\,(\omega_0^2 - \omega^2) \end{vmatrix} = 0$$

By solving we get

$$m^2\,(\omega_0^2 - \omega^2) - m^2\omega_0^4\,\mu^2 = 0$$

or $$\omega_0^4 - 2\omega_0^2\,\omega^2 + \omega^4 - \omega_0^4\,\mu^2 = 0$$

$$\omega^4 - 2\omega_0^2\,\omega^2 - (\mu^2 - 1)\,\omega_0^4 = 0$$

Quadratic in ω^2

Hence $$\omega^2 = \frac{2\omega_0^2 \pm \sqrt{4\omega_0^4 + 4\,(\mu^2 - 1)\,\omega_0^4}}{2}$$

or $$\omega^2 = \omega_0^2 \pm \sqrt{\mu^2\,\omega_0^4}$$

$$\omega^2 = \omega_0^2\,(1 \pm \mu)$$

$$= \omega_0^2\,(1 + \mu)\,;\ \omega_0^2\,(1 + \mu)$$

or $$\omega = \pm\,\omega_0\,\sqrt{(1 + \mu)}\,;\ \pm\,\omega_0\sqrt{1 - \mu}$$

Q. 37. (a) A Particle slides on the inside of a smooth vertical paraboloid of revolution $r^2 = az$. Show that the constraint force has a magnitude = constant . $\left(1 + \frac{4r^2}{a^2}\right)^{-3/2}$. What is its direction ?

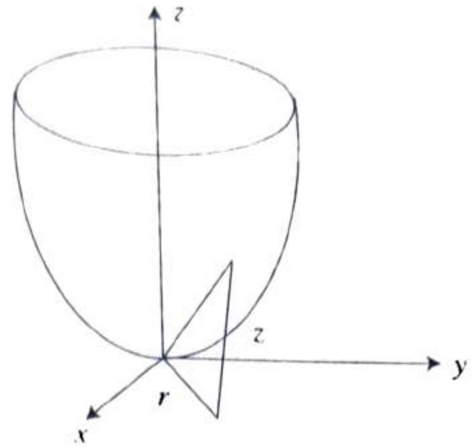
(b) A particle of mass m is acted on by a force whose potential is V(r).

(1) Set up the Lagrangian function in a spherical co-ordinate system which is rotating with angular velocity ω about the z-axis.

(2) Show that your Lagrangian has the same form as in a fixed co-ordinate system with the addition of a velocity dependent potential U(which gives the centrifugal and coriolis forces).

(3) Calculate from U the components of the centrifugal and coriolis forces in the radial (r) and azimuthal (ϕ) directions.

Sol. (a) Use cylindrical co-ordinates (r, ϕ, z) as shown in figure.



In cartesian co-ordinates the particle, mass m_1 has co-ordinate ($r\cos\phi$, $r\sin\phi$, z)

velocity ($\dot{r}\cos\phi - r\dot{\phi}\sin\phi$, $\dot{r}\sin\phi + r\dot{\phi}\cos\phi$, $\dot{z}$) and hence Lagrangian

$$L = T - V = \frac{1}{2}\,m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

The constraint equation is

$$f(r, \phi, z) = -r^2 + az = 0$$

or $$-2rdr + adz = 0$$

Lagrange's equations

$$\frac{\partial}{dt}\frac{\partial L}{d\dot{q}_i} - \frac{\partial L}{\partial q_j} = Q_i$$

Where Q_i are the generalised forces of constraint, then give, making use of Lagrange's undetermined multiplier λ.

$$m\ddot{r} - mr\dot{\phi}^2 = -2r\lambda \qquad \dots(1)$$

$$m\ddot{z} + mg = a\lambda \qquad \dots(2)$$

$$mr^2\dot{\phi} = \text{Constant} = J, \text{ say} \qquad \dots(3)$$

The equation of constraint $z = \frac{r^2}{a}$ gives

$$\dot{z} = \frac{2r\dot{r}}{a},\ \ddot{z} = \frac{2r\ddot{r}}{a} + \frac{2\dot{r}^2}{a} \qquad \dots(4)$$

Using equation (3) and (4), we rewrite the total energy

$$E = \frac{1}{2}\,m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) + mgz$$

which is conserved, as

$$\dot{r}^2 = \left(\frac{2E}{m} - \frac{J^2}{m^2r^2} - \frac{2gr^2}{a}\right)\left(1 + \frac{4r^2}{a^2}\right)^{-1} \quad ...(5)$$

and equation (2) as

$$\frac{m}{a}(2r\ddot{r} + 2\dot{r}^2) + mg = a\lambda$$

Making use of equations (1) & (3), this becomes

$$a\lambda\left(1 + \frac{4r^2}{a^2}\right) = \frac{2m\dot{r}^2}{a} + mg + \frac{2J^2}{mar^2}$$

Expression (5) then reduces it to

$$\lambda = \left(\frac{4E}{a^2} + \frac{8J^2}{ma^4} + \frac{mg}{a}\right)\left(1 + \frac{4r^2}{a^2}\right)^{-2}$$

$$= \text{constant.}\left(1 + \frac{4r^2}{a^2}\right)^{-2}$$

The force of constraint is thus

$$f = -2r\lambda c_r + a\lambda c_z$$

of magnitude

$$f = a\lambda\sqrt{1 + \frac{4r^2}{a^2}}$$

$$= \text{constant.}\left(1 + \frac{4r^2}{a^2}\right)^{-3/2}$$

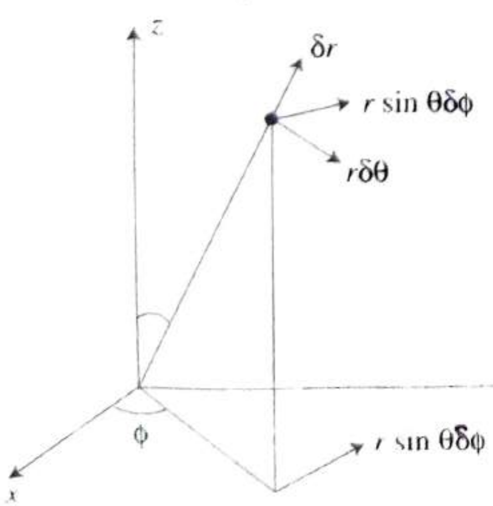
This force is in the *rz*-plane and is perpendicular to the inside surface of the paraboloid.

[It makes an angle *arc* tan $\left(\frac{-a}{2r}\right)$ with the *r*-axis while the slope of the parabola is $\frac{2r}{a}$].

(b) Now spherical co-ordinates (r, θ, ϕ), an infinitesimal displacement of the particle can be resolved as

$$\delta\bar{r} = (\delta r, r\delta\theta, r\delta\phi \sin\theta) \text{ and its velocity}$$

as $\dot{r} = (\dot{r}, r\dot{\theta}, r\dot{\phi}\sin\theta)$



(i) Suppose the co-ordinate frame rotates with angular velocity ω about the *z*-axis. Then the velocity of the particle with respect to a fixed frame is

$$v' = \dot{r} + \omega \times r$$

So, the kinetic energy of the particle is

$$T = \frac{1}{2}m\,[\dot{r}^2 + 2\dot{r}.\omega \times r + (\omega \times r)^2]$$

Referring to the rotating frame and using spherical co-ordinates we have

$$\bar{r} = (r, 0, 0)$$

$$\omega = (\omega\cos\theta, -\omega\sin\theta, 0)$$

$$\bar{\omega} \times \bar{r} = (0, 0, \omega r\sin\theta)$$

$$2\dot{r}\,.\,\bar{\omega} \times \bar{r} = 2\omega r^2\dot{\phi}\sin^2\theta$$

$$(\bar{\omega} \times \bar{r})^2 = \omega^2 r^2 \sin^2\theta$$

$$\dot{\bar{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 + \sin^2\theta$$

Hence $L = T - V$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta + 2\omega r^2\dot{\phi}\sin^2\theta + \omega^2 r^2\sin^2\theta) - V(r)$$

Note that this is the Lagrangian of the particle with respect to a fixed frame, which is to be used in Lagrange's equations, using co-ordinates referring to the rotating frame.

(2) The Lagrangian can be written as

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - U - V$$

with $U = -\frac{1}{2}m(2\omega r^2\dot{\phi}\sin^2\theta + \omega^2 r^2\sin^2\theta)$

Thus L has the form of the Lagrangian the particle would have if the co-ordinate frame referred to were fixed and the particle were under a potential U + V *i.e.* with an additional velocity-dependent potential U.

(3) With the Lagrangian as

$$L = T' - U - V = L' - U$$

where $T' = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$

$$L' = T' - V$$

are the kinetic and Lagrangian the particle would have if the co-ordinate frame referred to were fixed. Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

can be written as

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial U'}{\partial \dot{q}_i}\right) - \frac{\partial U'}{\partial q_i} = Q_i'$$

Q_i' are the generalized forces that have to be introduced because of the fact that the frame referred to is rotating. Differentiating U we find

$$Q_r' = 2m\omega r \dot{\phi} \sin^2 \theta + m\omega^2 r \sin^2\theta$$

$$Q_\theta' = 2m\omega r^2 \dot{\phi} \sin\theta \cos\theta + m\omega^2 r^2 \sin\theta \cos\theta$$

$$Q_\phi' = -2m\omega r \dot{r} \sin^2\theta - 2m\omega r^2 \dot{\theta} \sin\theta \cos\theta$$

The generalised components Q_j' of a force F′ are defined by

$$F'.\delta r = \sum_j Q_j' \delta q_j$$

i.e. $F_r \delta r + F_\theta r \delta_\theta + F_\phi r \sin\theta \delta\phi$

$$= Q_r \delta r + Q_\theta \delta\theta + Q_\phi \delta\phi$$

Hence

$$F_r = Q_r' = 2m\omega r \dot{\phi} \sin^2\theta + m\omega^2 r \sin^2\theta$$

$$F_\theta = \frac{Q_\theta'}{r} = 2m\omega r \dot{\phi} \sin\theta \cos\theta + m\omega^2 r \sin\theta \cos\theta$$

$$F_\phi = \frac{Q_\phi'}{r \sin\theta} = -2m\omega \dot{r} \sin\theta - 2m\omega r \dot{\theta} \cos\theta$$

are the components of the centrifugal and coriolis forces in the directions of e_r, e_θ, e_ϕ. Note that the velocity dependent terms are due to the coriolis force while the remaining terms are due to the centrifugal force.

Q. 38. Consider the Lagrangian

$$L = \frac{1}{2} m(\dot{x}^2 - \omega^2 x^2) e^{\gamma t}$$

for the motion of a particle of mass m in one dimension (x). The constant m, γ and ω are real and positive.

(a) Find the equation of motion.

(b) Interpret the equation of motion by stating the kinds of force to which the particle is subject.

(c) Find the canonical momentum and from this construct the Hamiltonian function.

(d) Is the Hamiltonian a constant of motion ? Is the energy conserved ? Explain.

(e) For the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$. What is $x(t)$ asymptotically as $t \to \infty$.

Sol. (a) Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

gives the equation of motion

$$\ddot{x} + \omega^2 x = -\gamma \dot{x}$$

(b) The particle moves as a damped harmonic oscillator. It is subject to an restoring force $-m\omega^2 x$ and a damping force $-m\gamma \dot{x}'$ proportional to its speed.

(c) The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = me^{\gamma t}\dot{x}$$

and the Hamiltonian is

$$H = p\dot{x} - L$$

$$= me^{\gamma t}\dot{x}^2 - \frac{1}{2} me^{\gamma t} \dot{x}^2 + \frac{1}{2} me^{\gamma t} \omega^2 x^2$$

$$= \frac{p^2 e^{-\gamma t}}{2m} + \frac{1}{2} m \omega^2 x^2 e^{\gamma t}$$

(d) Since H depends explicitly on time, it is not a constant of motion. It follows that energy is not conserved also. Physically, in the course of the motion, the damping force continually does negative fork causing dissipation of energy.

(e) Try a solution of type $x = e^{i\Omega t}$ substitution in the equation of motion gives

$$\Omega^2 - i\gamma\Omega - \omega^2 = 0$$

Which has solutions

$$\Omega = \frac{i}{2}(\gamma \pm \sqrt{\gamma^2 - 4\omega^2})$$

Hence $x = A \exp\left[-\frac{1}{2}(\gamma + \sqrt{\gamma^2 - 4\omega^2})t\right] + B \exp\left[-\frac{1}{2}(\gamma - \sqrt{\gamma^2 - 4\omega^2})t\right]$

The initial conditions $x = 0, \dot{x} = v_0$ at $t = 0$

give $B = -A, \quad A = -\frac{v_0}{\sqrt{\gamma^2 - 4\omega^2}}$

If $\gamma < 2\omega$, let $\frac{1}{2}\sqrt{\gamma^2 - 4\,\omega^2} = i\omega_1$, then

$$x = -\frac{v_0}{2i\omega_1} e^{-\frac{\gamma t}{2}} (e^{-i\omega_1 t} - e^{i\omega_1 t})$$

$$= \frac{v_0}{\omega_1} e^{-\gamma t/2} \sin \omega_1 t$$

So that $x \to 0$ as $t \to \infty$.

If $\gamma > 2\omega$, both $\gamma \pm \sqrt{\gamma^2 - 4\omega^2}$ are real and positive.

So that there will be no oscillation and x will decrease monotonically to zero as $t \to \infty$.